Critical Elliptic Boundary Value Problems with Singular Trudinger-Moser Nonlinearities

by

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ABSTRACT

Title:

Critical Elliptic Boundary Value Problems with Singular Trudinger-Moser Nonlinearities

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In this dissertation, we prove the existence of solutions for two classes of eliptic problems that are critical with respect to singular Trudinger-Moser embedding. The proofs are based on compactness and regularity arguments.

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Dedication

I dedicate my dissertation to my family. A special feeling of gratitude to my parents, Jin Fu and Guo Guo whose words of encouragement and guide me through my life. My grandma , Guizhe Liu, the first woman suggests me to study abroad to widen my horizon. My uncle, Ding Guo who gives financial aid to support me during these years. My cousin, Difei Guo who always tells me never say give up no matter what happens. I also dedicate this dissertation to my best friends and Melbourne Chinese Church who have encouraged and supported me throughout my study jorney. I appreciate for everything they have done for me, especially Grace Yeh for teaching me how to talk with local people, Yang Li for helping me develop my technology skills and Grant Yeh for helping me understand the truth of being a man.

Chapter 1

Introduction

Elliptic problems with critical Trudinger-Moser nonlinearities have been widely studied in the literature. The well-known Sobolev embedding theorem says that

$$W_0^{1,p}(\Omega) \hookrightarrow L^q(\Omega)$$

if

$$1 \leq q \leq \frac{Np}{N-p},$$

where Ω is a bounded domain in $\mathbb{R}^N$, $N \geq 2$, $p < N$, and $W_0^{1,p}(\Omega)$ is the standard Sobolev space of L^p-functions whose weak derivatives also belong to $L^p(\Omega)$. The Trudinger-Moser inequality concerns the borderline cases $N = p$, where $\frac{Np}{N-p} \sim +\infty$ and $W_0^{1,N}(\Omega)$ is not embedded in $L^\infty(\Omega)$. Trudinger [15] showed that

$$\int_\Omega e^{\,|u|^{N/(N\;\;1)}}\, dx < \infty \quad \forall u \in W_0^{1,N}(\Omega)$$

by developing the exponential in a power series and controlling the embedding constants of $W_0^{1,N}(\Omega) \hookrightarrow L^m(\Omega)$, $m \in \mathbb{N}$.

Moser [13] improved the above result by showing that

$$\int_\Omega e^{\alpha |u|^{N/(N-1)}}\, dx < \infty \quad \forall u \in W_0^{1,N}(\Omega)$$

for all $\alpha > 0$, and

$$\sup_{\|u\|_{W_0^{1,N}(\Omega)} \leq 1} \int_\Omega e^{\alpha |u|^{N/(N-1)}}\, dx < \infty$$

if and only if

$$\alpha \leq N\omega_{N-1}^{1/(N-1)},$$

where ω_{N-1} is the area of the unit sphere in $\mathbb{R}^N$.

For the case $N = 2$, we obtain that

$$\int_\Omega e^{\alpha u^2}\, dx < \infty \quad \forall u \in H_0^1(\Omega)$$

for all $\alpha > 0$, and

$$\sup_{\|u\|_{H_0^1(\Omega)} \leq 1} \int_\Omega e^{\alpha u^2}\, dx < \infty$$

if and only if

$$\alpha \leq 4\pi.$$

The following generalization of this embedding was obtained in Adimurthi and Sandeep [14]:

$$\int_\Omega \frac{e^{\alpha u^2}}{|x|^\gamma}\, dx < \infty \quad \forall u \in H_0^1(\Omega)$$

for all $\alpha > 0$ and $0 \leq \gamma < 2$, and

$$\sup_{\|u\|_{H_0^1(\Omega)} \leq 1} \int_\Omega \frac{e^{\alpha u^2}}{|x|^\gamma}\, dx < \infty \tag{1.1}$$

if and only if

$$\frac{\alpha}{4\pi} + \frac{\gamma}{2} \leq 1.$$

In this dissertation we prove the existence of solutions to two classes of elliptic problems that are critical with respect to this singular Trudinger-Moser embedding. As is usually the case with critical growth problems, the main difficulty here is the lack of compactness of the associated variational functionals.

First we establish an existence result for the class of singular elliptic problems with exponential nonlinearities

$$\begin{cases} -\Delta u = h(u)\, \dfrac{e^{\alpha u^2}}{|x|^\gamma} & \text{in } \Omega \\[2ex] \quad u = 0 & \text{on } \partial\Omega, \end{cases} \tag{1.2}$$

where Ω is a bounded domain in $\mathbb{R}^2$ containing the origin, $\alpha > 0$, $0 \leq \gamma < 2$, and h is a continuous function for which the limit

$$\beta = \lim_{|t| \to \infty} t h(t) \tag{1.3}$$

exists. The case $\beta = \infty$ was considered in [14], so we focus on the case $0 < \beta < \infty$ here. The nonsingular case $\gamma = 0$ has been widely studied in the literature (see, e.g., Adimurthi [1], Adimurthi and Yadava [2], de Figueiredo et al. [11, 9], Marcos B. do Ó [12], de Figueiredo et al. [10], Perera and Yang [16], and their references).

Let $\lambda_1(\gamma) > 0$ be the first eigenvalue of the singular eigenvalue problem

$$\begin{cases} -\Delta u = \lambda \dfrac{u}{|x|^\gamma} & \text{in } \Omega \\[2ex] u = 0 & \text{on } \partial\Omega, \end{cases}$$

given by

$$\lambda_1(\gamma) = \inf_{u \in H_0^1(\Omega)\setminus\{0\}} \frac{\displaystyle\int_\Omega |\nabla u|^2\, dx}{\displaystyle\int_\Omega \frac{u^2}{|x|^\gamma}\, dx}. \tag{1.4}$$

Set

$$G(t) = \int_0^t h(s)\, e^{\alpha s^2}\, ds.$$

Our main result for problem (1.2) is the following theorem.

Theorem 1.1. *Assume that $\alpha > 0$ and $0 \leq \gamma < 2$ satisfy*

$$\frac{\alpha}{4\pi} + \frac{\gamma}{2} \leq 1,$$

G satisfies

$$G(t) \geq 0 \quad \text{for } t \geq 0, \tag{1.5}$$

$$G(t) \leq \frac{1}{2}\,(\lambda_1(\gamma) - \sigma)\, t^2 \quad \text{for } |t| \leq \delta \tag{1.6}$$

for some $\sigma, \delta > 0$, and

$$\frac{(2-\gamma)^2}{2\alpha d^{2-\gamma}} < \beta < \infty, \tag{1.7}$$

where d is the radius of the largest open ball centered at the origin that is contained in Ω. Then problem (1.2) has a nontrivial solution.

This theorem is new even in the nonsingular case $\gamma = 0$. Indeed, the corresponding result for the nonsingular case is proved in de Figueiredo et al. [11, 9] and Marcos B. do Ó [12] only assuming that $h(t) \geq 0$ for all $t \geq 0$. This implies our assumption (1.5), but (1.5) is weaker. The proof of the theorem will be given in Section 2.2, after proving a suitable compactness property of the associated variational functional in Section 2.1.

Theorem 1.2. *Assume that $\alpha > 0$ and $0 \leq \gamma < 2$ satisfy*

$$\frac{\alpha}{4\pi} + \frac{\gamma}{2} \leq 1,$$

G satisfies

$$G(t) \leq \frac{1}{2}\left(\lambda_1(\gamma) - \sigma\right) t^2 \quad \textit{for } |t| \leq \delta$$

for some $\sigma, \delta > 0$, and

$$\frac{(2-\gamma)^2}{\alpha d^{2-\gamma}} < \beta < \infty. \tag{1.8}$$

Then problem (1.2) has a nontrivial solution.

Proofs of Theorems 1.1 and 1.2 will be given in Section 2.2, after proving a suitable compactness property of the associated variational functional in Section 2.1.

Our second result concerns a class of semipositone problems with singular exponential nonlinearities. We recall that the semilinear elliptic boundary value problem

$$\begin{cases} -\Delta u = f(x,u) & \text{in } \Omega \\ u > 0 & \text{in } \Omega \\ u = 0 & \text{on } \partial\Omega, \end{cases}$$

where f is a Carathéodory function on $\Omega \times [0, \infty)$, is said to be of semipositone type if $f(\cdot, 0) < 0$ on a set of positive measure. It is notoriously difficult to find positive solutions of this class of problems due to the fact that $u = 0$ is not a subsolution (see, e.g., Castro and Shivaji [6], Ali et al. [3], Ambrosetti et al. [4], Chhetri et al. [7], Castro et al. [5], Costa et al. [8], and their references). We consider the problem

$$\begin{cases} -\Delta u = \lambda u \, \dfrac{e^{\alpha u^2}}{|x|^\gamma} + \mu \, g(u) & \text{in } \Omega \\ u > 0 & \text{in } \Omega \\ u = 0 & \text{on } \partial\Omega, \end{cases} \tag{1.9}$$

where Ω is a smooth bounded domain in $\mathbb{R}^2$ containing the origin, $\alpha > 0$, $0 \le \gamma < 2$, $\lambda, \mu > 0$ are parameters, and g is a continuous function on $[0, \infty)$ satisfying

$$\lim_{t \to \infty} \frac{g(t)}{e^{\beta t^2}} = 0 \quad \forall \beta > 0 \tag{1.10}$$

and

$$\sup_{t \in [0, \infty)} \big(2G(t) - t g(t)\big) < \infty, \tag{1.11}$$

where $G(t) = \int_0^t g(s) \, ds$. We make no assumptions about the sign of $g(0)$ and hence allow the semipositone case $g(0) < 0$. For example, the functions $g(t) = -1$, $g(t) = t^p - 1$, where $p \ge 1$, and $g(t) = e^t - 2$ all satisfy (1.10), (1.11), and $g(0) < 0$.

We will show that problem (1.9) has a positive solution for all $0 < \lambda < \lambda_1(\gamma)$ and $\mu > 0$ sufficiently small. We have the following theorem.

Theorem 1.3. *Assume that* $\alpha > 0$ *and* $0 \leq \gamma < 1$ *satisfy*

$$\frac{\alpha}{4\pi} + \frac{\gamma}{2} \leq 1,$$

$0 < \lambda < \lambda_1(\gamma)$, *and* g *satisfies* (1.10) *and* (1.11). *Then there exists a* $\mu^* > 0$ *such that for all* $0 < \mu < \mu^*$, *problem* (1.9) *has a solution* u_μ.

We note that this result does not follow from standard arguments based on the maximum principle since $g(0)$ is not assumed to be nonnegative. Our proof is based on regularity arguments and will be given in Section 3.2, after establishing a suitable compactness property of an associated variational functional in Section 3.1.

Chapter 2

Proof of Theorem 1.1 and 1.2

2.1 Compactness

Weak solutions of problem (1.2) coincide with critical points of the C^1-functional

$$E(u) = \frac{1}{2}\int_\Omega |\nabla u|^2\, dx - \int_\Omega \frac{G(u)}{|x|^\gamma}\, dx, \quad u \in H_0^1(\Omega).$$

We recall that a $(\mathrm{PS})_c$ sequence of E is a sequence $(u_j) \subset H_0^1(\Omega)$ such that $E(u_j) \to c$ and $E'(u_j) \to 0$. Proof of Theorem 1.1 and 1.2 will be based on the following compactness result.

Proposition 2.1. *Assume that $\alpha > 0$ and $0 \le \gamma < 2$ satisfy*

$$\frac{\alpha}{4\pi} + \frac{\gamma}{2} \le 1$$

and $0 < \beta < \infty$. *Then for all* $c \neq 0$ *with*

$$c < \frac{2\pi}{\alpha}\left(1 - \frac{\gamma}{2}\right),$$

every $(\mathrm{PS})_c$ *sequence of* E *has a subsequence that converges weakly to a nontrivial solution of problem* (1.2).

Proof. Let $(u_j) \subset H^1_0(\Omega)$ be a $(\mathrm{PS})_c$ sequence of E. Then

$$E(u_j) = \frac{1}{2}\,\|u_j\|^2 - \int_\Omega \frac{G(u_j)}{|x|^\gamma}\,dx = c + \mathrm{o}(1) \tag{2.1}$$

and

$$E'(u_j)\,u_j = \|u_j\|^2 - \int_\Omega u_j\,h(u_j)\,\frac{e^{\alpha u_j^2}}{|x|^\gamma}\,dx = \mathrm{o}(\|u_j\|). \tag{2.2}$$

First we show that (u_j) is bounded in $H^1_0(\Omega)$. Multiplying (2.1) by 4 and subtracting (2.2) gives

$$\|u_j\|^2 + \int_\Omega \Big(u_j\,h(u_j)\,e^{\alpha u_j^2} - 4\,G(u_j)\Big)\,\frac{dx}{|x|^\gamma} = 4c + \mathrm{o}(\|u_j\| + 1),$$

so it suffices to show that $th(t)\,e^{\alpha t^2} - 4G(t)$ is bounded from below. Let $\varepsilon > 0$. By (1.3), $\exists M_\varepsilon > 0$ such that $|th(t) - \beta| < \varepsilon$ for $|t| > M_\varepsilon$. So, for some constant $C_\varepsilon > 0$,

$$th(t)\,e^{\alpha t^2} \geq (\beta - \varepsilon)\,e^{\alpha t^2} - C_\varepsilon \quad \forall t \tag{2.3}$$

and

$$|G(t)| \leq \begin{cases} (\beta + \varepsilon)\displaystyle\int_{M_\varepsilon}^{|t|} \frac{e^{\alpha s^2}}{s}\,ds + C_\varepsilon & \text{if } |t| > M_\varepsilon \\[2ex] C_\varepsilon & \text{otherwise.} \end{cases} \tag{2.4}$$

Taking M_ε larger if necessary, we may assume that $(\beta+\varepsilon)/s \le 2\varepsilon\alpha s$ for all $s \ge M_\varepsilon$, so (2.4) gives

$$|G(t)| \le \varepsilon\, e^{\alpha t^2} + C_\varepsilon \quad \forall t, \tag{2.5}$$

which together with (2.3) gives the desired conclusion if $\varepsilon < \beta/5$.

Since (u_j) is bounded in $H^1_0(\Omega)$, a renamed subsequence converges to some u weakly in $H^1_0(\Omega)$, strongly in $L^p(\Omega)$ for all $p \in [1, \infty)$, and a.e. in Ω. We have

$$E'(u_j)\, v = \int_\Omega \nabla u_j \cdot \nabla v\, dx - \int_\Omega v\, h(u_j)\, \frac{e^{\alpha u_j^2}}{|x|^\gamma}\, dx \to 0 \tag{2.6}$$

for all $v \in H^1_0(\Omega)$. By (1.3), given any $\varepsilon > 0$, there exists a constant $C_\varepsilon > 0$ such that

$$|h(t)\, e^{\alpha t^2}| \le \varepsilon\, e^{\alpha t^2} + C_\varepsilon \quad \forall t. \tag{2.7}$$

By (2.2),

$$\sup_j \int_\Omega u_j\, h(u_j)\, \frac{e^{\alpha u_j^2}}{|x|^\gamma}\, dx < \infty,$$

which together with (2.3) gives

$$\sup_j \int_\Omega \frac{e^{\alpha u_j^2}}{|x|^\gamma}\, dx < \infty. \tag{2.8}$$

For $v \in C^\infty_0(\Omega)$, it follows from (2.7) and (2.8) that the sequence $(v\, h(u_j)\, e^{\alpha u_j^2}/|x|^\gamma)$ is uniformly integrable and hence

$$\int_\Omega v\, h(u_j)\, \frac{e^{\alpha u_j^2}}{|x|^\gamma}\, dx \to \int_\Omega v\, h(u)\, \frac{e^{\alpha u^2}}{|x|^\gamma}\, dx$$

by Vitali's convergence theorem, so it follows from (2.6) that

$$\int_\Omega \nabla u \cdot \nabla v \, dx - \int_\Omega v \, h(u) \, \frac{e^{\alpha u^2}}{|x|^\gamma} \, dx = 0.$$

Then this holds for all $v \in H_0^1(\Omega)$ by density, so the weak limit u is a solution of problem (1.2).

Suppose that $u = 0$. Then

$$\int_\Omega \frac{G(u_j)}{|x|^\gamma} \, dx \to 0$$

since (2.5) and (2.8) imply that the sequence $(G(u_j)/|x|^\gamma)$ is uniformly integrable, so (2.1) gives $c \geq 0$ and

$$\|u_j\| \to (2c)^{1/2}. \tag{2.9}$$

Let $2c < \nu < 4\pi \, (1 - \gamma/2)/\alpha$. Then $\|u_j\| \leq \nu^{1/2}$ for all $j \geq j_0$ for some j_0. Let $q = 4\pi \, (1 - \gamma/2)/\alpha\nu > 1$ and let $1/(1 - 1/q) < r < 2/\gamma \, (1 - 1/q)$. By the Hölder inequality,

$$\left| \int_\Omega u_j \, h(u_j) \, \frac{e^{\alpha u_j^2}}{|x|^\gamma} \, dx \right| \leq \left(\int_\Omega |u_j \, h(u_j)|^p \, dx \right)^{1/p} \left(\int_\Omega \frac{e^{q\alpha u_j^2}}{|x|^\gamma} \, dx \right)^{1/q} \left(\int_\Omega \frac{dx}{|x|^{\gamma r \, (1 - 1/q)}} \right)^{1/r},$$

where $1/p + 1/q + 1/r = 1$. The first integral on the right-hand side converges to zero since $th(t)$ is bounded and $u = 0$, the second integral is bounded for $j \geq j_0$ by (1.1) since $q\alpha u_j^2 = 4\pi \, (1 - \gamma/2) \, \widetilde{u}_j^2$, where $\widetilde{u}_j = u_j/\nu^{1/2}$ satisfies $\|\widetilde{u}_j\| \leq 1$, and the last integral is finite since $\gamma r \, (1 - 1/q) < 2$, so

$$\int_\Omega u_j \, h(u_j) \, \frac{e^{\alpha u_j^2}}{|x|^\gamma} \, dx \to 0.$$

Then $u_j \to 0$ by (2.2) and hence $c = 0$ by (2.9), contrary to assumption. So u is nontrivial. $\square$

2.2 Proof of Theorem 1.1 and 1.2

In this section we prove Theorem 1.1 and 1.2. We will show that the functional E has the mountain pass geometry with the mountain pass level $c \in (0, 2\pi\,(1 - \gamma/2)/\alpha)$ and apply Proposition 2.1.

Lemma 2.2. *If* (1.6) *holds, then there exists a* $\rho > 0$ *such that*

$$\inf_{\|u\|=\rho} E(u) > 0.$$

Proof. Since (1.3) implies that h is bounded, there exists a constant $C_\delta > 0$ such that

$$|G(t)| \le C_\delta\, |t|^3\, e^{\alpha t^2} \quad \text{for } |t| > \delta,$$

which together with (1.6) gives

$$\int_\Omega \frac{G(u)}{|x|^\gamma}\, dx \le \frac{1}{2}\,(\lambda_1(\gamma) - \sigma) \int_\Omega \frac{u^2}{|x|^\gamma}\, dx + C_\delta \int_\Omega |u|^3\, \frac{e^{\alpha u^2}}{|x|^\gamma}\, dx. \tag{2.10}$$

By (1.4),

$$\int_\Omega \frac{u^2}{|x|^\gamma}\, dx \le \frac{\rho^2}{\lambda_1(\gamma)}, \tag{2.11}$$

where $\rho = \|u\|$. Let $2 < r < 4/\gamma$. By the Hölder inequality,

$$\int_\Omega |u|^3\, \frac{e^{\alpha u^2}}{|x|^\gamma}\, dx \le \left(\int_\Omega |u|^{3p}\, dx\right)^{1/p} \left(\int_\Omega \frac{e^{2\alpha u^2}}{|x|^\gamma}\, dx\right)^{1/2} \left(\int_\Omega \frac{dx}{|x|^{\gamma r/2}}\right)^{1/r}, \tag{2.12}$$

where $1/p+1/r = 1/2$. The first integral on the right-hand side is bounded by $C\rho^{3p}$ for some constant $C > 0$ by the Sobolev embedding. Since $2\alpha u^2 = 2\alpha\rho^2\,\widetilde{u}^2$, where $\widetilde{u} = u/\rho$ satisfies $\|\widetilde{u}\| = 1$, the second integral is bounded when $\rho^2 \le 2\pi\,(1-\gamma/2)/\alpha$ by (1.1). The last integral is finite since $\gamma r < 4$. So combining (2.10)–(2.12) gives

$$\int_\Omega \frac{G(u)}{|x|^\gamma}\,dx \le \frac{1}{2}\left(1 - \frac{\sigma}{\lambda_1(\gamma)}\right)\rho^2 + \mathrm{O}(\rho^3) \quad \text{as } \rho \to 0.$$

Then

$$E(u) \ge \frac{1}{2}\,\frac{\sigma}{\lambda_1(\gamma)}\,\rho^2 + \mathrm{O}(\rho^3),$$

and the desired conclusion follows from this for sufficiently small $\rho > 0$. □

We have $B_d(0) \subset \Omega$. For $j \ge 2$, let

$$v_j(x) = \frac{1}{\sqrt{2\pi}}\begin{cases} \sqrt{\log j} & \text{if } |x| \le d/j \\ \dfrac{\log(d/|x|)}{\sqrt{\log j}} & \text{if } d/j < |x| < d \\ 0 & \text{otherwise.} \end{cases}$$

It is easily seen that $v_j \in H_0^1(\Omega)$ with $\|v_j\| = 1$.

Lemma 2.3. *Assume that* $0 < \beta < \infty$.

(*i*) *For all* $j \ge 2$, $E(tv_j) \to -\infty$ *as* $t \to \infty$.

(*ii*) *If* (1.6) *holds, then there exists* $j_0 \ge 2$ *such that*

$$\sup_{t \ge 0} E(tv_{j_0}) < \frac{2\pi}{\alpha}\left(1 - \frac{\gamma}{2}\right) \tag{2.13}$$

in each of the following cases:

(a) (1.5) and (1.7) hold,

(b) (1.8) holds.

Proof. Fix $\varepsilon > 0$. By (1.3), $\exists M_\varepsilon > 0$ such that

$$th(t)\, e^{\alpha t^2} > (\beta - \varepsilon)\, e^{\alpha t^2} \quad \text{for } |t| > M_\varepsilon. \tag{2.14}$$

Since $e^{\alpha t^2} > \alpha^2\, t^4/2$ for all t, then there exists a constant $C_\varepsilon > 0$ such that for all $t \geq 0$,

$$h(t)\, e^{\alpha t^2} \geq \frac{1}{2}\, (\beta - \varepsilon)\, \alpha^2\, t^3 - C_\varepsilon \tag{2.15}$$

and hence

$$G(t) \geq \frac{1}{8}\, (\beta - \varepsilon)\, \alpha^2\, t^4 - C_\varepsilon\, t. \tag{2.16}$$

Since $\|v_j\| = 1$ and $v_j \geq 0$, then

$$E(tv_j) \leq \frac{t^2}{2} - \frac{1}{8}\, (\beta - \varepsilon)\, \alpha^2\, t^4 \int_\Omega \frac{v_j^4}{|x|^\gamma}\, dx + C_\varepsilon\, t \int_\Omega \frac{v_j}{|x|^\gamma}\, dx$$

and (i) follows.

Set

$$H_j(t) = E(tv_j) = \frac{t^2}{2} - \int_\Omega \frac{G(tv_j)}{|x|^\gamma}\, dx, \quad t \geq 0.$$

If (ii) is false, then it follows from Lemma 2.2 and (i) that for all j, $\exists t_j > 0$ such that

$$H_j(t_j) = \frac{t_j^2}{2} - \int_\Omega \frac{G(t_j v_j)}{|x|^\gamma}\, dx = \sup_{t\geq 0}\, H_j(t) \geq \frac{2\pi}{\alpha}\left(1 - \frac{\gamma}{2}\right), \tag{2.17}$$

$$H_j'(t_j) = t_j - \int_\Omega v_j\, h(t_j v_j)\, \frac{e^{\alpha t_j^2 v_j^2}}{|x|^\gamma}\, dx = 0. \tag{2.18}$$

Since $G(t) \geq -C_\varepsilon\, t$ for all $t \geq 0$ by (2.16), (2.17) gives

$$t_j^2 \geq t_0^2 - 2\delta_j\, t_j, \tag{2.19}$$

where

$$t_0 = \sqrt{\frac{4\pi}{\alpha}\left(1 - \frac{\gamma}{2}\right)}$$

and

$$\delta_j = C_\varepsilon \int_{B_d(0)} \frac{v_j}{|x|^\gamma}\, dx = \frac{C_\varepsilon\, d^{2-\gamma}}{(2-\gamma)^2} \sqrt{\frac{2\pi}{\log j}} \left(1 - \frac{1}{j^{2-\gamma}}\right) \to 0 \quad \text{as } j \to \infty. \tag{2.20}$$

First we will show that $t_j \to t_0$.

By (2.19), $t_j \geq \sqrt{t_0^2 + \delta_j^2} - \delta_j$ and hence

$$\liminf_{j\to\infty}\, t_j \geq t_0. \tag{2.21}$$

Write (2.18) as

$$t_j^2 = \int_{\{t_j v_j > M_\varepsilon\}} t_j v_j\, h(t_j v_j)\, \frac{e^{\alpha t_j^2 v_j^2}}{|x|^\gamma}\, dx + \int_{\{t_j v_j \leq M_\varepsilon\}} t_j v_j\, h(t_j v_j)\, \frac{e^{\alpha t_j^2 v_j^2}}{|x|^\gamma}\, dx =: I_1 + I_2. \tag{2.22}$$

Set $r_j = d e^{-M_\varepsilon \sqrt{2\pi \log j}/t_j}$. Since $\liminf t_j > 0$, for all sufficiently large j, $d/j < r_j <$

d and hence $t_j v_j(x) > M_\varepsilon$ if and only if $|x| < r_j$. So (2.14) gives

$$I_1 \geq (\beta - \varepsilon) \int_{\{|x|<r_j\}} \frac{e^{\alpha t_j^2 v_j^2}}{|x|^\gamma}\, dx = (\beta - \varepsilon) \left(\int_{\{|x| \leq d/j\}} \frac{e^{\alpha t_j^2 v_j^2}}{|x|^\gamma}\, dx \right.$$

$$\left. + \int_{\{d/j<|x|<r_j\}} \frac{e^{\alpha t_j^2 v_j^2}}{|x|^\gamma}\, dx \right) =: (\beta - \varepsilon)\,(I_3 + I_4). \quad (2.23)$$

We have

$$I_3 = e^{\alpha t_j^2 \log j/2\pi} \int_{\{|x| \leq d/j\}} \frac{dx}{|x|^\gamma} = \frac{2\pi}{2-\gamma} \left(\frac{d}{j}\right)^{2-\gamma} j^{\alpha t_j^2/2\pi} = \frac{2\pi d^{2-\gamma}}{2-\gamma}\, j^{\alpha\,(t_j^2 - t_0^2)/2\pi}. \quad (2.24)$$

Since $th(t)\, e^{\alpha t^2} \geq -C_\varepsilon\, t$ for all $t \geq 0$ by (2.15),

$$I_2 \geq -C_\varepsilon\, t_j \int_{\{t_j v_j \leq M_\varepsilon\}} \frac{v_j}{|x|^\gamma}\, dx \geq -\delta_j t_j. \quad (2.25)$$

Combining (2.22)–(2.25) and noting that $I_4 \geq 0$ gives

$$t_j^2 \geq (\beta - \varepsilon)\, \frac{2\pi d^{2-\gamma}}{2-\gamma}\, j^{\alpha\,(t_j^2 - t_0^2)/2\pi} - \delta_j t_j.$$

It follows from this that

$$\limsup_{j \to \infty}\, t_j \leq t_0,$$

which together with (2.21) shows that $t_j \to t_0$.

Next we estimate I_4. We have

$$\begin{aligned} I_4 &= \int_{\{d/j<|x|<r_j\}} \frac{e^{\alpha t_j^2\,[\log(d/|x|)]^2/2\pi\log j}}{|x|^\gamma}\,dx \\ &= 2\pi\left(\int_{d/j}^{d} e^{\alpha t_j^2\,[\log(d/r)]^2/2\pi\log j}\,r^{1-\gamma}\,dr - \int_{r_j}^{d} e^{\alpha t_j^2\,[\log(d/r)]^2/2\pi\log j}\,r^{1-\gamma}\,dr\right) \\ &= 2\pi d^{2-\gamma}\left(\log j\int_0^1 e^{-(2-\gamma)\,t\,[1-(t_j/t_0)^2\,t]\log j}\,dt - \int_{s_j}^1 s^{1-\gamma}\,e^{\alpha t_j^2\,(\log s)^2/2\pi\log j}\,ds\right), \end{aligned} \tag{2.26}$$

where $t = \log(d/r)/\log j$, $s = r/d$, and $s_j = r_j/d = e^{-M_\varepsilon\sqrt{2\pi\log j}/t_j} \to 0$. For $s_j \le s \le 1$, $\alpha t_j^2\,(\log s)^2/2\pi\log j$ is bounded by αM_ε^2 and goes to zero as $j\to\infty$, so the last integral converges to

$$\int_0^1 s^{1-\gamma}\,ds = \frac{1}{2-\gamma}.$$

So combining (2.22)–(2.26), letting $j\to\infty$, and noting that

$$\lim_{j\to\infty} j^{\alpha\,(t_j^2-t_0^2)/2\pi} \ge \lim_{j\to\infty} j^{-\alpha\delta_j\,t_j/\pi} = \lim_{j\to\infty} e^{-\alpha\delta_j\,t_j\log j/\pi} = 1$$

by (2.19) and (2.20) gives

$$t_0^2 \ge 2\pi\,(\beta-\varepsilon)\,d^{2-\gamma}L,$$

where

$$L = \lim_{j\to\infty}\,\log j\int_0^1 e^{-(2-\gamma)\,t\,[1-(t_j/t_0)^2\,t]\log j}\,dt = \frac{1}{2-\gamma}\,\lim_{j\to\infty}\int_0^1 n e^{-nt\,[1-(t_j/t_0)^2\,t]}\,dt$$

and $n = (2-\gamma)\log j \to \infty$. Letting $\varepsilon \to 0$ in this inequality gives

$$\beta \le \frac{2-\gamma}{L\alpha d^{2-\gamma}}. \tag{2.27}$$

We will show that this leads to a contradiction if (a) or (b) holds.

(a) By (1.5), $G(t_j v_j) \ge 0$ and hence (2.17) gives $t_j \ge t_0$, so

$$L \ge \frac{1}{2-\gamma} \lim_{n\to\infty} \int_0^1 ne^{-nt\,(1-t)}\, dt = \frac{2}{2-\gamma}$$

(see de Figueiredo et al. [11, 9]). Then (2.27) gives $\beta \le (2-\gamma)^2/2\alpha d^{2-\gamma}$, contradicting (1.7).

(b) Let $\kappa > 0$. For all sufficiently large j, $(t_j/t_0)^2 \ge 1-\kappa$ and hence

$$L \ge \frac{1}{2-\gamma} \lim_{n\to\infty} \int_0^1 ne^{-nt\,[1-(1-\kappa)\,t]}\, dt,$$

and letting $\kappa \to 0$ gives $L \ge 1/(2-\gamma)$ (see de Figueiredo et al. [11, 9]). Then $\beta \le (2-\gamma)^2/\alpha d^{2-\gamma}$ by (2.27), contradicting (1.8). $\square$

We are now ready to prove Theorem 1.1 and 1.2.

Proof of Theorem 1.1 and 1.2. The proofs are identical. Let j_0 be as in Lemma 2.3 (ii). By Lemma 2.3 (i), $\exists R > \rho$ such that $E(Rv_{j_0}) \le 0$. Let

$$\Gamma = \left\{\gamma \in C([0,1], H_0^1(\Omega)) : \gamma(0) = 0,\, \gamma(1) = Rv_{j_0}\right\}$$

be the class of paths joining the origin to Rv_{j_0}, and set

$$c := \inf_{\gamma\in\Gamma} \max_{u\in\gamma([0,1])} E(u).$$

By Lemma 2.2, $c > 0$. Since the path $\gamma_0(t) = tRv_{j_0}$, $t \in [0,1]$ is in Γ,

$$c \leq \max_{u \in \gamma_0([0,1])} E(u) \leq \sup_{t \geq 0} E(tv_{j_0}) < \frac{2\pi}{\alpha}\left(1 - \frac{\gamma}{2}\right)$$

by (2.13). If there are no $(\mathrm{PS})_c$ sequences of E, then E satisfies the $(\mathrm{PS})_c$ condition vacuously and hence has a critical point u at the level c by the mountain pass theorem. Then u is a solution of problem (1.2) and u is nontrivial since $c > 0$. So we may assume that E has a $(\mathrm{PS})_c$ sequence. Then this sequence has a subsequence that converges weakly to a nontrivial solution of problem (1.2) by Proposition 2.1. □

Chapter 3

Proof of Theorem 1.3

3.1 Compactness

In this section we consider the modified problem

$$\begin{cases} -\Delta u = \lambda u^+ \dfrac{e^{\alpha\,(u^+)^2}}{|x|^\gamma} + \mu\,\widetilde{g}(u) & \text{in } \Omega \\ \quad u = 0 & \text{on } \partial\Omega, \end{cases} \tag{3.1}$$

where $u^+(x) = \max\{u(x), 0\}$ and

$$\widetilde{g}(t) = \begin{cases} 0, & t \leq -1 \\ (1+t)\,g(0), & -1 < t < 0 \\ g(t), & t \geq 0. \end{cases}$$

Weak solutions of this problem coincide with critical points of the C^1-functional

$$E_\mu(u) = \int_\Omega \left[\frac{1}{2}\,|\nabla u|^2 - \frac{\lambda}{2\alpha}\,\frac{e^{\alpha\,(u^+)^2} - 1}{|x|^\gamma} - \mu\,\widetilde{G}(u)\right] dx, \quad u \in H^1_0(\Omega),$$

where $\widetilde{G}(t) = \int_0^t \widetilde{g}(s)\,ds$. The main result of this section is the following compactness result.

Theorem 3.1. *Assume that $\alpha > 0$ and $0 \le \gamma < 2$ satisfy $\alpha/4\pi + \gamma/2 \le 1$ and g satisfies (1.10) and (1.11). If $\mu_j > 0$, $\mu_j \to \mu \ge 0$, $(u_j) \subset H^1_0(\Omega)$, and*

$$E_{\mu_j}(u_j) \to c, \qquad E'_{\mu_j}(u_j) \to 0$$

for some $c \ne 0$ satisfying

$$c < \frac{2\pi}{\alpha}\left(1 - \frac{\gamma}{2}\right) - \frac{\mu\theta}{2}\,|\Omega|\,, \tag{3.2}$$

where

$$\theta = \sup_{t\in\mathbb{R}}\,\left(2\widetilde{G}(t) - t\widetilde{g}(t)\right)$$

and $|\cdot|$ denotes the Lebesgue measure in $\mathbb{R}^2$, then a subsequence of (u_j) converges to a critical point of E_μ at the level c. In particular, E_μ satisfies the $(\mathrm{PS})_c$ condition for all $c \ne 0$ satisfying (3.2).

First we prove the following lemma.

Lemma 3.2. *If (u_j) is a sequence in $H^1_0(\Omega)$ converging a.e. to $u \in H^1_0(\Omega)$ and*

$$\sup_j \int_\Omega (u_j^+)^2\,\frac{e^{\alpha\,(u_j^+)^2}}{|x|^\gamma}\,dx < \infty, \tag{3.3}$$

then

$$\int_\Omega \frac{e^{\alpha\,(u_j^+)^2}}{|x|^\gamma}\,dx \to \int_\Omega \frac{e^{\alpha\,(u^+)^2}}{|x|^\gamma}\,dx.$$

Proof. For $M > 0$, write

$$\int_\Omega \frac{e^{\alpha\,(u_j^+)^2}}{|x|^\gamma}\,dx = \int_{\{u_j^+ < M\}} \frac{e^{\alpha\,(u_j^+)^2}}{|x|^\gamma}\,dx + \int_{\{u_j^+ \ge M\}} \frac{e^{\alpha\,(u_j^+)^2}}{|x|^\gamma}\,dx.$$

By (3.3),

$$\int_{\{u_j^+ \ge M\}} \frac{e^{\alpha\,(u_j^+)^2}}{|x|^\gamma}\,dx \le \frac{1}{M^2} \int_\Omega (u_j^+)^2\, \frac{e^{\alpha\,(u_j^+)^2}}{|x|^\gamma}\,dx = \mathrm{O}\!\left(\frac{1}{M^2}\right) \text{ as } M \to \infty.$$

Hence

$$\int_\Omega \frac{e^{\alpha\,(u_j^+)^2}}{|x|^\gamma}\,dx = \int_{\{u_j^+ < M\}} \frac{e^{\alpha\,(u_j^+)^2}}{|x|^\gamma}\,dx + \mathrm{O}\!\left(\frac{1}{M^2}\right),$$

and the conclusion follows by first letting $j \to \infty$ and then letting $M \to \infty$. □

We will also need the following result from Adimurthi and Sandeep [14, Theorem 2.3].

Lemma 3.3. *Let $0 \le \gamma < 2$. If (u_j) is a sequence in $H^1_0(\Omega)$ with $\|u_j\| = 1$ for all j and converging weakly to a nonzero function u, then*

$$\sup_j \int_\Omega \frac{e^{\beta u_j^2}}{|x|^\gamma}\,dx < \infty$$

for all $\beta < 4\pi(1 - \gamma/2)/(1 - \|u\|^2)$.

We are now ready to prove Theorem 3.1.

Proof of Theorem 3.1. We have

$$E_{\mu_j}(u_j) = \frac{1}{2}\,\|u_j\|^2 - \frac{\lambda}{2\alpha}\int_\Omega \frac{e^{\alpha\,(u_j^+)^2} - 1}{|x|^\gamma}\,dx - \mu_j \int_\Omega \widetilde{G}(u_j)\,dx = c + \mathrm{o}(1) \tag{3.4}$$

and

$$E'_{\mu_j}(u_j)\,u_j = \|u_j\|^2 - \lambda \int_\Omega (u_j^+)^2\,\frac{e^{\alpha\,(u_j^+)^2}}{|x|^\gamma}\,dx - \mu_j \int_\Omega u_j\,\widetilde{g}(u_j)\,dx = \mathrm{o}(\|u_j\|). \tag{3.5}$$

Multiplying (3.4) by 4 and subtracting (3.5) gives

$$\|u_j\|^2 + \lambda \int_\Omega \left(\left[(u_j^+)^2 - \frac{2}{\alpha}\right] e^{\alpha\,(u_j^+)^2} + \frac{2}{\alpha}\right)\frac{dx}{|x|^\gamma} + \mu_j \int_\Omega \big(u_j\,\widetilde{g}(u_j) - 4\widetilde{G}(u_j)\big)\,dx$$

$$= 4c + \mathrm{o}(\|u_j\| + 1),$$

and this together with (1.10) implies that (u_j) is bounded in $H^1_0(\Omega)$. Hence a renamed subsequence converges to some u weakly in $H^1_0(\Omega)$, strongly in $L^p(\Omega)$ for all $p \in [1, \infty)$, and a.e. in Ω. Moreover,

$$\sup_j \int_\Omega e^{\beta u_j^2}\,dx < \infty$$

for all $\beta \le 4\pi/(\sup_j \|u_j\|)$ by (1.1), and hence $\int_\Omega u_j\,\widetilde{g}(u_j)\,dx$ is bounded by (1.10). Then

$$\sup_j \int_\Omega (u_j^+)^2\,\frac{e^{\alpha\,(u_j^+)^2}}{|x|^\gamma}\,dx < \infty \tag{3.6}$$

by (3.5), and hence

$$\int_\Omega \frac{e^{\alpha\,(u_j^+)^2}}{|x|^\gamma}\,dx \to \int_\Omega \frac{e^{\alpha\,(u^+)^2}}{|x|^\gamma}\,dx \tag{3.7}$$

by Lemma 3.2. Denoting by C a generic positive constant,

$$|u_j\, \widetilde{g}(u_j)| \le |u_j| \left(e^{\alpha\, (u_j^+)^2/2} + C\right) \le \frac{e^{\alpha\, (u_j^+)^2}}{|x|^\gamma} + C\left(u_j^2 + 1\right)$$

by (1.10), so it follows from (3.7) and the dominated convergence theorem that

$$\int_\Omega u_j\, \widetilde{g}(u_j)\, dx \to \int_\Omega u\, \widetilde{g}(u)\, dx. \tag{3.8}$$

Similarly,

$$\int_\Omega \widetilde{G}(u_j)\, dx \to \int_\Omega \widetilde{G}(u)\, dx. \tag{3.9}$$

We claim that the weak limit u is nonzero. Suppose $u = 0$. Then

$$\int_\Omega \frac{e^{\alpha\, (u_j^+)^2}}{|x|^\gamma}\, dx \to \int_\Omega \frac{dx}{|x|^\gamma}, \qquad \int_\Omega u_j\, \widetilde{g}(u_j)\, dx \to 0, \qquad \int_\Omega \widetilde{G}(u_j)\, dx \to 0 \tag{3.10}$$

by (3.7)–(3.9). So (3.4) implies that $c \ge 0$ and

$$\|u_j\| \to (2c)^{1/2}. \tag{3.11}$$

Noting that $c < 2\pi\, (1 - \gamma/2)/\alpha$ by (3.2), let $2c < \nu < 4\pi\, (1 - \gamma/2)/\alpha$. Then (3.11) implies that $\|u_j\| \le \nu^{1/2}$ for all $j \ge j_0$ for some j_0. Let $q = 4\pi\, (1 - \gamma/2)/\alpha\nu > 1$ and let $1/(1 - 1/q) < r < 2/\gamma\, (1 - 1/q)$. By the Hölder inequality,

$$\int_\Omega (u_j^+)^2\, \frac{e^{\alpha\, (u_j^+)^2}}{|x|^\gamma}\, dx \le \left(\int_\Omega |u_j|^{2p}\, dx\right)^{1/p} \left(\int_\Omega \frac{e^{q\alpha u_j^2}}{|x|^\gamma}\, dx\right)^{1/q} \left(\int_\Omega \frac{dx}{|x|^{\gamma r\, (1 - 1/q)}}\right)^{1/r},$$

where $1/p + 1/q + 1/r = 1$. The first integral on the right-hand side converges to zero since $u = 0$, the second integral is bounded for $j \ge j_0$ by (1.1) since

$q\alpha u_j^2 = 4\pi\,(1-\gamma/2)\,\widetilde{u}_j^2$, where $\widetilde{u}_j = u_j/\nu^{1/2}$ satisfies $\|\widetilde{u}_j\| \le 1$, and the last integral is finite since $\gamma r\,(1 - 1/q) < 2$, so

$$\int_\Omega (u_j^+)^2\, \frac{e^{\alpha\,(u_j^+)^2}}{|x|^\gamma}\, dx \to 0.$$

Then $u_j \to 0$ by (3.5) and (3.10), and hence $c = 0$ by (3.11), a contradiction. So u is nonzero.

Since $E'_{\mu_j}(u_j) \to 0$,

$$\int_\Omega \nabla u_j \cdot \nabla v\, dx - \lambda \int_\Omega u_j^+\, \frac{e^{\alpha\,(u_j^+)^2}}{|x|^\gamma}\, v\, dx - \mu_j \int_\Omega \widetilde{g}(u_j)\, v\, dx \to 0 \tag{3.12}$$

for all $v \in H_0^1(\Omega)$. For $v \in C_0^\infty(\Omega)$, an argument similar to that in the proof of Lemma 3.2 using the estimate

$$\left| \int_{\{u_j^+ \ge M\}} u_j^+\, \frac{e^{\alpha\,(u_j^+)^2}}{|x|^\gamma}\, v\, dx \right| \le \frac{\sup |v|}{M} \int_\Omega (u_j^+)^2\, \frac{e^{\alpha\,(u_j^+)^2}}{|x|^\gamma}\, dx$$

and (3.6) shows that $\displaystyle\int_\Omega u_j^+\, \frac{e^{\alpha\,(u_j^+)^2}}{|x|^\gamma}\, v\, dx \to \int_\Omega u^+\, \frac{e^{\alpha\,(u^+)^2}}{|x|^\gamma}\, v\, dx$. Moreover, denoting by C a generic positive constant,

$$|\widetilde{g}(u_j)\, v| \le \sup |v| \left(e^{\alpha\,(u_j^+)^2} + C\right) \le C\, \sup |v| \left(\frac{e^{\alpha\,(u_j^+)^2}}{|x|^\gamma} + 1\right)$$

by (1.10), so it follows from (3.7) and the dominated convergence theorem that

$$\int_\Omega \widetilde{g}(u_j)\, v\, dx \to \int_\Omega \widetilde{g}(u)\, v\, dx.$$

So it follows from (3.12) that

$$\int_\Omega \nabla u \cdot \nabla v \, dx = \lambda \int_\Omega u^+ \, \frac{e^{\alpha \, (u^+)^2}}{|x|^\gamma} \, v \, dx + \mu \int_\Omega \widetilde{g}(u) \, v \, dx.$$

Then this holds for all $v \in H^1_0(\Omega)$ by density, and taking $v = u$ gives

$$\|u\|^2 = \lambda \int_\Omega (u^+)^2 \, \frac{e^{\alpha \, (u^+)^2}}{|x|^\gamma} \, dx + \mu \int_\Omega u \, \widetilde{g}(u) \, dx. \tag{3.13}$$

Next we claim that

$$\int_\Omega (u_j^+)^2 \, \frac{e^{\alpha \, (u_j^+)^2}}{|x|^\gamma} \, dx \to \int_\Omega (u^+)^2 \, \frac{e^{\alpha \, (u^+)^2}}{|x|^\gamma} \, dx. \tag{3.14}$$

We have

$$(u_j^+)^2 \, \frac{e^{\alpha \, (u_j^+)^2}}{|x|^\gamma} \le u_j^2 \, \frac{e^{\alpha u_j^2}}{|x|^\gamma} = u_j^2 \, \frac{e^{\alpha \, \|u_j\|^2 \, \widetilde{u}_j^2}}{|x|^\gamma}, \tag{3.15}$$

where $\widetilde{u}_j = u_j / \|u_j\|$. Setting

$$\kappa = \frac{\lambda}{2\alpha} \int_\Omega \frac{e^{\alpha \, (u^+)^2} - 1}{|x|^\gamma} \, dx + \mu \int_\Omega \widetilde{G}(u) \, dx,$$

we have

$$\|u_j\|^2 \to 2 \, (c + \kappa)$$

by (3.4), (3.7), and (3.9), so $\widetilde{u}_j$ converges weakly and a.e. to $\widetilde{u} = u/[2 \, (c + \kappa)]^{1/2}$. Then

$$\|u_j\|^2 \left(1 - \|\widetilde{u}\|^2\right) \to 2 \, (c + \kappa) - \|u\|^2 . \tag{3.16}$$

Since $te^t \geq e^t - 1$ for all $t \geq 0$,

$$\int_\Omega (u^+)^2 \, \frac{e^{\alpha\,(u^+)^2}}{|x|^\gamma}\, dx \geq \frac{1}{\alpha} \int_\Omega \frac{e^{\alpha\,(u^+)^2} - 1}{|x|^\gamma}\, dx,$$

and

$$\int_\Omega u\, \widetilde{g}(u)\, dx \geq 2 \int_\Omega \widetilde{G}(u)\, dx - \theta\, |\Omega|$$

since $\theta \geq 2\widetilde{G}(t) - t\widetilde{g}(t)$ for all $t \in \mathbb{R}$, so it follows from (3.13) that $\|u\|^2 \geq 2\kappa - \mu\theta\, |\Omega|$. Hence

$$2\,(c + \kappa) - \|u\|^2 \leq 2c + \mu\theta\, |\Omega| < \frac{4\pi}{\alpha} \left(1 - \frac{\gamma}{2}\right) \tag{3.17}$$

by (3.2). We are done if $\|\widetilde{u}\| = 1$, so suppose $\|\widetilde{u}\| < 1$ and let

$$\frac{2c + \mu\theta\, |\Omega|}{1 - \|\widetilde{u}\|^2} < \widetilde{\nu} - 2\varepsilon < \widetilde{\nu} < \frac{4\pi\,(1 - \gamma/2)/\alpha}{1 - \|\widetilde{u}\|^2}.$$

Then $\|u_j\|^2 \leq \widetilde{\nu} - 2\varepsilon$ for all $j \geq j_0$ for some j_0 by (3.16) and (3.17), and

$$\sup_j \int_\Omega \frac{e^{\alpha \widetilde{\nu}\, \widetilde{u}_j^2}}{|x|^\gamma}\, dx < \infty \tag{3.18}$$

by Lemma 3.3. For $M > 0$ and $j \geq j_0$, (3.15) then gives

$$\begin{aligned}
&\int_{\{u_j^+ \geq M\}} (u_j^+)^2 \, \frac{e^{\alpha\,(u_j^+)^2}}{|x|^\gamma}\, dx \\
&\leq \int_{\{u_j^+ \geq M\}} u_j^2 \, \frac{e^{\alpha\,(\widetilde{\nu} - 2\varepsilon)\, \widetilde{u}_j^2}}{|x|^\gamma}\, dx \\
&= \|u_j\|^2 \int_{\{u_j^+ \geq M\}} \widetilde{u}_j^2\, e^{-\varepsilon\alpha\, \widetilde{u}_j^2}\, e^{-\varepsilon\alpha\,(u_j/\|u_j\|)^2}\, \frac{e^{\alpha\widetilde{\nu}\, \widetilde{u}_j^2}}{|x|^\gamma}\, dx \\
&\leq \left(\max_{t \geq 0} te^{-\varepsilon\alpha\, t}\right) \|u_j\|^2\, e^{-\varepsilon\alpha\,(M/\|u_j\|)^2} \int_\Omega \frac{e^{\alpha\widetilde{\nu}\, \widetilde{u}_j^2}}{|x|^\gamma}\, dx.
\end{aligned}$$

The last expression goes to zero as $M \to \infty$ uniformly in j since $\|u_j\|$ is bounded and (3.18) holds, so (3.14) now follows as in the proof of Lemma 3.2.

Now it follows from (3.5), (3.14), (3.8), and (3.13) that

$$\|u_j\|^2 \to \lambda \int_\Omega (u^+)^2 \, \frac{e^{\alpha\,(u^+)^2}}{|x|^\gamma} \, dx + \mu \int_\Omega u \, \widetilde{g}(u) \, dx = \|u\|^2$$

and hence $\|u_j\| \to \|u\|$, so $u_j \to u$. Clearly, $E_\mu(u) = c$ and $E'_\mu(u) = 0$. □

3.2 Proof of Theorem 1.3

In this section we prove our main result. By Theorem 3.1, E_μ satisfies the $(\mathrm{PS})_c$ condition for all $c \neq 0$ satisfying

$$c < \frac{2\pi}{\alpha} \left(1 - \frac{\gamma}{2}\right) - \frac{\mu\theta}{2} \, |\Omega| \, .$$

First we show that E_μ has a uniformly positive mountain pass level below this threshold for compactness for all sufficiently small $\mu > 0$. Take $r > 0$ so small that $\overline{B_r(0)} \subset \Omega$ and let

$$v_j(x) = \frac{1}{\sqrt{2\pi}} \begin{cases} \sqrt{\log j}, & |x| \le r/j \\[2ex] \dfrac{\log(r/|x|)}{\sqrt{\log j}}, & r/j < |x| < r \\[2ex] 0, & |x| \ge r. \end{cases}$$

It is easily seen that $v_j \in H_0^1(\Omega)$ with $\|v_j\| = 1$ and

$$\int_\Omega v_j^2\,dx = \mathrm{O}(1/\log j) \quad \text{as } j \to \infty. \tag{3.19}$$

Lemma 3.4. *There exist $\mu_0, \rho, c_0 > 0$, $j_0 \ge 2$, $R > \rho$, and $\vartheta < \dfrac{2\pi}{\alpha}\left(1 - \dfrac{\gamma}{2}\right)$ such that the following hold for all $\mu \in (0, \mu_0)$:*

(i) $\|u\| = \rho \implies E_\mu(u) \ge c_0$,

(ii) $E_\mu(Rv_{j_0}) \le 0$,

(iii) denoting by $\Gamma = \{\gamma \in C([0,1], H_0^1(\Omega)) : \gamma(0) = 0,\, \gamma(1) = Rv_{j_0}\}$ the class of paths joining the origin to Rv_{j_0},

$$c_0 \le c_\mu := \inf_{\gamma \in \Gamma}\ \max_{u \in \gamma([0,1])} E_\mu(u) \le \vartheta + C\mu^2 \tag{3.20}$$

for some constant $C > 0$,

(iv) E_μ has a critical point u_μ at the level c_μ.

Proof. Set $\rho = \|u\|$ and $\widetilde{u} = u/\rho$. Since $e^t - 1 \le t + t^2 e^t$ for all $t \ge 0$,

$$\frac{1}{\alpha} \int_\Omega \frac{e^{\alpha\,(u^+)^2} - 1}{|x|^\gamma}\,dx \le \int_\Omega \frac{u^2}{|x|^\gamma}\,dx + \alpha \int_\Omega u^4\,\frac{e^{\alpha u^2}}{|x|^\gamma}\,dx. \tag{3.21}$$

By (1.4),

$$\int_\Omega \frac{u^2}{|x|^\gamma}\,dx \le \frac{\rho^2}{\lambda_1(\gamma)}. \tag{3.22}$$

Let $2 < r < 4/\gamma$. By the Hölder inequality,

$$\int_\Omega u^4\,\frac{e^{\alpha u^2}}{|x|^\gamma}\,dx \le \left(\int_\Omega u^{4p}\,dx\right)^{1/p} \left(\int_\Omega \frac{e^{2\alpha u^2}}{|x|^\gamma}\,dx\right)^{1/2} \left(\int_\Omega \frac{dx}{|x|^{\gamma r/2}}\right)^{1/r}, \tag{3.23}$$

where $1/p + 1/r = 1/2$. The first integral on the right-hand side is bounded by $C\rho^4$ for some constant $C > 0$ by the Sobolev embedding. Since $2\alpha u^2 = 2\alpha\rho^2\,\widetilde{u}^2$ and $\|\widetilde{u}\| = 1$, the second integral is bounded when $\rho^2 \le 2\pi\,(1 - \gamma/2)/\alpha$ by (1.1). The last integral is finite since $\gamma r < 4$. So combining (3.21)–(3.23) gives

$$\frac{1}{\alpha} \int_\Omega \frac{e^{\alpha\,(u^+)^2} - 1}{|x|^\gamma}\, dx \le \frac{\rho^2}{\lambda_1(\gamma)} + \mathrm{O}(\rho^4) \quad \text{as } \rho \to 0.$$

On the other hand, it follows from (1.10) that $\displaystyle\int_\Omega \widetilde{G}(u)\, dx$ is bounded on bounded subsets of $H^1_0(\Omega)$. So

$$E_\mu(u) \ge \frac{1}{2} \left(1 - \frac{\lambda}{\lambda_1(\gamma)}\right) \rho^2 + \mathrm{O}(\rho^4) - C\mu \quad \text{as } \rho \to 0$$

for some constant $C > 0$. Since $\lambda(\gamma) < \lambda_1$, (i) follows from this for sufficiently small $\rho, \mu, c_0 > 0$.

Since $\|v_j\| = 1$ and $v_j \ge 0$,

$$E_\mu(tv_j) = \frac{t^2}{2} - \int_\Omega \left[\frac{\lambda}{2\alpha}\, \frac{e^{\alpha t^2 v_j^2} - 1}{|x|^\gamma} + \mu\, G(tv_j)\right] dx$$

for $t \ge 0$. For $\mu \le \lambda/2$, this gives

$$E_\mu(tv_j) \le \frac{t^2}{2} - \int_\Omega \left[\frac{\lambda}{4\alpha}\, \frac{e^{\alpha t^2 v_j^2} - 1}{|x|^\gamma} + \mu\, F(x, tv_j)\right] dx,$$

where

$$F(x,t) = \frac{1}{2\alpha}\, \frac{e^{\alpha t^2} - 1}{|x|^\gamma} + G(t) = \int_0^t \left(s\, \frac{e^{\alpha s^2}}{|x|^\gamma} + g(s)\right) ds \ge -Ct$$

for some generic positive constant C by (1.10), so

$$E_\mu(tv_j) \leq \frac{t^2}{2} - \frac{\lambda}{4\alpha} \int_\Omega \frac{e^{\alpha t^2 v_j^2} - 1}{|x|^\gamma}\, dx + C\mu t \int_\Omega v_j\, dx.$$

Since

$$C\mu t \int_\Omega v_j\, dx \leq C\mu t \left(\int_\Omega v_j^2\, dx\right)^{1/2} \leq C\mu^2 + \frac{t^2}{2} \int_\Omega v_j^2\, dx,$$

then

$$E_\mu(tv_j) \leq H_j(t) + C\mu^2,$$

where

$$H_j(t) = \frac{t^2}{2} \left(1 + \int_\Omega v_j^2\, dx\right) - \frac{\lambda}{4\alpha} \int_\Omega \frac{e^{\alpha t^2 v_j^2} - 1}{|x|^\gamma}\, dx \to -\infty \quad \text{as } t \to \infty.$$

So to prove (ii) and (iii), it suffices to show that $\exists j_0 \geq 2$ such that

$$\vartheta := \sup_{t \geq 0} H_{j_0}(t) < \frac{2\pi}{\alpha} \left(1 - \frac{\gamma}{2}\right).$$

Suppose $\sup_{t \geq 0} H_j(t) \geq 2\pi\,(1 - \gamma/2)/\alpha$ for all j. Since $H_j(t) \to -\infty$ as $t \to \infty$, there exists $t_j > 0$ such that

$$H_j(t_j) = \frac{t_j^2}{2}\,(1 + \varepsilon_j) - \frac{\lambda}{4\alpha} \int_\Omega \frac{e^{\alpha t_j^2 v_j^2} - 1}{|x|^\gamma}\, dx = \sup_{t \geq 0} H_j(t) \geq \frac{2\pi}{\alpha} \left(1 - \frac{\gamma}{2}\right) \tag{3.24}$$

and

$$H_j'(t_j) = t_j \left(1 + \varepsilon_j - \frac{\lambda}{2} \int_\Omega v_j^2\, \frac{e^{\alpha t_j^2 v_j^2}}{|x|^\gamma}\, dx\right) = 0, \tag{3.25}$$

where $\varepsilon_j = \int_\Omega v_j^2 \, dx \to 0$ by (3.19). The inequality in (3.24) gives

$$\alpha t_j^2 \geq \frac{4\pi}{1+\varepsilon_j} \left(1 - \frac{\gamma}{2}\right),$$

and then (3.25) gives

$$\frac{2}{\lambda} (1+\varepsilon_j) = \int_\Omega v_j^2 \, \frac{e^{\alpha t_j^2 v_j^2}}{|x|^\gamma} \, dx \geq \int_{B_{r/j}(0)} v_j^2 \, \frac{e^{4\pi \, (1-\gamma/2) \, v_j^2/(1+\varepsilon_j)}}{|x|^\gamma} \, dx$$
$$= \frac{r^{2\,(1-\gamma/2)}}{2\,(1-\gamma/2)} \, \frac{\log j}{j^{2\,(1-\gamma/2)\,\varepsilon_j/(1+\varepsilon_j)}}.$$

This is impossible for large j since

$$j^{2\,(1-\gamma/2)\,\varepsilon_j/(1+\varepsilon_j)} \leq j^{2\,(1-\gamma/2)\,\varepsilon_j} = e^{2\,(1-\gamma/2)\,\varepsilon_j \log j} = \mathrm{O}(1)$$

by (3.19).

By (i)–(iii), E_μ has the mountain pass geometry and the mountain pass level c_μ satisfies

$$0 < c_\mu \leq \vartheta + C\mu^2 < \frac{2\pi}{\alpha} \left(1 - \frac{\gamma}{2}\right) - \frac{\mu\theta}{2} \, |\Omega|$$

for all sufficiently small $\mu > 0$, so E_μ satisfies the $(\mathrm{PS})_{c_\mu}$ condition. So E_μ has a critical point u_μ at this level by the mountain pass theorem. □

Next we prove the following lemma.

Lemma 3.5. *If (u_j) is a convergent sequence in $H^1_0(\Omega)$, then*

$$\sup_j \int_\Omega \frac{e^{\beta u_j^2}}{|x|^\gamma} \, dx < \infty$$

for all $\beta > 0$ and $0 \le \gamma < 2$.

Proof. Let $u \in H_0^1(\Omega)$ be the limit of (u_j). Since $u_j^2 \le (|u| + |u_j - u|)^2 \le 2u^2 + 2\,(u_j - u)^2$,

$$\int_\Omega \frac{e^{\beta u_j^2}}{|x|^\gamma}\,dx \le \left(\int_\Omega \frac{e^{4\beta u^2}}{|x|^\gamma}\,dx\right)^{1/2} \left(\int_\Omega \frac{e^{4\beta\,(u_j - u)^2}}{|x|^\gamma}\,dx\right)^{1/2}.$$

The first integral on the right-hand side is finite, and the second integral equals

$$\int_\Omega \frac{e^{4\beta\,\|u_j - u\|^2 w_j^2}}{|x|^\gamma}\,dx,$$

where $w_j = (u_j - u)/\,\|u_j - u\|$. Since $\|w_j\| = 1$ and $\|u_j - u\| \to 0$, this integral is bounded by (1.1). □

Now we show that u_μ is positive in Ω, and hence a solution of problem (1.9), for all sufficiently small $\mu \in (0, \mu_0)$. It suffices to show that for every sequence $\mu_j > 0$, $\mu_j \to 0$, a subsequence of $u_j = u_{\mu_j}$ is positive in Ω. By (3.20), a renamed subsequence of c_{μ_j} converges to some c satisfying

$$0 < c < \frac{2\pi}{\alpha}\left(1 - \frac{\gamma}{2}\right).$$

Then a renamed subsequence of (u_j) converges in $H_0^1(\Omega)$ to a critical point u of E_0 at the level c by Theorem 3.1. Since $c > 0$, u is nontrivial.

Since u_j is a critical point of E_{μ_j},

$$-\Delta u_j = \lambda u_j^+\,\frac{e^{\alpha\,(u_j^+)^2}}{|x|^\gamma} + \mu_j\,\widetilde{g}(u_j)$$

in Ω. Let $2 < p < 2/\gamma$ and $1 < r < 2/\gamma p$. By the Hölder inequality,

$$\int_\Omega \left| u_j^+ \, \frac{e^{\alpha\,(u_j^+)^2}}{|x|^\gamma} \right|^p dx \le \left(\int_\Omega |u_j|^{pq}\, dx \right)^{1/q} \left(\int_\Omega \frac{e^{pr\alpha u_j^2}}{|x|^{\gamma pr}}\, dx \right)^{1/r},$$

where $1/q + 1/r = 1$. The first integral on the right-hand side is bounded by the Sobolev embedding, and so is the second integral by Lemma 3.5 since $\gamma pr < 2$, so $u_j^+ \, e^{\alpha\,(u_j^+)^2}/|x|^\gamma$ is bounded in $L^p(\Omega)$. By (1.10) and Lemma 3.5 again, $\widetilde{g}(u_j)$ is also bounded in $L^p(\Omega)$. By the Calderon-Zygmund inequality, then (u_j) is bounded in $W^{2,p}(\Omega)$. Since $W^{2,p}(\Omega)$ is compactly embedded in $C^1(\overline{\Omega})$ for $p > 2$, it follows that a renamed subsequence of u_j converges to u in $C^1(\overline{\Omega})$.

Since u is a nontrivial solution of the problem

$$\begin{cases} -\Delta u = \lambda u^+ \, \dfrac{e^{\alpha\,(u^+)^2}}{|x|^\gamma} & \text{in } \Omega \\[2ex] \quad u = 0 & \text{on } \partial\Omega, \end{cases}$$

$u > 0$ in Ω by the strong maximum principle and its interior normal derivative $\partial u/\partial \nu > 0$ on $\partial\Omega$ by the Hopf lemma. Since $u_j \to u$ in $C^1(\overline{\Omega})$, then $u_j > 0$ in Ω for all sufficiently large j. This concludes the proof of Theorem 1.3.

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