Computing the Elementary Symmetric Polynomials of the Multiplier Spectra of z^2 + c

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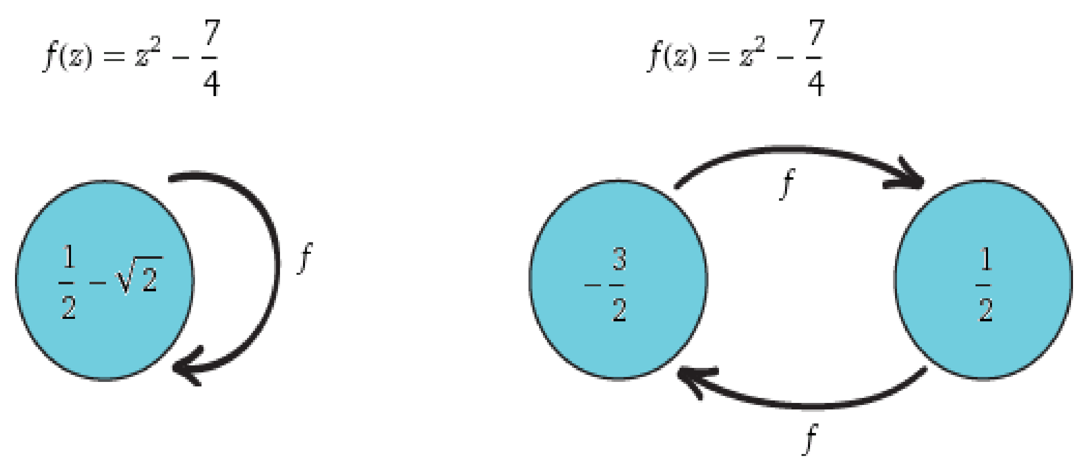
Introduction

This project investigates a family of polynomials that occur in the field of arithmetic dynamics. These polynomials correspond to quadratic rational maps and are difficult to compute in general. We focused on the subfamily of maps $z^2 + c$. Our goal was to create and implement an algorithm to compute the corresponding polynomials and to use the resulting data to hypothesize about patterns in their general forms.

Background

A discrete dynamical system consists of a set and a mapping from that set to itself. With such a map, we can consider compositions of it with itself, a process called **iteration**. One of the primary goals of arithmetic dynamics is to classify the behavior of these iterates and their actions on points. We use the notation f^n to denote the map f composed with itself $n-1$ times. If n is zero, we define f^0 to be the identity mapping. We say a point P is a **periodic point of period n** if it is fixed by the nth iterate of our map, that is, if $f^n(P) = P$.



We call the set of forward images of P its forward **orbit**. If P is periodic, this orbit will have finitely many elements, each of which will also be a periodic point of the same period. Periodic points of period n are always periodic points of period nm for any positive integer m. Because of this, we can further classify periodic points based on their smallest periods If a periodic point P has period n and is not a periodic point for any positive period less than n, we call P a **periodic point of minimal period n**.

If we have a periodic point of minimal period n of a map f, we call $(f^n)'(P)$ the **multiplier** of P.

The polynomials we study are elementary symmetric polynomials of a set of multipliers. The **elementary symmetric polynomials** for a set of N variables are small polynomials whose values are invariant under permutation of their variables. For N variables, there are N elementary symmetric polynomials where the ith elementary symmetric polynomial is the sum of the products over the subsets of size i of the set of N variables.

We define the **n-multiplier spectra** of $z^2 + c$ to be the set of multipliers counted with appropriate multiplicity of the periodic points of minimal period n of $z^2 + c$. All of the elements of the forward orbit of a minimal periodic point will have the same multiplier, so we include only one multiplier per forward orbit.

We define $\boldsymbol{\sigma_i^{(n)}}$ to be the ith elementary symmetric polynomial of the n-multiplier spectra of $z^2 + c$. These are the objects of our study.

We have a powerful result due to Silverman [3] that $\sigma_i^{(n)}$ is a polynomial with integer coefficients in the variable σ_2, the 2nd elementary symmetric polynomial of the multipliers of the fixed points of $z^2 + c$.

From Milnor [1], and Silverman [2], σ_1, σ_2 define an isomorphism from the space of all rational quadratic maps modulo the action by conjugation of the group of Möbius transformations to affine space of dimension 2.

Method

Our first goal was to devise an algorithm to compute these polynomials. To compute an unknown polynomial, we first must determine its degree and then interpolate its coefficients. To compute the degree of $\sigma_i^{(n)}$ we analyze its observed growth rate and compare it to the growth rates of powers of σ_2.

Suppose we are able to compute the value of $\sigma_i^{(n)}$ corresponding to a given value of σ_2. Our algorithm to compute the degree of $\sigma_i^{(n)}$ is as follows:

1. Start with a guess of $t = 1$ for the degree of $\sigma_i^{(n)}$
2. Create a sequence of exponentially increasing values of σ_2: $(\sigma_2)_1, \ldots, (\sigma_2)_k$
3. Compute the respective sequence of values of $\sigma_i^{(n)}(\sigma_2)$: $\sigma_i^{(n)}((\sigma_2)_1), \ldots, \sigma_i^{(n)}((\sigma_2)_k)$
4. Compute the sequence of ratios $\frac{((\sigma_2)_1)^t}{\sigma_i^{(n)}((\sigma_2)_1)}, \ldots, \frac{((\sigma_2)_k)^t}{\sigma_i^{(n)}((\sigma_2)_k)}$
5. If this sequence appears to grow exponentially, then $t - 1$ is the degree of the polynomial. Otherwise increment t and repeat steps 4 - 5.

Suppose we know the degree of $\sigma_i^{(n)}$ to be m, we can use Lagrange interpolation to compute the coefficients of the polynomial:

1. Create a sequence of $m+1$ distinct values of σ_2: $(\sigma_2)_1, \ldots, (\sigma_2)_{m+1}$
2. Compute the respective sequence of values of $\sigma_i^{(n)}(\sigma_2)$: $\sigma_i^{(n)}((\sigma_2)_1), \ldots, \sigma_i^{(n)}((\sigma_2)_{m+1})$
3. Perform Lagrange interpolation on the $m+1$ data points $((\sigma_2)_j, \sigma_i^{(n)}((\sigma_2)_j))$
4. Approximate the coefficients with integers

Both of these methods rely on the ability to compute the value of $\sigma_i^{(n)}$ corresponding to a given value of σ_2. It turns out that for $z^2 + c$, we can write c in terms of σ_2 as $c = \frac{\sigma_2}{4}$. Thus the periodic points of period n, and also the n-multiplier spectra of $z^2 + c$, are functions of σ_2. To compute the value of $\sigma_i^{(n)}$ for a given value of σ_2:

1. Compute the periodic points of minimal period n of $\varphi(z) = z^2 + \frac{\sigma_2}{4}$
2. Compute the multipliers of these periodic points
3. Remove all duplicates in the list of multipliers to emulate the n-multiplier spectra of φ
4. If there aren't enough multipliers, choose a different value of σ_2
5. Otherwise, compute $\sigma_i^{(n)}$ directly by evaluating the ith elementary symmetric polynomial of the list of multipliers

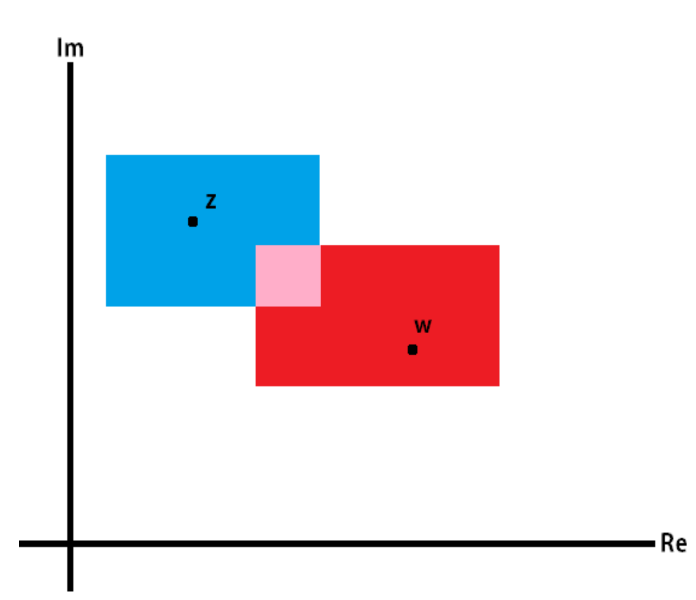
There are cases where we have too few multipliers in our list after we remove duplicates. However, these are rare occurrences, and thus we can safely ignore values of σ_2 that cause a deficiency.

To compute the minimal periodic points of period n of a polynomial map, we find the roots of its nth dynatomic polynomial [2].

Implementation with the Sage computer algebra system

For this project we used the Sage computer algebra system to compute the polynomials [4]. Sage has many tools for computing iterates of maps, multipliers, and dynatomic polynomials.

With any computer algebra system, computing the roots of polynomials of degree greater than four is problematic since most often these roots can only be approximated. Since our algorithm for computing the polynomials requires the removal of duplicates in our list of multipliers, we need duplicate roots to be treated as equal by Sage. With approximations, duplicate roots may be computed differently and thus will not pass equality tests.



A way around this is to treat the roots of a polynomial as intervals in the complex plane. To do this, only the general location within some error bound of the root needs to be known. We can treat these intervals as points themselves and define equality of two points as the intersection of their respective intervals.

With high enough precision, duplicate roots will be computed accurately enough so that their respective intervals intersect, passing required equality tests.

Results

We were able to compute up to the polynomials of the 7-multiplier spectra. Here are the polynomials for the first 4-multiplier spectra:

2-multiplier Spectra

$$\sigma_1^{(2)} = \sigma_2 + 4$$

3-multiplier Spectra

$$\sigma_1^{(3)} = 2\sigma_2 + 16$$
$$\sigma_2^{(3)} = \sigma_2^3 + 8\sigma_2^2 + 16\sigma_2 + 64$$

4-multiplier Spectra

$$\sigma_1^{(4)} = -\sigma_2^2 + 48$$
$$\sigma_2^{(4)} = -\sigma_2^4 - 4\sigma_2^3 + 16\sigma_2^2 + 768$$
$$\sigma_3^{(4)} = \sigma_2^6 + 12\sigma_2^5 + 48\sigma_2^4 + 192\sigma_2^3 + 512\sigma_2^2 + 4096$$

We were also able to hypothesize a formula for the degrees of these polynomials from the data we computed. We have proven one direction to show that our hypothesized formula is always an upper bound for their degrees:

<u>Conjecture</u>: *For all positive integers n, and for all i from 1 to the number of elements of the n-multiplier spectra of $z^2 + c$,* $\deg(\sigma_i^{(n)}) \leq \left\lfloor \frac{in}{2} \right\rfloor$.

<u>Justification</u>: We analyze the growth rates of the periodic points of minimal period n of $z^2 + c = z^2 + \frac{\sigma_2}{4}$ for real values of σ_2. We can determine that they grow asymptotically at the same rate as ${\sigma_2}^{\frac{1}{2}}$. Through the chain rule, the multiplier of a periodic point of minimal period n is the product of the first derivative of $z^2 + c$, which is linear, evaluated at each minimal periodic point in the forward orbit of the point. Thus the multiplier grows asymptotically at the same rate as ${\sigma_2}^{\frac{n}{2}}$. Since the ith elementary symmetric polynomial is the sum of products of i multipliers from the n-multiplier spectra of $z^2 + c$, its degree is at most $\frac{in}{2}$, and since we know $\sigma_i^{(n)}$ is a polynomial, $\deg(\sigma_i^{(n)}) \leq \left\lfloor \frac{in}{2} \right\rfloor$. This argument cannot conclude equality since it is possible for cancellation of the terms in the summation to occur.

References

[1] Milnor, John, *Geometry and Dynamics of Quadratic Rational Maps*. Experiment. Math. 2, 1993.

[2] Silverman, Joseph, *The Arithmetic of Dynamical Systems*. Graduate Texts in Mathematics, vol. 241, Springer, New York, 2007.

[3] Silverman, Joseph, *The Space of Rational Maps on P^1*. Duke Math. J., 1998.

[4] Stein, William and David Joyner, *SAGE: System for Algebra and Geometry Experimentation*. Communications in Computer Algebra (SIGSAM Bulletin), 2005, http://www.sagemath.org