Diffusion of relativistic runaway electrons and implications for lightning initiation

Joseph R. Dwyer

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[1] Using detailed Monte Carlo simulations, the diffusion coefficients for relativistic runaway electron avalanches in air are found for the range of electric field strengths applicable to thunderclouds. Diffusion causes runaway electron avalanches to spread perpendicular to and parallel to the avalanche direction, resulting in much smaller peak conductivities than would be inferred otherwise. The idea that runaway electron avalanches seeded by extensive cosmic ray air showers may initiate lightning has gained considerable popularity in recent years. However, using the diffusion coefficients calculated in this paper along with the avalanche multiplication limit from X-ray and positron feedback, it is found that even $10^{17}$ eV cosmic ray air showers do not produce high enough conductivities to significantly alter the electric field inside a thundercloud. As a result, it will be shown that at present, no compelling theoretical argument exists to suggest that cosmic ray extensive air showers initiate lightning.


1. Introduction

[2] When relativistic electrons move through air with an electric field above $284 \text{kV/m} \times n$, where $n$ is the density of air relative to that at sea level, the rate that the electrons gain energy from the electric field exceeds the rate that the electrons lose energy through interactions with the air [Dwyer, 2004; Wilson, 1925]. As these so-called runaway electrons propagate through air, they produce secondary electrons by hard elastic scattering with atomic electrons. These secondary electrons may also gain energy from the electric field, resulting in an avalanche of runaway electrons that grows exponentially with both time and distance [Gurevich et al., 1992]. Sufficiently strong electric fields for runaway electron avalanche multiplication have been reported inside thunderstorms [e.g., Marshall et al., 2005]. Furthermore, terrestrial gamma ray flash (TGFR) observations [Fishman et al., 1994; Smith et al., 2005; Dwyer and Smith, 2005; Dwyer, 2008, 2009] show that such runaway electron avalanches do indeed occur in our atmosphere.

[3] It has been suggested that runaway electron avalanches seeded by cosmic ray extensive air showers (EASs) could result in enough ionization to initiate lightning [Gurevich et al., 1999]. This hypothesis, which attempts to address the long-standing mystery of how lightning initiates in the relatively low electric fields observed inside thunderstorms [Rakov and Uman, 2003, pp. 82–84], has gained a great deal of attention in recent years, both from the scientific community and from the popular press [Dwyer, 2005a; Gurevich and Zybin, 2005]. Unfortunately, so far, there is little observational evidence and only limited theoretical work to support this idea. In particular, the lack of detailed modeling has hampered observational studies, since it is not clear what ranges of parameters, such as the total air shower energy, are to be investigated.

[4] In order to create detailed models of runaway electron production by cosmic ray extensive air showers and their possible role in lightning initiation, it is necessary to first determine the basic properties of runaway electron avalanches. To date, the avalanche lengths, avalanche speeds, and positron and X-ray feedback rates, all as a function of electric field strengths, have been well studied [Lehtinen et al., 1999; Babich et al., 2001; Dwyer, 2003, 2007, 2008; Coleman and Dwyer, 2006]. However, the diffusion coefficients of the runaway electrons in the avalanche are important parameters that have not yet been determined. In particular, since lateral diffusion causes the avalanches to spread out in the directions perpendicular to the avalanche axis, the larger the lateral diffusion, the lower the density of energetic electrons and the lower the resulting conductivity. In this paper, the lateral (perpendicular to the avalanche axis) and longitudinal (parallel to the avalanche axis) diffusion coefficients of runaway electrons experiencing avalanche multiplication are determined as a function of electric field strength using detailed Monte Carlo simulations. It will be shown that based upon these results, lightning initiation by cosmic ray extensive air showers as...
region used in this study and about 1/100 the number of runaway electrons used at each field value. Examples of runaway electron avalanches for other electric fields are presented by Dwyer [2003, 2004, 2007]. Note the spreading of the avalanche along the $x$ direction as the avalanche propagates in the $+z$ direction. This lateral diffusion is due primarily to elastic scattering of the runaway electrons with air atoms. For higher altitudes, the avalanche would appear the same except the lengths would be scale by $1/n$, where $n$ is the density of air relative to that at sea level.

[7] Although not apparent in Figure 1, runaway electrons also experience a spreading in the $z$ direction, i.e., the avalanche axis, due to elastic scattering and velocity dispersion. Elastic scattering produces variations in the $z$ component of the runaway electron’s velocity, causing a spread in the electrons positions as the runaway electrons propagate. In addition, as new runaway electrons are created in the avalanche and are added to the population, these electrons (which are often injected in the perpendicular direction) will take some time to accelerate and gain energy, allowing the runaway electrons already moving near the speed of light to travel some distance ahead of the slower, newly formed electrons. Finally, runaway electrons that are produced by photon interactions may be created over a wide range of positions and times. It is not immediately obvious that the evolution of the runaway electron distribution in the $z$ direction is well described by diffusion, especially the contribution from the velocity dispersion. However, it will be shown with Monte Carlo calculations that the diffusion approximation works reasonably well in most cases, and longitudinal diffusion coefficients are useful for describing the runaway electron distributions.

[8] There exists more than one approach to modeling elastic scattering [e.g., Lehtinen et al., 1999]. In the Monte Carlo simulations used for this work, elastic scattering of the runaway electrons (and runaway positrons) with atoms with atomic number $Z$ are calculated using the shielded-Coulomb potential

$$V(r) = \frac{Ze}{4\pi \varepsilon_0 a} \exp(-r/a),$$

where in this equation $r$ is the spherical radius. This potential is an approximate expression for the potential derived from the Thomas-Fermi model of the atom when $a = 183.8 \lambda Z^{1/3}$ [Mott and Massey, 1965, pp. 460–464], where $\lambda$ is the Compton wavelength. Elastic scattering of the runaway electrons with the individual atomic electrons (Møller scattering) is also included in the simulation. Møller scattering is important for the development of the avalanche and the production of energetic “knock-on” electrons but is not a major contributor to the spatial diffusion.

[9] As derived by Dwyer [2007], the elastic scattering differential cross section per solid angle is given by

$$\frac{d\sigma}{d\Omega} = \frac{1}{4} \left( \frac{Z_a}{\beta^2 \gamma^2} \right)^2 \left( 1 - \beta^2 \sin^2(\theta/2) \right) \left( \sin^2(\theta/2) + \frac{\beta^2}{\beta^2 \gamma^2} \right)^2,$$

where all the symbols have their usual meaning. The Monte Carlo simulation fully models the runaway electron propagation.
gation by allowing the electrons to scatter according to this cross section.

3. Method of Determining the Diffusion Coefficients

[10] For an arbitrary electric field, the density of runaway electrons (electrons per unit volume) is approximately given by the diffusion-convection transport equation

$$\frac{\partial n_{re}}{\partial t} + \vec{v} \cdot (\vec{v} n_{re}) - \vec{v} \cdot \left( \vec{D} \cdot \vec{v} n_{re} \right) - \frac{n_{re}}{\tau} = n_{s},$$

(3)

where $\vec{v}$ is the average velocity of the runaway electrons [Gurevich and Zybin, 2001; Dwyer, 2005b]. In equation (3), $n_{s}$ is the source function describing the injection rate of energetic electrons, i.e., the number of seed particles per second per cubic meter. The second term on the left-hand side describes the convection of the particles due to their motion in the electric field. The average speed of the avalanche was found by Coleman and Dwyer [2006]. The speed changes slightly with electric field strength but is always fairly close to the value $0.89c$. As a result, $\vec{v} \approx -0.89c\vec{E}/E$, where $\vec{E}$ is the electric field vector. The third term on the left side describes diffusion of the electrons, and the last term on the left describes avalanche multiplication, with $\tau$ being the avalanche e-folding time. The avalanche e-folding time and the related avalanche length are given by the empirical equation [Dwyer, 2003; Coleman and Dwyer, 2006]

$$v_{\tau} \approx \lambda = \frac{7300 \text{ kV}}{(E - 276 \text{ kV/m} \times n)},$$

(4)

where $E$ is measured in kV/m and $n$ is the density of air relative to that at sea level. Equation (4) is valid over the range 300 kV/m $\times n$ to 3000 kV/m $\times n$.

[11] For a uniform electric field in the $-z$ direction, the runaway electron avalanche moves in the $+z$ direction and equation (3) can be simplified somewhat to give

$$\frac{\partial n_{re}}{\partial t} + v \frac{\partial n_{re}}{\partial z} - D_x \frac{\partial^2 n_{re}}{\partial x^2} - D_y \frac{\partial^2 n_{re}}{\partial y^2} - D_z \frac{\partial^2 n_{re}}{\partial z^2} - \frac{n_{re}}{\tau} = n_{s},$$

(5)

where $D_x$ and $D_y$ are the lateral diffusion coefficients and $D_z$ describes the diffusion along the direction of propagation of the avalanche.

[12] If we consider a Dirac delta function source, which corresponds to injecting one seed electron at position and time, $x_o$, $y_o$, $z_o$, and $t_o$, i.e.,

$$n_{s}(x, y, z, t) = \delta(x - x_o)\delta(y - y_o)\delta(z - z_o)\delta(t - t_o),$$

(6)

then the solution is the Green’s function

$$G_{re}(x, y, z; t; x_o, y_o, z_o, t_o) = \frac{1}{(4\pi(t - t_o))^{3/2} D_x^{1/2} D_y^{1/2} D_z^{1/2}} \times \exp \left( \frac{(t - t_o)}{\tau} - \frac{(x - x_o)^2}{4D_x(t - t_o)} - \frac{(y - y_o)^2}{4D_y(t - t_o)} - \frac{(z - z_o)^2}{4D_z(t - t_o)} \right) \frac{S(t - t_o)}{S(t - t_o)},$$

(7)

where the step function, $S$, insures that only times, $t > t_o$, after the injection of the seed particles, are considered. The step function is defined to be 0 for $t < t_o$ and 1 otherwise. By multiplying equation (7) by $z$ and integrating over all positions, the average $z$ position of the runaway electrons at time $t$ is found to be $z = \langle t - t_o \rangle + z_o$, which justifies the interpretation of $v$ in equation (5) as the average component of the velocity of the runaway electrons in the $z$ direction.

[13] The solution for an arbitrary distribution of seed particles, $n_{s}$, may be found by integrating the Green’s function given in equation (7) as follows:

$$n_{re}(x, y, z, t) = \int \int \int G_{re}(x, y, z; t; x_o, y_o, z_o, t_o) \cdot n_{s}(x_o, y_o, z_o, t_o) \, dx_o \, dy_o \, dz_o \, dt_o.$$ 

(8)

For the remainder of this paper we shall consider only a Dirac delta-function source (equation (6)) with the understanding that all quantities can be calculated for an arbitrary source distribution by integrating as in equation (8).

[14] In this paper, Monte Carlo simulations will be used to calculate the runaway electron diffusion coefficients as a function of electric field strength. It will be shown that the distributions of runaway electrons produced by the simulations are well described by the following two equations (equations (9) and (10)), which are then fit to the Monte Carlo data to extract the diffusion coefficients. These equations are derived and discussed in detail in Appendix A.

[15] The first equation is for the fluence (electrons/m$^3$) of runaway electrons passing through a plane, perpendicular to the avalanche direction ($z$ direction). If we assume no external magnetic fields, then there is a natural cylindrical symmetry in the $x$-$y$ directions. As result, we define $D_z = D_x = D_y$, and the cylindrical radius $r^2 = (x - x_o)^2 + (y - y_o)^2$. The fluence is then given by the equation

$$\Phi_{re}(r, z) = \int_{-\infty}^{\infty} F_r(x, y, z, t) \, dt \approx \frac{1}{4\pi (D_z/v)(z - z_o)} \cdot \exp \left( \frac{(z - z_o)^2}{4(D_z/v)(z - z_o)} \right) S(z - z_o),$$

(9)

where $F_r$ is the net runaway electron flux in the $+z$-direction (see equation (A1)). Since the avalanche time,$\tau$, and the average speed of the avalanche, $v$, have been found in previous studies (e.g., see equation (4) above) [Dwyer, 2003; Babich et al., 2005; Coleman and Dwyer, 2006], the only free parameter in equation (9) is the lateral diffusion coefficient, $D_x$. Using the Monte Carlo simulation to record the positions of runaway electrons passing through a plane near the end of the avalanche region, the cylindrical radial distribution of runaway electrons is found, which is then fit to equation (9) to find $D_z$.

[16] In order to find the longitudinal diffusion coefficient, $D_z = D_x$, we shall use the equation for the rate (electrons/s) of runaway electrons passing through a plane, perpendicular to the avalanche direction:

$$R_{re}(z, t) = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} F_r(x, y, z, t) \, dx \, dy \approx \frac{v}{(4\pi(t - t_o)D_z)^{1/2}} \cdot \exp \left( \frac{(t - t_o)}{\tau} - \frac{(v(t - t_o) - (z - z_o))^2}{4D_z(t - t_o)} \right) S(t - t_o).$$

(10)
Calculated runaway electron fluence (electrons/m$^2$) for (top) $E = 350$ kV/m and (bottom) $E = 2500$ kV/m. The diamonds are the calculated data that exclude runaway electron avalanches generated by photon interactions. The triangles are the calculated data for only the runaway electron avalanche involving photon interactions. The solid curves are the fit of the $r$-dependent part of the function in equation (9). The dashed curve, just visible under the solid curve, is the result of numerically integrating equation (7) with respect to time as in equation (A14) without approximations.

Figure 2. Calculated runaway electron fluence (electrons/m$^2$) at the end of the avalanche with length $6A$ for (top) $E = 350$ kV/m and (bottom) $E = 2500$ kV/m. The diamonds are the calculated data that exclude runaway electron avalanches generated by photon interactions. The triangles are the calculated data for only the runaway electron avalanche involving photon interactions. The solid curves are the fit of the $r$-dependent part of the function in equation (9). The dashed curve, just visible under the solid curve, is the result of numerically integrating equation (7) with respect to time as in equation (A14) without approximations.

Note that as with the Green’s function, $R_{re}$ and $\Phi_{re}$ are also functions of the initial position and time of the seed particle. However, for brevity, we shall no longer explicitly list the initial positions and time. As in equation (9), the only free parameter in equation (10) is the longitudinal diffusion coefficient, $D_l$. For a fixed position, $z$, equation (10) describes the rate that runaway electrons reach the $z$ position as a function of time, $t$. Using the Monte Carlo simulation to record the times that the runaway electrons pass through a plane near the end of the avalanche region, the arrival time distribution of runaway electrons is found, which is then fit to equation (10) to find $D_l$.

[17] The diffusion coefficient divided by the speed, which appears in several equations in this paper has units of meters. It is convenient both to calculate and for use in calculations. From equation (7), it follows that the diffusion coefficients divided by $v$ is related to the root-mean-square (RMS) of the $x$, $y$, and $z$ distributions, $\sigma_v$, as follows:

$$
\sigma_v^2 = 2(t - t_o)D_l = 2(z - z_o)D_l/v.
$$

where the index $i$ stands for either $x$, $y$, or $z$ and $\sigma$ is the average position of the runaway electrons at a given time. Here $\sigma_v^2$ is also called the variance.

[18] In order to evaluate how runaway electron avalanches spread laterally and longitudinally, it is necessary to distinguish two main mechanisms: The first is diffusion resulting primarily from scattering of the electrons with air. The second is due to secondary electron production from bremsstrahlung X rays, either by photoelectric production, Compton scattering, or pair production (see Dwyer [2003, 2007] for illustrations of these processes).

[19] Because the mean free path of the photons may be long compared with the avalanche length, this process is often not well described using diffusion equations. Furthermore, secondary avalanches due to X rays can be created both inside and outside the initial avalanche, and so there is no clear dividing line between the initial avalanche and the secondary avalanches. As a result, in this paper, the diffusion coefficients are calculated for the runaway electrons that are not associated with avalanches produced by X rays (or positrons). This topic will be addressed further later in this paper.

[20] Simulations were done by injecting energetic seed electrons into the start of the avalanche region with position $x_o = y_o = z_o = 0$. The energetic seed electrons were drawn randomly from a large sample of runaway electrons, calculated in previous simulations at the same electric field strengths. For this sample, the electron momenta were recorded at the end of the avalanche region, after six avalanche e-folding lengths, where the electrons reached an approximate steady state (exponentially growing) distribution. As a result, in the simulations in which the diffusion coefficients are determined, the initial seed electrons are injected with the approximately the same distribution of energies and directions as the steady state runaway electron avalanches. It is found that with this method, the diffusion coefficients become nearly independent of $z$, i.e., they are the steady state values independent of the details of the seed electron distribution (see Figure 4 and accompanying discussion below).

[21] To extract the diffusion coefficients, the positions of the runaway electrons passing through a plane six avalanche e-folding lengths from the start of the avalanche are recorded. The runaway electrons versus radial distance from the avalanche axis are binned to produce the fluence of runaway electrons (electrons/m$^2$) as a function of $r$. These calculated data are then fit with the radial function given in equation (9) to determine the diffusion coefficient, $D_r/v$. To prevent small fluctuation in the tail of the distribution from adversely affecting the fit (see Figure 2), only values of the runaway electron fluences greater than 1% of the peak values are included in the fit. Similarly, the arrival times of the runaway electrons reaching the plane are binned in time and fit to equation (10) to determine the diffusion coefficient, $D_l/v$. Again, only values of the runaway electron rates greater than 1% of the peak values are included in the fit to prevent the small tail on the side of the distribution (see Figure 3) from overly affecting the fits. With this method, it was found that the values of $\sigma$ found by fitting equations (9) and (10) and using equation (11) were nearly identical to the RMS of the distributions produced by the Monte Carlo simulations.
distribution at longer times. The dashed curve, just visible under the solid curve, uses the rate calculated from the net flux (see equation (A7) and associated discussion).

[24] The development of the avalanche over time and distance can also be used to test equations (9) and (10) versus time and distance. Figure 4 shows the variance, $\sigma^2$, versus time for the longitudinal and lateral directions. From equation (11) above, it can be seen that if equation (5) is correct and the diffusion approximation is valid then we expect that the variance increases linearly with time (and distance). Any deviation from linearity would indicate that equations (9) and (10) are not valid. To create Figure 4, the RMS of the runaway electron arrival times and the RMS of the radial distribution were calculated for a series of planes along the avalanche axis. To reduce any possible effect due to the injection of the seeds into the avalanche region, the RMS from the first plane was subtracted in quadrature from those of the other planes and the times from that first plane were then used. In Figure 4, 19 different electric field strengths are plotted, showing that the treatment of the runaway electron distributions used in this paper are accu-

Figure 3. Calculated runaway electron rate (electrons/s) at the end of the avalanche with length $6\lambda$ for (top) $E = 350$ kV/m and (bottom) $E = 2500$ kV/m. The diamonds are the calculated data that exclude runaway electron avalanches generated by photon interactions. The triangles are the calculated data for only the runaway electron avalanches involving photon interactions. The solid curves are the fit of the function in equation (10). The dashed curve, just visible under the solid curve, is the plot of equation (A7), which used the net flux instead of the convective flux.

Figure 4. Variance, $\sigma^2$, of the runaway electron distribution versus avalanche propagation time (proportional to propagation distance) showing (top) the variance in the longitudinal direction (along the avalanche axis) and (bottom) the variance is the lateral direction. Each includes the data for 19 electric field magnitudes ranging from 300 kV/m up to 3000 kV/m. So that all the data may be plotted together, the variances and the propagation times are all normalized so that their maximum values are all 1. The solid lines indicate a linear relationship, which is expected for the variance resulting from diffusion.
Lateral and longitudinal diffusion coefficients

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at other air densities by converting the actual E to the sea

level equivalent field and then scaling D/v in Figure 5 and

equations (12) and (13) by 1/n, where n is the density of air

relative to that at sea level.

4. Results

[25] Figure 5 shows Dl/v and Dl/v versus electric field

strength found by the Monte Carlo simulations. In Figure 5,

the solid lines are empirical fits to the data:

\[ D_l/v = (5.86 \times 10^3)E^{-1.79}[m], \]  

(12)

which agrees with all the calculated points to within 6%, and

\[ D_l/v = (3.80 \times 10^3)E^{-1.57}[m], \]  

(13)

which agrees with all the calculated points to within 12%. In

equations (12) and (13), E is in units of kV/m and Dl/v and

Dl/v are in meters. To calculate the diffusion coefficients,

Dl and Dl, Dl/v and Dl/v should be multiplied by the

avalanche speed presented by Coleman and Dwyer [2006]

or by the approximate value v = 0.89c. The dotted line in

Figure 5 shows the lateral diffusion coefficient calculated

from the RMS of all runaway electrons including those produced

by photon interactions and those in the tail shown in Figure 2.

However, these values that include the photon interactions

are dependent upon the geometry of the field region

and the number of avalanche lengths present, and so they
do not necessarily represent an intrinsic property of the

runaway electron avalanches. Indeed, relativistic feedback

effects, which will be discussed in more detail below, will

cause an additional lateral diffusion over longer timescales

as more and more avalanches are generated. The investiga-
tion of the diffusion caused by relativistic feedback will

be left to future work.

5. Applications

[26] It is not surprising that the diffusion coefficients
decrease rapidly with increasing electric field strength,

since the electric field tends to reduce the angle between

the electron trajectories and the field lines, counteracting

the tendency of the elastic scattering to move the electrons

off the electric field lines. However, why they should obey

power laws in the electric field strength is not obvious.

[27] Because runaway electron avalanches follow the

similarity rule with reduced atmospheric density, Figure 5

and equations (12) and (13) can be used to find the values

at other air densities by converting the actual E to the sea

deck level equivalent field and then scaling D/v in Figure 5 and

equations (12) and (13) by 1/n, where n is the density of air

relative to that at sea level.

Figure 5. Lateral and longitudinal diffusion coefficients
divided by the avalanche speed versus electric field strength.
The largest electric field plotted corresponds to the conven-
tional breakdown field in air. The vertical dashed line is the

runaway electron avalanche threshold. The solid lines are

power law fits to the calculated data. The dotted line shows

the lateral diffusion coefficient calculated from the RMS of

all runaway electrons including those produced by photon

interactions and those in the tail shown in Figure 2.

rate over a wide range of conditions, i.e., 300 kV/m to

3000 kV/m.

[29] Thundercloud discharge models that involve runaway
electrons are also sensitive to the lateral diffusion coeffi-
cients. For example, Dwyer [2005b] introduced a lightning

initiation model that involved the steady state cosmic ray

background plus effects of positron and X-ray feedback. In

this model, the large amount of ionization produced by the

runaway electrons discharges some regions in the thunder-

cloud while locally enhancing the electric field in other

regions. The amount of electric field enhancement is sen-
tive to the amount of diffusion experienced by the run-

away electrons. Note that the runaway electrons generated

by this model and the accompanying X rays and gamma

rays may be relevant to TGFs as well.

[30] In addition, the lateral and the longitudinal diffusion

coefficients presented in the paper are important for calcu-
lating the radio frequency emissions resulting from a cosmic

ray extensive air shower passing through a runaway electron

avalanche region. Dwyer et al. [2009] performed detailed

analytical and numerical calculations of the electromagnetic

fields from such air showers and found that both the lateral

and longitudinal diffusion of the runaway electrons signifi-
cantly alters the shape and the pulse height of the RF

emission. Consequently, the results presented in this paper

should be useful for future studies of such RF emissions.

[31] On a related topic, if air showers are to play any role

in lightning initiation, then in order to achieve large enough

numbers of energetic particles, many runaway electron

avalanche lengths must be present. As a result, the number

of runaway electrons will exceed the original number of

particles in the cosmic ray air shower by many orders of

magnitude. Therefore regardless of the width of the air

shower, the charged particle distribution at the end of the

avalanche region will be broad, as has been shown above.

6 of 11
The possibility of the initiation of lightning by cosmic ray air shower will be considered in more detail in the following section.

6. Lightning Initiation by Cosmic Ray Air Shower

[32] Gurevich et al. [1999] estimated the fluence of runaway electrons created by a cosmic ray extensive air shower by taking (their estimate of) the core density of the air shower and ignoring the effects of the lateral diffusion of the runaway electrons. From equation (9), it is seen that the peak fluence of runaway electrons is given by

\[
\Phi(0, L) = \frac{N_o}{4\pi(D_L/v)L} \exp\left(\frac{L}{\sqrt{\tau}}\right), \tag{14}
\]

where \(D_L/v\) is given in Figure 5 and equation (12); \(L\) is the length of the avalanche region and \(N_o\) is the number of seed electrons injected by the shower. Owing to the effects of the X-ray interactions, the width of the air shower, and the tail of the distribution at high fields, as discussed above, equation (14) should be considered an upper limit on this peak runaway electron fluence.

[33] Equation (14) may be rewritten as the maximum possible runaway electron fluence as a function of the electric field strength when considering the effects of relativistic feedback, which places strict limits on the maximum possible \(L\) for a given \(E\) (see below):

\[
\Phi^{\text{max}}(E) = \frac{N_o}{4\pi(D_L/E/v)L_{\text{max}}} \exp\left(\frac{L_{\text{max}}}{\sqrt{\tau(E)}}\right), \tag{15}
\]

In order for an air shower to have a chance of initiating lightning, the number of avalanche lengths must be large. As an example, if we assume that 10 avalanche lengths are present, corresponding to an avalanche multiplication factor of \(2.3 \times 10^5\), then the lateral \(\sigma\) of the runaway electron distribution is found from equations (4), (11), and (12). This \(\sigma\), which is a measure of the width of the lateral distribution, is plotted versus electric field in Figure 6 for an altitude of 8 km, corresponding to a typical height for lightning initiation. For very strong fields, above about \(E/n > 1500 \text{ kV/m}\), lightning probably can initiate without the help of air showers. Below this field strength, the runaway electron distribution will be at least tens of meters across near the end of the avalanche region where the number of runaway electrons is greatest. Indeed, for realistic thundercloud fields, the runaway electrons distribution will be hundreds of meters across.

[34] Another measure of the effect of diffusion is the length and width of the runaway electron avalanche due to diffusion compared to the avalanche length, which sets the length scale for the avalanche. Figure 7 shows \(\sigma/\lambda\) for both the longitudinal and lateral directions at the end of 10 avalanche lengths versus the sea level equivalent electric field strength. As can be seen, the spreading of the avalanche is significant compared to the avalanche length.

[35] In equations (14) and (15) and in Figures 6 and 7, only the spreading of the runaway electrons is included. For a realistic case, the original distribution of the energetic seed particles, e.g., from the extensive air shower, must also be included by convolving that distribution with the distribution of the runaway electrons calculated in this paper for a delta function source. Because the lateral distribution of the cosmic ray air showers can be substantial (see Dwyer et al. [2009] for more details), the width shown in Figure 6 will be a lower limit and the fluence presented in equations (14) and (15) will again be upper limits. We note that the propagation of low-energy drifting ions and electrons will not contribute significantly to the distributions, since those length scales are on the order of centimeters and the length scale under discussion here due to the runaway electrons are on the order of many meters. In summary, for electric fields relevant to lightning initiation, \(\sigma\) is found to be much larger than the 1–10 cm values used by Gurevich et al. [1999]. As a
result, the true fluence of runaway electrons will be much smaller than their estimate.

[36] Because of the low occurrence rate of high-energy air showers [see Dwyer, 2008, and references therein], it is unlikely that air showers above $10^{17}$ eV play an important role in most lightning initiation. For example, air showers with energies above $10^{17}$ eV only pass through a 10 km by 10 km region once every 20 s, too long to account for the high flash rates often seen in some thunderstorm cells [Rakov and Uman, 2003, p. 25].

[37] Dwyer [2003, 2007, 2008] found that X-ray and positron feedback effects limit the electric field and hence the maximum amount of avalanche multiplication that is likely to occur inside thunderclouds. As the electric field inside a thundercloud is increased, a limit is reached, above which the electric field becomes unstable and discharges very rapidly due to the large amount of ionization produced by the runaway electrons. B. E. Carlson et al. (Simulations of terrestrial gamma-ray flashes from storm to satellite, Chapman Conference on the Effects of Thunderstorms and Lightning in the Upper Atmosphere, AGU, University Park, Pa., 10–14 May 2009) referred to this condition as the “Dwyer instability.” This can be viewed as placing a limit on $L$ for a given $E$, which is plotted in Figure 8. Figure 8 is the same data plotted in a slightly different way as Figure 1 ($R = L/2$) in the work of Dwyer [2008]. This maximum avalanche region length, $L_{\text{max}}$, may be used in equation (15) to calculate the maximum possible fluence.

[38] Now, let us consider the number of low-energy secondary electrons produced by ionization by the runaway electrons. Each runaway electron will produce at most $l_e = 10^4 \times n$ low-energy electrons per meter, where $l_e$ is the avalanche multiplication factor by relativistic feedback, even for a very large air shower, the effects of the self-generated electric and magnetic fields from the runaway electrons and associated ionization are negligible and so will not affect the conclusions in this paper.

[39] It should also be noted that $10^4 \times n$ low-energy electrons per meter per runaway electron, used in this work, is consistent with the standard ionization rates for air, used for many years, for example, in work with ionization chambers and gas filled proportional counters, which also involve energetic particles propagating through gases with large electric fields. The energy required to liberate one low-energy electron-ion pair in air is 33.8 eV for an energetic electron. Runaway electrons are minimum ionizing (7.3 MeV average energy) [Dwyer, 2004]. This means that they deposit $(2.2 \times 10^5$ eV/m in air at standard conditions. Consequently, $(2.2 \times 10^5$ eV/m)/(33.8 eV = 6500 electrons/m, which is then rounded up to $10^4$ (to be conservative). Previously, Gurevich...
et al. [2004] derived a much larger value than this. However, their results do not agree with the standard values that appear in the literature [e.g., see Knoll, 2000, Table 5.1, p. 130], so they will not be used here.

[41] In order to initiate lightning, Gurevich et al. [1999] and Solomon et al. [2001] calculated that at least $3 \times 10^8$ electron/cm$^3$ must be created and that this highly conductive region must have a thin pointed shape so that the electric field would be enhanced at its tip. An alternative calculation using a one-dimensional plane geometry gives $n_e > \frac{z_{re}}{\mu v e}$ as the required electron density to reduce the electric field, where $\mu = 0.094 \times (1/n)$ m$^2$/(V s) is the mobility of low-energy electrons and $\tau_a \sim 10^{-7}$ s is the attachment time of the electrons at thunderstorm altitudes. This gives $n_e > 3 \times 10^9$ cm$^{-3}$. Even the more optimistic value of $3 \times 10^8$ cm$^{-3}$ is more than an order of magnitude higher than the largest density possibly produced by a $10^{17}$ eV air shower as seen in Figure 9. Furthermore, the assumption of a sharp, pointed region of high conductivity is problematic given the results of this paper (e.g., Figure 7).

[42] In summary, lateral and longitudinal diffusion coefficients for runaway electron avalanches have been calculated and simple empirical formulas have been presented. The diffusion described by these coefficients is an intrinsic property of all runaway electron avalanches and so these results should be useful for future work. In addition, according to the criteria for lightning initiation calculated by Gurevich et al. [1999] and Solomon et al. [2001], when lateral diffusion of the runaway electron avalanches is included, extensive air showers do not play a role in most lightning initiation. Although it cannot be ruled out that lightning initiation is sometimes caused by cosmic ray extensive air showers, exactly how this might occur remains to be established.

Appendix A

[43] Because many different quantities are found in the literature for describing runaway electrons, it is instructive to consider several variations of equation (5) and its solution, equation (7). In this appendix, equations (9) and (10) are calculated, along with several related quantities found in other work.

[44] As can be seen from equation (3), the net flux vector for runaway electrons is given by

$$\vec{F} = \vec{v} n_{re} - \vec{D} \cdot \nabla n_{re}, \quad (A1)$$

where the first term on the right side is the convective flux and the second term is the flux due to diffusion.

[45] The net flux (number/m$^3$/s) of runaway electrons passing through the plane perpendicular to the avalanche direction, i.e., the $+z$ direction, is then

$$F_z = v n_{re} - D_z \frac{\partial n_{re}}{\partial z} = v n_{re} \left( 1 - \frac{v(t - t_o) - (z - z_o)}{2v(t - t_o)} \right), \quad (A2)$$

where equation (7) is used to calculate the right side of equation (A2).

[46] Because $v(t - t_o) \gg vo(t - t_o) - (z - z_o)$, as will be shown below, for the cases investigated in this paper, the contribution of diffusion to the flux often can be ignored and we have simply the convective flux

$$F_z \approx v n_{re}. \quad (A3)$$

If we integrate the net flux given by equation (A2) over all $x$ and $y$, then we obtain the rate (number/s) of runaway electrons.

$$R_w(z, t) = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} F_z(x, y, z, t) \, dx \, dy = R_{we} - \frac{D_z}{v} \frac{\partial R_{we}}{\partial z} \quad (A4)$$

where $R_{we}$ is the rate calculated from just the convective flux (equation (A3)). Using equation (7) for the density, we obtain:

$$R_{we}(z, t) = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} v n_{re}(x, y, z, t) \, dx \, dy = \frac{v}{(4\pi(t - t_o)D_z)^{1/2}} \cdot \exp \left( \frac{(t - t_o)}{\tau} - \frac{(v(t - t_o) - (z - z_o))^2}{4D_z(t - t_o)} \right) S(t - t_o). \quad (A5)$$

Note that when integrating, the runaway electron density goes to zero when any one variable goes to $\pm \infty$ while holding the other variables finite. This rate, $R_{we}$, is the solution to

$$\frac{\partial R_{we}}{\partial t} + v \frac{\partial R_{we}}{\partial z} - D_z \frac{\partial^2 R_{we}}{\partial z^2} - \frac{R_{we}}{\tau} = \delta(z - z_o) \delta(t - t_o). \quad (A6)$$

obtained by integrating equation (5) over all $x$ and $y$. 

Figure A1. $D_w/\nu^2 \tau$ versus the sea level equivalent electric field strength. This quantity gives the relative importance of the longitudinal diffusion term and the avalanche multiplication term in the transport equations. Since the value is always small, the longitudinal diffusion does not significantly alter the number of runaway electrons produced versus distance along the avalanche axis, although it does affect the spreading of the runaway electrons in the avalanche.
[47] Plugging equation (A5) into (A4) gives

\[
R_{re}(z, t) = \frac{v}{(4\pi(t - t_0)D_t)^{1/2}} \left(1 - \frac{v^2}{2v(t - t_0)} - \frac{(v(t - t_0) - (z - z_0))^2}{4D_t(t - t_0)}\right) S(t - t_0).
\]

(A7)

Note that since \( v(t - t_0) \gg 1\) \( v(t - t_0) - (z - z_0) \), the rate calculated using the net flux and the rate calculated using the convective flux are very close in value. This is illustrated in Figure 3. Because \( R_{re}(z, t) \) is simpler, equation (A5) is used as the rate in section 3 (equation (10)). The rate, \( R_{re} \), describes the arrival time distribution of runaway electrons for a fixed \( z \) position. Equivalently, \( R_{re}/v \) gives the distribution of \( z \) positions at a fixed time.

[48] If we, instead, integrate the density (equation (7)) over all \( z \), then we obtain the column density of runaway electrons

\[
I_{re}(x, y, z) = \int_{-\infty}^{\infty} n_{re}(x, y, z, t) dz = \frac{1}{4\pi D_{\perp}(t - t_0)} \exp\left(\frac{(t - t_0)}{\tau} \cdot \frac{r^2}{4D_{\perp}(t - t_0)}\right) S(t - t_0).
\]

(A8)

As discussed in section 3, if we assume no external magnetic fields, then there is a natural symmetry in the \( x-y \) directions. As a result, defining \( D_1 = D_\perp = D_\parallel \) and the cylindrical radius \( r^2 = (x - x_0)^2 + (y - y_0)^2 \), then equation (A8) reduces to

\[
I_{re}(r, t) = \frac{1}{4\pi D_\perp(t - t_0)} \exp\left(\frac{(t - t_0)}{\tau} \cdot \frac{r^2}{4D_\perp(t - t_0)}\right) S(t - t_0).
\]

(A9)

Equation (A8) is a solution of the transport equation

\[
\frac{\partial I_{re}}{\partial t} - D_\parallel \nabla_z^2 I_{re} - \frac{I_{re}}{\tau} = \delta(t - t_0)\delta(x - x_0)\delta(y - y_0),
\]

(A10)

which is obtained by integrating both sides of equation (5) (using equation (6) for \( n_z \)) over all \( z \). In equation (A10), we define \( \nabla_z^2 I_{re} = \frac{\partial^2 I_{re}}{\partial z^2} + \frac{\partial I_{re}}{\partial z} = \frac{1}{\tau} \frac{\partial}{\partial z} (r \frac{\partial I_{re}}{\partial r}) \), assuming a cylindrical symmetry.

[49] If we further integrate the column density (equation (A8)) over all \( x \) and \( y \), or equivalently multiply equation (A8) by \( 2\pi r \) and integrate over all \( r \), then we obtain

\[
N_{re}(t) = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} n_{re}(x, y, z, t) dx dy dz = \exp\left(\frac{(t - t_0)}{\tau} \cdot \frac{r^2}{4D_{\perp}(t - t_0)}\right) S(t - t_0)
\]

(A11)

the total number of runaway electrons at time \( t \) that results from injecting one seed electron at time \( t_0 \). This equation is the solution to the equation

\[
\frac{dN_{re}}{dt} - \frac{N_{re}}{\tau} = \delta(t - t_0),
\]

(A12)

which describes the avalanche growth of the number of runaway electrons with time. Although equations (A8) through (A12) are mathematically well defined, generally, they are physically not very useful, since they describe the number of particles at a given time for all \( z \) positions. More physical quantities can be obtained by considering the number of particles at a position \( z \) integrated over all times. Specifically, if we integrate the flux given by equation (A2) over all times, then we obtain the fluence (number/m²) of runaway electrons that pass through the plane at position, \( z \):

\[
\Phi_{re}(z, t) = \int_{-\infty}^{\infty} F_r(x, y, z, t) dt = \Phi_{re} v - \frac{D_{\perp}}{v} \frac{\partial \Phi_{re}}{\partial z},
\]

(A13)

where

\[
\Phi_{re}(x, y, z) = \int_{-\infty}^{\infty} v n_{re}(x, y, z, t) dt.
\]

(A14)

Equation (A14) is the solution of the equation

\[
\frac{\partial \Phi_{re}}{\partial z} - \frac{D_{\perp}}{v} \frac{\partial^2 \Phi_{re}}{\partial z^2} - \frac{\Phi_{re}}{\tau} = \delta(x - x_0)\delta(y - y_0)\delta(z - z_0),
\]

(A15)

obtained by integrating equation (5) over all times. As can be seen, equation (A15) is slightly unwieldy. However, because the fluence increases approximately as \( \exp(z/v\tau) \), the third term on the left in equation (A15) is approximately \( \frac{D_{\perp}/v}{\tau} \) times smaller than the fourth term on the left. As is shown in Figure A1, \( \frac{D_{\perp}}{v} \) is in all cases less than 0.01. As a result, equation (A15) can be simplified to

\[
\frac{\partial \Phi_{re}}{\partial z} - \frac{D_{\perp}}{v} \frac{\partial^2 \Phi_{re}}{\partial z^2} = \delta(x - x_0)\delta(y - y_0)\delta(z - z_0),
\]

(A16)

which has the solution

\[
\Phi_{re}(r, z) = \frac{1}{4\pi (D_{\perp}/v)(z - z_0)} \exp\left(\frac{(z - z_0)^2}{4(D_{\perp}/v)(z - z_0)}\right) S(z - z_0).
\]

(A17)

Plugging in equation (A17) into (A13) gives the fluence

\[
\Phi_{re}(r, z) = \frac{1}{4\pi (D_{\perp}/v)(z - z_0)} \left(1 - \frac{1}{\tau} \left[1 - \frac{1}{(z - z_0)^2} + \frac{r^2}{4(D_{\perp}/v)(z - z_0)}\right]\right) \int_{-\infty}^{\infty} v n_{re}(x, y, z, t) dt \]

(A18)

As with the rate, the difference between using the net flux and the convective flux is very small for the cases of interest in this paper and so the simpler equation (A17) is used for the fluence in section 3 (equation (9)). Note that equations (A16) and (A17) are exactly the same equation that we would obtain from equations (A9) and (A10) with the change of variable \( z = v(t - t_0) + z_0 \). The approximation used to
derive equations (A16) and (A17) does not result in a significant error in most cases. For instance, exactly integrating equation (7) numerically according to equation (A14) gives the curves shown as the dashed line in Figure 2, which are almost exactly the same as equation (A17) plotted as the solid curves. Also, using equation (A18) instead of (A17) would not make a visible difference in the plot.

[50] For completeness, if we integrate the net flux over \(x\), \(y\) and \(t\), we obtain the avalanche multiplication factor

\[
M_{av}(z) = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} F_z(x, y, z, t) dx dy dt
\]

\[
= M_{re} \frac{D_I}{\nu} \frac{dM_{re}}{dz},
\]

(A19)

where \(M_{re}\) is the multiplication factor calculated from the convective flux, which satisfies

\[
\frac{dM_{re}}{dz} - \frac{D_I}{\nu} \frac{d^2M_{re}}{dz^2} + \frac{M_{re}}{\nu \tau} = \delta(z - z_0).
\]

(A20)

The avalanche multiplication factor is the total number of runaway electrons passing through a plane at position \(z\) after injecting one seed electron at \(z_0\).

[51] The solution of equation (A20) is

\[
M_{av}(z) \approx \left(1 - \frac{D_I}{\nu \lambda} \right) \left(1 - 4D_I/v^2 \tau \right)^{-1/2} \exp \left(\frac{z - z_0}{\lambda}\right)
\]

for \(z > z_0\),

(A21)

where

\[
\lambda = \nu \tau \left(1 + \sqrt{1 - 4D_I/v^2 \tau}\right)
\]

(A22)

In equations (A21) and (A22) the unitless quantity \(D_I/v^2 \tau\) is very small (see Figure A1). As a result, we make little error in using the simpler equation

\[
\frac{dM_{re}}{dt} = \frac{M_{re}}{\lambda} = \delta(z - z_0),
\]

(A23)

with \(\lambda \approx \nu \tau\). For the avalanche multiplication factor, \(\lambda\) is a quantity that is easy to calculate directly from Monte Carlo simulations [e.g., see Dwyer, 2003; Coleman and Dwyer, 2006], and equation (A22) is only necessary when converting \(\nu \tau\) from other studies [e.g., Babich et al., 2005] to \(\lambda\).

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References


J. R. Dwyer, Department of Physics and Space Sciences, Florida Institute of Technology, Melbourne, FL 32901, USA. (jdwyer@fit.edu)