Boundary Value Problems in a Multidimensional Box for Higher Order Linear and Quasi-Linear Hyperbolic Equations

By

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ABSTRACT

Title: *Multi dimensional Boundary Value Problems for Linear and Quasi-Linear Hyperbolic Equations of Higher Order*

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Boundary value problems in a multidimensional box for higher order linear hyperbolic equations are considered. The concept of associated problems are introduced.

For general boundary value problems there are established:

(i) Necessary and sufficient conditions for a linear problem to have the Fredholm property in two–dimensional case;

(ii) Necessary and sufficient conditions of well–posedness in two–dimensional case;

(iii) Unimprovable sufficient conditions for a linear problem to have the Fredholm property;

(iv) Unimprovable sufficient conditions of well–posedness and α–well–posedness;

(v) Effective sufficient conditions of unqie solvability of two–point, periodic and Dirichlet type problems.

(iv) Unimprovable conditions of unique solvability of two dimensional ill–posed periodic problems.

For the Dirichlet type problem in a two–dimensional smooth convex domain:

(i) Sufficient conditions for a linear problem to have the Fredholm property;

(ii) sufficient conditions of unique solvability.

For quasi–linear boundary value problems there are established:

(i) Optimal sufficient conditions of solvability and unique solvability;

(ii) Effective sufficient conditions of solvability of periodic and Dirichlet type problems in case, where the righthand side of the equation has arbitrary growth order with respect to some phase variables.

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List of Notations

$\mathbf{m} = (m_1, \dots, m_n)$, $\boldsymbol{\alpha} = (\alpha_1, \dots, \alpha_n)$.

$\boldsymbol{\alpha} = (\alpha_1, \dots, \alpha_n) < \boldsymbol{\beta} = (\beta_1, \dots, \beta_n) \Longleftrightarrow \alpha_i \le \beta_i$ $(i = 1, \dots, n)$ and $\boldsymbol{\alpha} \ne \boldsymbol{\beta}$.

$\boldsymbol{\alpha} = (\alpha_1, \dots, \alpha_n) \le \boldsymbol{\beta} = (\beta_1, \dots, \beta_n) \Longleftrightarrow \boldsymbol{\alpha} < \boldsymbol{\beta}$, or $\boldsymbol{\alpha} = \boldsymbol{\beta}$.

$\mathbf{0} = (0, \dots, 0)$, $\mathbf{1} = (1, \dots, 1)$, $\mathbf{1}_i = (0, \dots, 0, 1, 0, \dots, 0)$.

$\operatorname{supp} \boldsymbol{\alpha} = \{i \mid \ \alpha_i > 0\}$, $\|\boldsymbol{\alpha}\| = |\alpha_1| + \dots + |\alpha_n|$.

$\boldsymbol{\Xi} = \big\{\boldsymbol{\sigma} \big| \ \mathbf{0} < \boldsymbol{\sigma} < \mathbf{1}\big\}$.

$\widehat{\boldsymbol{\alpha}} = \mathbf{m} - \boldsymbol{\alpha}$. If $\boldsymbol{\sigma} \in \boldsymbol{\Xi}$, then $\widehat{\boldsymbol{\sigma}} = \mathbf{1} - \boldsymbol{\sigma}$.

$\mathbf{m}_{\boldsymbol{\sigma}} = (\sigma_1 m_1, \dots, \sigma_n m_n)$. It is clear that $\widehat{\mathbf{m}}_{\boldsymbol{\sigma}} = \mathbf{m} - \mathbf{m}_{\boldsymbol{\sigma}} = \mathbf{m}_{\widehat{\boldsymbol{\sigma}}}$.

$\boldsymbol{\Upsilon}_{\mathbf{m}} = \big\{\boldsymbol{\alpha} \le \mathbf{m} : \alpha_i = m_i \ \text{ for some } \ i \in \{1, \dots, n\}\big\}$.

$\boldsymbol{\omega} = (\omega_1, \dots, \omega_n)$, $\boldsymbol{\omega}_{\mathbf{i}} = (0, \dots, \omega_i, \dots, 0)$.

$\mathbf{x}_{\boldsymbol{\sigma}} = (\sigma_1 x_1, \dots, \sigma_n x_n)$. $\mathbf{x}_{\boldsymbol{\sigma}}$ will be identified with $(x_{i_1}, \dots, x_{i_l})$, as well as the set $\Omega_{\boldsymbol{\sigma}} = [0, \sigma_1 \omega_1] \times \dots \times [0, \sigma_n \omega_n]$ will be identified with the set $[0, \omega_{i_1}] \times \dots \times [0, \omega_{i_l}]$, where $\{i_1, \cdots i_l\} = \operatorname{supp} \boldsymbol{\sigma}$.

$$u^{(\boldsymbol{\alpha})}(\mathbf{x}) = \frac{\partial^{\alpha_1 + \dots + \alpha_n} u(\mathbf{x})}{\partial x_1^{\alpha_1} \cdots \partial x_n^{\alpha_n}}.$$

$C^{\mathbf{m}}(\Omega)$ is the Banach space of functions $u : \Omega \to \mathbb{R}$, having continuous partial derivatives $u^{(\boldsymbol{\alpha})}$ $(\boldsymbol{\alpha} \le \mathbf{m})$, endowed with the norm

$$\|u\|_{C^{\mathbf{m}}(\Omega)} = \sum_{\boldsymbol{\alpha} \le \mathbf{m}} \|u^{(\boldsymbol{\alpha})}\|_{C(\Omega)}.$$

$C_0^{\mathbf{m}}(\Omega)$ is subspace $C^{\mathbf{m}}(\Omega)$ consisting of functions having a compact support, i.e.

$$u^{(\boldsymbol{\alpha})}(\mathbf{x})\Big|_{\mathbf{x} \in \partial\Omega} = 0 \qquad (\boldsymbol{\alpha} \le \mathbf{m}).$$

$C_{\boldsymbol{\omega}}^{\mathbf{m}}(\mathbb{R}^n)$ is the Banach space of $\boldsymbol{\omega}$-periodic continuous functions, i.e. functions that are ω_i-periodic with respect to the variable x_i $(i = 1, \dots, n)$, having continuous partial derivatives $u^{(\boldsymbol{\alpha})}$ $(\boldsymbol{\alpha} \le \mathbf{m})$, endowed with the norm

$$\|u\|_{C_{\boldsymbol{\omega}}^{\mathbf{m}}} = \sum_{\boldsymbol{\alpha} \le \mathbf{m}} \|u^{(\boldsymbol{\alpha})}\|_{C(\Omega)}.$$

$L(\Omega)$ is the Banach space of Lebesgue integrable functions $u : \Omega \to \mathbb{R}$ endowed with the norm

$$\|u\|_L = \iint\limits_{\Omega} |u(\mathbf{x})| \, d\mathbf{x}.$$

$L_{\boldsymbol{\omega}}(\Omega)$ is the Banach space of Lebesgue integrable $\boldsymbol{\omega}$-periodic functions $u : \mathbb{R}^n \to \mathbb{R}$ endowed with the norm

$$\|u\|_{L_{\boldsymbol{\omega}}} = \iint\limits_{\Omega} |u(\mathbf{x})| \, d\mathbf{x}.$$

$L^2(\Omega)$ is the Banach space of Lebesgue square integrable functions $u : \Omega \to \mathbb{R}$ endowed with the norm

$$\|u\|_L = \left(\iint\limits_{\Omega} u^2(\mathbf{x}) \, d\mathbf{x} \right)^{\frac{1}{2}}.$$

$L^2_{\boldsymbol{\omega}}(\Omega)$ is the Banach space of locally square Lebesgue integrable $\boldsymbol{\omega}$-periodic functions $u : \mathbb{R}^n \to \mathbb{R}$ endowed with the norm

$$\|u\|_{L^2_{\boldsymbol{\omega}}} = \left(\iint\limits_{\Omega} u^2(\mathbf{x}) \, d\mathbf{x} \right)^{\frac{1}{2}}.$$

$L^\infty(\Omega)$ is the spaces of essentially bounded measurable functions $u : \Omega \to \mathbb{R}$ endowed with the norm

$$\|u\|_{L^\infty} = \operatorname*{ess\,sup}_{\mathbf{x} \in \Omega} |u(\mathbf{x})|.$$

$L^\infty_{\boldsymbol{\omega}}(\Omega)$ is the spaces of essentially bounded $\boldsymbol{\omega}$-periodic functions $u : \mathbb{R}^n \to \mathbb{R}$ endowed with the norm

$$\|u\|_{L^\infty_{\boldsymbol{\omega}}} = \operatorname*{ess\,sup}_{\mathbf{x} \in \Omega} |u(\mathbf{x})|.$$

$AC([0,\omega])$ is the Banach space of absolutely continuous functions $u : [0,\omega] \to \mathbb{R}$ endowed with the norm

$$\|u\|_{AC} = |u(0)| + \int\limits_0^\omega \left| u'(x) \right| dx.$$

$AC(\Omega)$ is the Banach space of absolutely continuous functions $u : \Omega \to \mathbb{R}$ endowed with the norm

$$\|u\|_{AC(\Omega)} = |u(\mathbf{0})| + \sum_{\boldsymbol{\sigma} \leq \mathbf{1}} \iint\limits_{\Omega_{\boldsymbol{\sigma}}} |u^{(\boldsymbol{\sigma})}(\mathbf{x}_{\boldsymbol{\sigma}})| \, d\mathbf{x}_{\boldsymbol{\sigma}}$$

$AC^{\mathbf{m-1}}(\Omega)$ is the Banach space of functions $u \in C^{(\mathbf{m-1})}(\Omega)$, having absolutely continuous derivative $u^{(\mathbf{m-1})}$, endowed with the norm

$$\|u\|_{AC^{\mathbf{m-1}}(\Omega)} = \|u\|_{C^{\mathbf{m-1}}(\Omega)} + \|u^{(\mathbf{m-1})}\|_{AC(\Omega)}.$$

Introduction

In the present dissertation for the higher order linear hyperbolic equation

$$u^{(\mathbf{m})} = \sum_{\boldsymbol{\alpha}<\mathbf{m}} p_{\boldsymbol{\alpha}}(\mathbf{x})u^{(\boldsymbol{\alpha})} + q(\mathbf{x}) \tag{0.1}$$

we investigate boundary conditions

$$h_{ik}\big(u^{(\mathbf{m}^{i-1})}(\bullet, \widehat{\mathbf{x}}_i)\big) = \varphi_{ik}^{(\mathbf{m}^{i-1})}(\widehat{\mathbf{x}}_i) \ \text{ for } \ \widehat{\mathbf{x}}_i \in \widehat{\Omega}_i \ \ (k = 1, \dots, m_i; \ \ i = 1, \dots, n), \tag{0.2}$$

where $\mathbf{x} = (x_1, \dots, x_n)$, $\widehat{\mathbf{x}}_i = (x_1, \dots, x_{i-1}, x_{i+1}, \dots, x_n)$, $\widehat{\Omega}_i = [0, \omega_1] \times \cdots \times [0, \omega_{i-1}] \times [0, \omega_{i+1}] \times \cdots \times [0, \omega_n]$, $\mathbf{m} = (m_1, \dots, m_n)$, $\boldsymbol{\alpha} = (\alpha_1, \dots, \alpha_n)$, $\mathbf{m}^i = (m_1, \dots, m_i, 0, \dots, 0)$ ($\mathbf{m}^i = (0, \dots, 0)$ if $i = 0$), $\mathbf{m}_i = (0, \dots, m_i, \dots, 0)$ and $\widehat{\mathbf{m}}_i = \mathbf{m} - \mathbf{m}_i$ are multi-indices,

$$u^{(\boldsymbol{\alpha})}(\mathbf{x}) = \frac{\partial^{\alpha_1+\cdots+\alpha_n} u(\mathbf{x})}{\partial x_1^{\alpha_1} \cdots \partial x_n^{\alpha_n}} \,,$$

$p_{\boldsymbol{\alpha}} \in C(\Omega)$ $(\boldsymbol{\alpha} < \mathbf{m})$, $q \in C(\Omega)$, $\varphi_{ik} \in C^{\widehat{\mathbf{m}}_i}(\Omega_i)$ $(k = 1, \dots, m_i;\ i = 1, \dots, n)$, and $h_{ik} : C^{m_i-1}([0, \omega_i]) \to \mathbb{R}$ $(k = 1, \dots, m_i;\ i = 1, \dots, n)$ are bounded linear functionals.

By a solution of problem (0.1),(0.2) we understand a classical solution, i.e., a function $u \in C^{\mathbf{m}}(\Omega)$ satisfying equation (0.1) and boundary conditions (0.2) everywhere in Ω.

Problem (0.1),(0.2) does not belong to the classical boundary value problems of mathematical physics, with the exception of Darboux and Goursat initial value problems for second order hyperbolic equations (the case $n = 2$, $\mathbf{m} = (1, 1)$).

Beginning from the 1960ies, two–dimensional problems on periodic solutions, as well as problems with boundary conditions connecting the values of an unknown solution in various characteristics have been intensively studied for partial differential equations of hyperbolic type (see [4-17, 41–55]). These problems naturally led to the initial–boundary value problems in a rectangle with general boundary conditions:

$$w^{(1,1)} = P_0(x, y)w + P_1(x, y)w^{(1,0)} + P_2(x, y)w^{(0,1)} + q(x, y), \tag{0.3}$$

$$w(0, y) = \varphi(y), \qquad\qquad h\big(w^{(1,0)}(x, \cdot)\big)(x) = \psi(x), \tag{0.4}$$

where $P_i \in C([0,a]\times[0,b];\mathbb{R}^{n\times n})$ $(i=0,1,2)$, $q \in C([0,a]\times[0,b];\mathbb{R}^n)$, $\varphi \in C^1([0,b];\mathbb{R}^n)$, $\varphi \in C([0,a];\mathbb{R}^n)$ and $h : C([0,b]) \to C([0,a])$ is a bounded linear operator. A complete theory of problem (0.3), (0.4) was constructed in [24].

Initial–boundary value problems with integral boundary conditions for quasi–linear systems were studied in [1–3].

The initial–periodic boundary value problems for quasi-linear and nonlinear systems

$$w^{(1,1)} = F\big(x, y, w^{(1,0)}, w^{(0,1)}, w\big), \tag{0.5}$$

$$w(0,y) = \varphi(y), \quad w^{(1,0)}(x,0) = w^{(1,0)}(x,b), \tag{0.6}$$

were studied in [25, 26, 33, 35].

Nonlocal boundary value problems, in particular the Dirichlet problem and problems on doubly periodic solutions, were studied in [22, 28, 29, 30, 34].

Two–dimensional initial–boundary value problems for linear equations

$$u^{(m,n)} = \sum_{k=0}^{n-1} p_{mk}(x,y)u^{(m,k)} + \sum_{j=0}^{m-1}\sum_{k=0}^{n} p_{jk}(x,y)u^{(j,k)} + q(x,y), \tag{0.7}$$

$$\begin{aligned} &u^{(j,0)}(0,y) = \varphi_j(y) \quad (j=0,\dots,m-1), \\ &h_k(u^{(m,0)}(x,\cdot))(x) = \psi_k(x) \quad (k=1,\dots,n). \end{aligned} \tag{0.8}$$

was studied in [31] and [32].

Same problems for the nonlinear hyperbolic equation

$$u^{(m,n)} = f\big(x, y, u^{(m,0)}, \dots, u^{(m,n-1)}, u^{(0,n)}, \dots, u^{(m-1,n)}, u^{(m-1,n-1)}, \dots, u\big), \tag{0.9}$$

was studied in [36].

Investigation of problem (0.1),(0.2) required a fundamental modification of methods developed in [31] and [32]. In particular, introduction of the concepts of *associated problems* and *α–well–posedness*.

The work is organized as follows: well–posedness of general boundary value problems are studied in Chapter I; problems on periodic solutions are studied in Chapter II; two–dimensional Dirichlet problem in a smooth bounded and convex domain is studied in Chapter III; and quasi–linear equations are studied in Chapter IV.

CHAPTER I
Well–Posed Boundary Value Problems

1. Formulation of the Main Results

Let $m_1,\dots,m_n$ be positive integers. In the n-dimensional box $\Omega=[0,\omega_1]\times\cdots\times[0,\omega_n]$ for the linear hyperbolic equation

$$u^{(\mathbf{m})}=\sum_{\boldsymbol{\alpha}<\mathbf{m}}p_{\boldsymbol{\alpha}}(\mathbf{x})u^{(\boldsymbol{\alpha})}+q(\mathbf{x}) \tag{1.1}$$

consider the boundary conditions

$$h_{ik}\big(u^{(\mathbf{m}^{i-1})}(\bullet,\widehat{\mathbf{x}}_i)\big)=\varphi_{ik}^{(\mathbf{m}^{i-1})}(\widehat{\mathbf{x}}_i)\ \text{ for }\ \widehat{\mathbf{x}}_i\in\widehat{\Omega}_i\ \ (k=1,\dots,m_i;\ \ i=1,\dots,n). \tag{1.2}$$

Here $\mathbf{x}=(x_1,\dots,x_n)$, $\widehat{\mathbf{x}}_i=(x_1,\dots,x_{i-1},x_{i+1},\dots,x_n)$, $\widehat{\Omega}_i=[0,\omega_1]\times\cdots\times[0,\omega_{i-1}]\times[0,\omega_{i+1}]\times\cdots\times[0,\omega_n]$, $\mathbf{m}=(m_1,\dots,m_n)$, $\boldsymbol{\alpha}=(\alpha_1,\dots,\alpha_n)$, $\mathbf{m}^i=(m_1,\dots,m_i,0,\dots,0)$ ($\mathbf{m}^i=(0,\dots,0)$ if $i=0$), $\mathbf{m}_i=(0,\dots,m_i,\dots,0)$ and $\widehat{\mathbf{m}}_i=\mathbf{m}-\mathbf{m}_i$ are multi-indices,

$$u^{(\boldsymbol{\alpha})}(\mathbf{x})=\frac{\partial^{\alpha_1+\cdots+\alpha_n}u(\mathbf{x})}{\partial x_1^{\alpha_1}\cdots\partial x_n^{\alpha_n}}\,,$$

$p_{\boldsymbol{\alpha}}\in C(\Omega)$ ($\boldsymbol{\alpha}<\mathbf{m}$), $q\in C(\Omega)$, $\varphi_{ik}\in C^{\widehat{\mathbf{m}}_i}(\Omega_i)$ ($k=1,\dots,m_i$; $i=1,\dots,n$), and $h_{ik}:C^{m_i-1}([0,\omega_i])\to\mathbb{R}$ ($k=1,\dots,m_i$; $i=1,\dots,n$) are bounded linear functionals.

By a solution of problem (1.1),(1.2) we understand a classical solution, i.e., a function $u\in C^{\mathbf{m}}(\Omega)$ satisfying equation (1.1) and boundary conditions (1.2) everywhere in Ω.

Remark 1.1. Conditions (1.2) are not equivalent to the conditions

$$h_{ik}\big(u(\bullet,\widehat{\mathbf{x}}_i)\big)=\varphi_{ik}(\widehat{\mathbf{x}}_i)\ \ (k=1,\dots,m_i;\ \ i=1,\dots,n), \tag{$\widetilde{1.2}$}$$

since the latter require the additional consistency conditions

$$h_{ik}(\varphi_{jl})=h_{jl}(\varphi_{ik})\ \ (k=1,\dots,m_i;\ \ l=1,\dots,m_j;\ \ i,j=1,\dots,n). \tag{1.3}$$

However, in the well–posed case the homogeneous conditions (1.2_0) are equivalent to the homogeneous conditions

$$h_{ik}\big(u(\bullet,\widehat{\mathbf{x}}_i)\big)=0\ \ (k=1,\dots,m_i;\ \ i=1,\dots,n). \tag{$\widetilde{1.2_0}$}$$

Along with problem (1.1), (1.2) consider its corresponding homogeneous problem

$$u^{(\mathbf{m})} = \sum_{\boldsymbol{\alpha}<\mathbf{m}} p_{\boldsymbol{\alpha}}(\mathbf{x}) u^{(\boldsymbol{\alpha})}, \tag{1.1$_0$}$$

$$h_{ik}\big(u^{(\mathbf{m}^{i-1})}(\bullet,\, \widehat{\mathbf{x}}_i)\big) = 0 \ \text{ for } \widehat{\mathbf{x}}_i \in \Omega_i \ (k = 1, \dots, m_i;\ i = 1, \dots, n). \tag{1.2$_0$}$$

Let $\boldsymbol{\sigma} \in \boldsymbol{\Xi}$. In the domain $\Omega_{\boldsymbol{\sigma}}$ consider the homogeneous boundary value problem depending on the parameter $\mathbf{x}_{\widehat{\boldsymbol{\sigma}}} \in \Omega_{\widehat{\boldsymbol{\sigma}}}$

$$v^{(\mathbf{m}_{\boldsymbol{\sigma}})} = \sum_{\boldsymbol{\alpha}<\mathbf{m}_{\boldsymbol{\sigma}}} p_{\boldsymbol{\alpha}+\widehat{\mathbf{m}}_{\boldsymbol{\sigma}}}(\mathbf{x}) v^{(\boldsymbol{\alpha})}, \tag{1.1$_{\boldsymbol{\sigma}}$}$$

$$h_{ik}\big(v^{(\mathbf{m}_{\boldsymbol{\sigma}}^{i-1})}(\bullet\, \widehat{\mathbf{x}}_i)\big) = 0 \qquad (k = 1, \dots, m_i;\ \ i \in \operatorname{supp} \boldsymbol{\sigma}). \tag{1.2$_{\boldsymbol{\sigma}}$}$$

Definition 1.1. Let $\boldsymbol{\sigma} \in \boldsymbol{\Xi} \cup \{\mathbf{1}\}$. Then problem $(1.1_{\boldsymbol{\sigma}}), (1.2_{\boldsymbol{\sigma}})$ is called $\boldsymbol{\sigma}$–associated problem, or associate problem of level $l = \|\sigma\|$.

If $\boldsymbol{\sigma} = \mathbf{1}$, then problem $(1.1_{\boldsymbol{\sigma}}), (1.2_{\boldsymbol{\sigma}})$ becomes problem $(1.1_0), (1.2_0)$. In other words problem $(1.1_0), (1.2_0)$ is the only associated problem of level n. We include the homogeneous problem $(1.1_0), (1.2_0)$ in the class of associated problems for the sake of convenience.

Associated problems of the level $n-1$ can be written in the relatively simpler form

$$v^{(\widehat{\mathbf{m}}_j)} = \sum_{\alpha<\widehat{\mathbf{m}}_j} p_{\mathbf{m}_j+\alpha}(\mathbf{x}) v^{(\alpha)}, \tag{1.1$_j$}$$

$$h_{ik}\big(v^{(\mathbf{m}^{i-1})}(\bullet, \widehat{\mathbf{x}}_i)\big) = 0 \ \text{ for } \ \widehat{\mathbf{x}}_i \in \Omega_i \ (k = 1, \dots, m_i,\ i \neq j). \tag{1.2$_j$}$$

Definition 1.2. Problem (1.1),(1.2) is called *well-posed*, if it is uniquely solvable for arbitrary $\varphi_{ik} \in C^{\widehat{\mathbf{m}}_i}(\Omega_i)$ $(k = 1, \dots, m_i;\, i = 1, \dots, n)$ and $q \in C(\Omega)$, and its solution u admits the estimate

$$\|u\|_{C^{\mathbf{m}}(\Omega)} \le M\Big(\sum_{i=1}^{n} \sum_{k=1}^{m_i} \|\varphi_{ik}\|_{C^{\widehat{\mathbf{m}}_\mathbf{i}}(\Omega_i)} + \|q\|_{C(\Omega)}\Big), \tag{1.4}$$

where M is a positive constant independent of q and φ_{ik} $(k = 1, \dots, m_i;\, i = 1, \dots, n)$.

Consider the equation

$$u^{(\mathbf{m})} = \sum_{\boldsymbol{\alpha}<\mathbf{m}} p_{\boldsymbol{\alpha}}(\mathbf{x}) u^{(\boldsymbol{\alpha})} + q^{(\mathbf{m})}(\mathbf{x}). \tag{1.5}$$

Definition 1.3. Problem (1.1), (1.2) is called *α–well-posed*, if it is well posed, and for an arbitrary $q \in C^{\mathbf{m}}(\Omega)$ the solution u of problem (1.5), (1.2_0) admits the estimates

$$\|u^{(\boldsymbol{\alpha})} - q^{(\boldsymbol{\alpha})}\|_{C(\Omega)} \le M \|q\|_{C^{\boldsymbol{m}-\mathbf{1}}(\Omega)} \qquad (\boldsymbol{\alpha} \in \boldsymbol{\Upsilon}_{\mathbf{m}}), \tag{1.6}$$
$$\|u\|_{C^{\boldsymbol{m}-\mathbf{1}}(\Omega)} \le M \|q\|_{C^{\boldsymbol{m}-\mathbf{1}}(\Omega)},$$

where M is a positive constant independent of q.

Definition 1.4. Problem (1.1),(1.2) is said to have *the Fredholm property*, if:
(i) the homogeneous problem $(1.1_0), (1.2_0)$ has a finite dimensional space of solutions;
(ii) problem (1.1),(1.2) is uniquely solvable if and only if problem $(1.1_0), (1.2_0)$ has only the trivial solution.

1.1. **Necessary Conditions of Well–Posedness.**

Theorem 1.1. *Let problem* (1.1), (1.2) *be solvable for arbitrary* $\varphi_{ik} \in C^{\widehat{\mathbf{m}}_{\mathbf{i}}}(\Omega_i)$ $(k = 1, \dots, m_i;\ i = 1, \dots n)$. *Then for each* $i \in \{1, \dots, n-1\}$ *the problem*

$$z^{(m_i)} = 0, \qquad h_{ik}(z) = 0 \ \ (k = 1, \dots, m_1) \tag{1.7}$$

has only the trivial solution.

Theorem 1.2. *Let all of the coefficients of equation* (1.1) *be constants, and let for some* $\boldsymbol{\sigma} \in \boldsymbol{\Xi}$ *associated problem* $(1.1_{\boldsymbol{\sigma}}), (1.2_{\boldsymbol{\sigma}})$ *have a nontrivial solution. Furthermore, let*

$$p_{\boldsymbol{\alpha}+\boldsymbol{\beta}} + p_{a+\mathbf{m}_{\widehat{\boldsymbol{\sigma}}}} p_{\mathbf{m}_{\boldsymbol{\sigma}}+\boldsymbol{\beta}} = 0 \quad \text{for} \quad \mathbf{0} < \boldsymbol{\alpha} < \mathbf{m}_{\boldsymbol{\sigma}},\ \mathbf{0} < \boldsymbol{\beta} < \mathbf{m}_{\widehat{\boldsymbol{\sigma}}}. \tag{1.8}$$

Then for solvability of problem (1.1), (1.2) *it is necessary that for every* $j \in \operatorname{supp} \widehat{\boldsymbol{\sigma}}$ *and* $l \in \{1, \dots, m_j\}$ *the problem*

$$v^{(\mathbf{m}_{\boldsymbol{\sigma}})} = \sum_{\boldsymbol{\alpha}<\mathbf{m}_{\boldsymbol{\sigma}}} p_{\boldsymbol{\alpha}+\mathbf{m}_{\widehat{\boldsymbol{\sigma}}}} v^{(\boldsymbol{\alpha})} + Q_{j\,l}(\widehat{\mathbf{x}}_j), \tag{1.9}$$

$$h_{ik}\big(v^{(\mathbf{m}^{i-1})}(\bullet, \widehat{\mathbf{x}}_i)\big) = \Psi_{ik}^{j,l}(\widehat{\mathbf{x}}_{ij}) \quad (k = 1, \dots, m_i;\ \ i \in \operatorname{supp} \boldsymbol{\sigma}), \tag{1.10}$$

where

$$Q_{j\,l}(\widehat{\mathbf{x}}_j) = (p_{\mathbf{0}} + p_{\mathbf{m}_{\widehat{\sigma}}} p_{\mathbf{m}_{\sigma}})\varphi_{jl}(\widehat{\mathbf{x}}_j) + h_{j\,l}(q)(\widehat{\mathbf{x}}_j) \tag{1.11}$$

and

$$\Psi_{ik}^{jl}(\widehat{\mathbf{x}}_{ij}) = h_{j\,l}\Big(\varphi_{ik}^{(\mathbf{m}^{i-1}+\mathbf{m}_{\widehat{\sigma}})}(\bullet, \widehat{\mathbf{x}}_{ij})\Big) - \sum_{\boldsymbol{\beta}<\mathbf{m}_{\widehat{\sigma}}} p_{\boldsymbol{\beta}+\mathbf{m}_{\sigma}} h_{j\,l}\Big(\varphi_{ik}^{(\mathbf{m}^{i-1}+\boldsymbol{\beta})}(\bullet, \widehat{\mathbf{x}}_{ij})\Big), \tag{1.12}$$

is solvable.

Remark 1.2. Solvability of ill–posed nonhomogenous associated problem (1.9),(1.10) is, in fact, additional consistency condition between the boundary values φ_{ik}, the coefficients $p_{\boldsymbol{\alpha}}$ and the free term q. Indeed, consider the problem

$$u^{(m,2)} = -u^{(m,0)} + p_0 u + p_1 u^{(0,1)} + p_2 u^{(0,2)} + q(x_1, x_2), \tag{1.13}$$

$$u^{(j-1,0)}(0, x_2) = \varphi_j(x_2)\ (j = 1, \dots, m), \quad u^{(m,0)}(x_1, 0) = 0,\ u^{(m,0)}(x_2, \pi) = 0, \tag{1.14}$$

where p_1 and p_2 are positive constants and $q \in C^{m,0}(\Omega)$. By Corollary 1.2 from [32] problem (1.13),(1.14) is solvable if and only if

$$\int_0^{\pi} \Big(\sum_{k=0}^{2} p_k\, \varphi_1^{(k)}(0) + q(0,t)\Big) \sin t\, dt = 0. \tag{1.15}$$

Thus, for problem (1.13),(1.14), solvability of ill–posed nonhomogenous associated problem (1.9),(1.10) is equivalent to the consistency condition (1.15).

Remark 1.3. Solvability of ill–posed nonhomogenous associated problem (1.9),(1.10) is necessary for solvability of problem (1.1),(1.2), and by no means sufficient, even if the homogeneous problem $(1.1_0), (1.2_0)$ has only the trivial solution. Indeed, consider the periodic problem

$$u^{(1,1,1)} = \cos^2 x_1 u - q(x_1), \tag{1.16}$$

$$u(\pi, x_2, x_3) = u(0, x_2, x_3), \quad u(x_1, \pi, x_3) = u(x_1, 0, x_3), \quad u(x_1, x_2, \pi) = u(x_1, x_2, 0), \tag{1.17}$$

where q is a continuous function such that $q(\pi) = q(0)$. Problem (1.16),(1.17) is ill–posed, and its corresponding homogeneous problem has only the trivial solution.

Furthermore, for problem (1.16),(1.17) all consistency conditions hold, Therefore, due to uniqueness, the only possible solution of problem (1.16),(1.17) should be

$$u(x_1) = \frac{q(x_1)}{\cos^2 x_1}.$$

On the other hand, it is clear that problem (1.16),(1.17)) has a solution if and only if

$$q(x_1) = \cos^2 x_1\, \widetilde{q}(x_1),$$

where $\widetilde{q} \in C^1([0,\pi])$. In particular, if $q(x_1) \equiv 1$, then problem (1.16),(1.17) has no solution despite the fact that all coefficients of equation (1.16) and boundary data are analytic functions.

1.2. **Case** $n = 2$**.**

Theorem 1.3. *Let* $n = 2$*, and let problem* $(1.1), (1.2)$ *be well–posed. Then each associated problem* $(1.1_j), (1.2_j)$ *has only the trivial solution for every* $x_j \in [0, \omega_j]$ $(j = 1, 2)$*.*

Theorem 1.4. *Let* $n = 2$*, and let:*

(A_1) *problem* (1.7) *have only the trivial solution for* $(i = 1)$*;*

(A_2) *each associated problem* $(1.1_j), (1.2_j)$ *have only the trivial solution for every* $x_j \in [0, \omega_j]$ $(j = 1, 2)$*.*

Then problem $(1.1), (1.2)$ *has the Fredholm property. Moreover, if problem* $(1.1_0), (1.2_0)$ *has only the trivial solution, then problem* $(1.1), (1.2)$ *is well–posed. Furthermore, if* $p_{\boldsymbol{\alpha}} \in C^{\boldsymbol{m}}(\Omega)$ $(\boldsymbol{\alpha} \in \Upsilon m)$*, then problem* $(1.1), (1.2)$ *is* α*–well–posed.*

Remark 1.4. Consider the problem

$$u^{(1,1)} = p_1(x_1, x_2)u^{(1,0)} + p_2(x_1, x_2)u^{(0,1)} + p_0(x_1, x_2)u + q(x_1, x_2), \tag{1.18}$$

$$u(0, x_2) = \varphi(x_2), \qquad u^{(1,0)}(x_1, 0) = u^{(1,0)}(x_1, \omega_2). \tag{1.19}$$

Set

$$r(x_1)=\int\limits_0^{\omega_2} p_1(x_1,t)\,dt, \qquad I_r=\{x_1\in[0,\omega_1]: r(x_1)=0\}.$$

and there exist $\widetilde{p}_0,\ \widetilde{p}_2; \widetilde{q}\in C([0,\omega_1]\times[0,\omega_2])$ such that

$$p_0(x_1,x_2)=r(x_1)\widetilde{p}_0(x_1,x_2); \quad p_2(x_1,x_2)=r(x_1)\widetilde{p}_2(x_1,x_2); \quad q(x_1,x_2)=r(x_1)\widetilde{q}(x_1,x_2).$$

Then:

(i) problem (1.18),(1.19) is well–posed if and only if $I_r=\emptyset$;

(ii) problem (1.18),(1.19) has a unique classical solution if $\operatorname{supp} r=\overline{[0,\omega_1]\backslash I_r}=[0,\omega_1]$, and has infinite dimensional set of classical solutions otherwise;

(iii) problem (1.18),(1.19) has a unique absolutely continuous (i.e. weak) solution if and only if $\operatorname{mes} I_r=0$; and has infinite dimensional set of absolutely continuous solutions otherwise;

(iv) problem (1.18),(1.19) has a unique classical solution and infinite dimensional set of absolutely continuous solutions if I_r is a nowhere dense set of a positive measure;

(v) if problem $I_r\neq\emptyset$ and $q(x_1,x_2)\equiv 1$, then problem (1.18),(1.19) has no classical solutions.

1.3. **Case $n\geq 2$.** Set:

$$\mathcal{L}_{\mathbf{m}}v=v^{(\mathbf{m})}-\sum_{\boldsymbol{\alpha}<\mathbf{m}} p_{\boldsymbol{\alpha}}(\mathbf{x})v^{(\boldsymbol{\alpha})}, \tag{1.20}$$

$$\mathcal{L}_{\boldsymbol{\sigma}}v=v^{(\mathbf{m}_{\boldsymbol{\sigma}})}-\sum_{\boldsymbol{\alpha}<\mathbf{m}_{\boldsymbol{\sigma}}} p_{\boldsymbol{\alpha}+\mathbf{m}_{\widehat{\boldsymbol{\sigma}}}}(\mathbf{x})v^{(\boldsymbol{\alpha})}. \tag{1.21}$$

Theorem 1.5. *Let problem* (1.7) *have the trivial solution for every* $i\in\{1,\dots,n-1\}$, *and let for some* $\boldsymbol{\sigma}\in\boldsymbol{\Xi}$ $\boldsymbol{\sigma}$ *and* $\widehat{\boldsymbol{\sigma}}$*–associated problems be well–posed. Moreover, let* $p_{\boldsymbol{\alpha}+\mathbf{m}_{\boldsymbol{\sigma}}}(\mathbf{x})=p_{\boldsymbol{\alpha}+\mathbf{m}_{\boldsymbol{\sigma}}}(\mathbf{x}_{\widehat{\boldsymbol{\sigma}}})$ $(\boldsymbol{\alpha}<\mathbf{m}_{\widehat{\boldsymbol{\sigma}}})$, *and*

$$\mathcal{L}_{\mathbf{m}}\,v=\mathcal{L}_{\boldsymbol{\sigma}}\circ\mathcal{L}_{\widehat{\boldsymbol{\sigma}}}\,v+\sum_{\boldsymbol{\alpha}\leq \boldsymbol{m}-\mathbf{1}} p_{\boldsymbol{\alpha}}(\mathbf{x})v^{(\boldsymbol{\alpha})}. \tag{1.22}$$

Then problem (1.1), (1.2) *has the Fredholm property.*

Along with the equation (1.1_0) consider the equation

$$u^{(\mathbf{m})} = \sum_{\boldsymbol{\alpha}<\mathbf{m}} p_{\lambda\,\boldsymbol{\alpha}}(\mathbf{x}) u^{(\boldsymbol{\alpha})} + q(\mathbf{x}), \tag{1.23}$$

where $\lambda \in [0,1]$, $p_{\lambda\,\boldsymbol{\alpha}}(\mathbf{x}) = (1-\lambda)p_{0\,\boldsymbol{\alpha}}(\mathbf{x}) + \lambda p_{\boldsymbol{\alpha}}(\mathbf{x})$, $p_{0\,\boldsymbol{\alpha}}(\mathbf{x})\ C(\Omega)$ $(\boldsymbol{\alpha} < \boldsymbol{m})$, and also its corresponding homogeneous and associated equations

$$u^{(\mathbf{m})} = \sum_{\boldsymbol{\alpha}<\mathbf{m}} p_{\lambda\,\boldsymbol{\alpha}}(\mathbf{x}) u^{(\boldsymbol{\alpha})} \tag{1.23$_0$}$$

and

$$v^{(\mathbf{m}_{\boldsymbol{\sigma}})} = \sum_{\boldsymbol{\alpha}<\mathbf{m}_{\boldsymbol{\sigma}}} p_{\lambda\,\boldsymbol{\alpha}+\mathbf{m}_{\widehat{\boldsymbol{\sigma}}}}(\mathbf{x}) v^{(\boldsymbol{\alpha})}, \tag{1.23$_{\boldsymbol{\sigma}}$}$$

Theorem 1.6. *Let $p_{\boldsymbol{\alpha}} \in C^{\boldsymbol{m}}(\Omega)$ $(\boldsymbol{\alpha} \in \boldsymbol{\Upsilon}_{\mathbf{m}})$, and let:*

(A_1) problem (1.7) have only the trivial solution for every $i \in \{1,\dots,n-1\}$;

(A_2) each $\boldsymbol{\sigma}$–associated problem $(1.23_{\boldsymbol{\sigma}}), (1.2_{\boldsymbol{\sigma}})$ be α–well–posed for $\lambda = 0$;

(A_3) problem $(1.23), (1.2_0)$ be α–well–posed for $\lambda = 0$.

(A_4) each $\boldsymbol{\sigma}$–associated problem $(1.23_{\boldsymbol{\sigma}}), (1.2_{\boldsymbol{\sigma}})$ have only the trivial solution for every $\mathbf{x}_{\widehat{\boldsymbol{\sigma}}} \in \Omega_{\widehat{\boldsymbol{\sigma}}}$ $(\boldsymbol{\sigma} \in \boldsymbol{\Xi})$ and every $\lambda \in [0,1]$;

(A_5) problem $(1.23_0), (1.2_0)$ have only the trivial solution for every $\lambda \in [0,1)$.

Then problem $(1.1), (1.2)$ has the Fredholm property. Moreover, if problem $(1.1_0), (1.2_0)$ has only the trivial solution, then problem $(1.1), (1.2)$ is α–well–posed.

1.4. **Dirichlet Type Boundary Value Problems.** For the equation

$$u^{(2\mathbf{m})} = \sum_{\boldsymbol{\alpha}+\boldsymbol{\beta}<2\mathbf{m}} \big(p_{\boldsymbol{\alpha}+\boldsymbol{\beta}}(\mathbf{x}) u^{(\boldsymbol{\alpha})}\big)^{(\boldsymbol{\beta})} + q(\mathbf{x}) \tag{1.24}$$

consider the boundary conditions

$$\begin{aligned} u^{(2\mathbf{m}^{i-1}+(k-1)\mathbf{1}_i)}(0,\widehat{\mathbf{x}}_{\mathbf{i}}) &= \varphi_{ik}^{2(\mathbf{m}^{i-1})}(\widehat{\mathbf{x}}_{\mathbf{i}}) \quad (k=1,\dots,m_i;\ i=1,\dots,n),\\ u^{(2\mathbf{m}^{i-1}+(k-1)\mathbf{1}_i)}(\omega_i,\widehat{\mathbf{x}}_{\mathbf{i}}) &= \psi_{ik}^{2(\mathbf{m}^{i-1})}(\widehat{\mathbf{x}}_{\mathbf{i}}) \quad (k=1,\dots,m_i;\ i=1,\dots,n), \end{aligned} \tag{1.25}$$

and

$$\begin{aligned} u^{(2\mathbf{m}^{i-1}+2(k-1)\mathbf{1}_i)}(0,\widehat{\mathbf{x}}_{\mathbf{i}}) &= \varphi_{ik}^{2(\mathbf{m}^{i-1})}(\widehat{\mathbf{x}}_{\mathbf{i}}) \quad (k=1,\dots,m_i;\ i=1,\dots,n),\\ u^{(2\mathbf{m}^{i-1}+2(k-1)\mathbf{1}_i)}(\omega_i,\widehat{\mathbf{x}}_{\mathbf{i}}) &= \psi_{ik}^{2(\mathbf{m}^{i-1})}(\widehat{\mathbf{x}}_{\mathbf{i}}) \quad (k=1,\dots,m_i;\ i=1,\dots,n), \end{aligned} \tag{1.26}$$

where $\varphi_{ik}, \psi_{ik} \in C^{\mathbf{2}\widehat{\mathbf{m}}_i}(\Omega_i)$ $(k = 1, \ldots, m_i;\ i = 1, \ldots, n)$.

Theorem 1.7. *Let* $p_{\boldsymbol{\alpha}+\boldsymbol{\beta}} \in C^{\boldsymbol{m}}(\Omega)$ $(\boldsymbol{\alpha}+\boldsymbol{\beta} \in \boldsymbol{\Upsilon}_{2\boldsymbol{m}})$, $p_{\boldsymbol{\alpha}+\boldsymbol{\beta}} \in C^{\boldsymbol{\beta}}(\Omega)$ $(\boldsymbol{\alpha}+\boldsymbol{\beta} \notin \boldsymbol{\Upsilon}_{2\boldsymbol{m}})$, *and let the quadratic form nonnegative defined:*

$$\sum_{\boldsymbol{\alpha}+\boldsymbol{\beta}<2\mathbf{m}} (-1)^{\|\boldsymbol{m}\|+\|\boldsymbol{\beta}\|-1} p_{\boldsymbol{\alpha}+\boldsymbol{\beta}}(\mathbf{x}) z_{\boldsymbol{\alpha}} z_{\boldsymbol{\beta}} \geq 0 \quad for \quad \mathbf{x} \in \Omega, \quad z_{\boldsymbol{\alpha}} \in \mathbb{R} \ (\boldsymbol{\alpha} < 2\mathbf{m}). \quad (1.27)$$

Then problem (1.24), (1.25) *is* α*–well–posed.*

Theorem 1.8. *Let* $p_{\boldsymbol{\alpha}+\boldsymbol{\beta}} \in C_0^{2(\boldsymbol{m}-\mathbf{1})}(\Omega)$ $(\boldsymbol{\alpha}+\boldsymbol{\beta} \in \boldsymbol{\Upsilon}_{2\boldsymbol{m}})$, $p_{\boldsymbol{\alpha}+\boldsymbol{\beta}} \in C_0^{\boldsymbol{\beta}}(\Omega)$ $(\boldsymbol{\alpha} + \boldsymbol{\beta} \notin \boldsymbol{\Upsilon}_{2\boldsymbol{m}})$, *and let the quadratic form* (1.27) *nonnegative defined. Then problem* (1.24), (1.26) *is* α*–well–posed.*

Consider the equation

$$u^{(2\mathbf{m})} = \sum_{\boldsymbol{\alpha} \leq \mathbf{m}} \left(p_{\boldsymbol{\alpha}}(\mathbf{x}) u^{(\boldsymbol{\alpha})}\right)^{(\boldsymbol{\alpha})} + q(\mathbf{x}). \quad (1.28)$$

Notice that due to the term $\left(p_{\boldsymbol{m}}(\mathbf{x}) u^{(\boldsymbol{m})}\right)^{(\boldsymbol{m})}$ equation (1.28) is not a particular case of equation (1.24).

Theorem 1.9. *Let* $p_{\boldsymbol{\alpha}} \in C^{\boldsymbol{m}}(\Omega)$ $(\boldsymbol{\alpha} \in \boldsymbol{\Upsilon}_{\boldsymbol{m}})$, $p_{\boldsymbol{\alpha}} \in C^{\boldsymbol{\alpha}}(\Omega)$ $(\boldsymbol{\alpha} \notin \boldsymbol{\Upsilon}_{\boldsymbol{m}})$, *and let the inequalities hold:*

$$(-1)^{\|\boldsymbol{m}\|+\|\boldsymbol{\alpha}\|-1} p_{\boldsymbol{\alpha}}(\mathbf{x}) \geq 0 \quad (\boldsymbol{\alpha} \leq \mathbf{m}). \quad (1.29)$$

Then problem (1.28), (1.25) *is* α*–well–posed.*

Theorem 1.10. *Let* $p_{\boldsymbol{\alpha}} \in C_0^{2(\boldsymbol{m}-\mathbf{1})}(\Omega)$ $(\boldsymbol{\alpha} \in \boldsymbol{\Upsilon}_{\boldsymbol{m}})$, $p_{\boldsymbol{\alpha}} \in C_0^{\boldsymbol{\alpha}}(\Omega)$ $(\boldsymbol{\alpha} \notin \boldsymbol{\Upsilon}_{\boldsymbol{m}})$, *and let the inequalities* (1.29) *hold. The problem* (1.28), (1.26) *is* α*–well–posed.*

Consider the particular case of equation (1.28)

$$u^{(2\mathbf{m})} = \sum_{\boldsymbol{\alpha} \leq \mathbf{m}} p_{\boldsymbol{\alpha}}(\widehat{\mathbf{x}}_{\alpha}) u^{(2\boldsymbol{\alpha})} + q(\mathbf{x}). \quad (1.30)$$

Theorem 1.11. *Let* $p_{\boldsymbol{\alpha}} \in C(\Omega)$ $(\boldsymbol{\alpha} \leq \mathbf{m})$, *and let the inequalities* (1.29) *hold. The problem* (1.30), (1.25) *is* α*–well–posed.*

Theorem 1.12. *Let* $p_{\boldsymbol\alpha} \in C(\Omega)$ $(\boldsymbol\alpha \le \mathbf{m})$, *and let the inequalities* (1.29) *hold. The problem* (1.30), (1.26) *is* α*–well–posed.*

1.5. **Periodic Type Boundary Value Problems.** For the equations (1.24) and (1.28) consider the following boundary conditions of periodic type

$$u^{(2\mathbf{m}^{i-1}+(k-1)\mathbf{1}_i)}(0,\widehat{\mathbf{x}}_\mathbf{i}) - a_{ik}u^{(2\mathbf{m}^{i-1}+(k-1)\mathbf{1}_i)}(\omega_i,\widehat{\mathbf{x}}_\mathbf{i}) = \varphi_{ik}^{(2\mathbf{m}^{i-1})}(\widehat{\mathbf{x}}_\mathbf{i}) \quad (k=1,\dots,2m_i;\ i=1,\dots,n), \quad (1.31)$$

where $a_{ik} \neq 0$ $(k=1,\dots,2m_i;\ i=1,\dots,n)$, and $p_{\boldsymbol\alpha+\boldsymbol\beta} \in C_{\boldsymbol\omega}(\mathbb{R}^n)$ $(\boldsymbol\alpha+\boldsymbol\beta < 2\boldsymbol{m})$, and $\varphi_{ik} \in C^{\mathbf{2}\widehat{\mathbf{m}}_i}(\Omega_i)$ $(k=1,\dots,2m_i;\ i=1,\dots,n)$.

If $a_{ik}=1$, then (1.31) are nonhomogeneous periodic conditions. On the other hand, problem (1.7) with periodic boundary conditions has nontrivial solutions for every $i \in \{1,\dots,n-1\}$. Consequently, problem (1.1),(1.30) with $a_{ik}=1$ $(k=1,\dots,m_i;\ i=1,\dots,n)$ is ***not*** well–posed in the sense of Definition 1.2. This is the reason why periodic problems are studied separately in Chapter II.

Theorem 1.13. *Let* $p_{\boldsymbol\alpha+\boldsymbol\beta} \in C_0^{2\boldsymbol{m}}(\Omega)$ $(\boldsymbol\alpha+\boldsymbol\beta \in \boldsymbol\Upsilon_{2\boldsymbol{m}})$, $p_{\boldsymbol\alpha+\boldsymbol\beta} \in C_0^{\boldsymbol\beta}(\Omega)$ $(\boldsymbol\alpha+\boldsymbol\beta \notin \boldsymbol\Upsilon_{2\boldsymbol{m}})$,

$$a_{ik} \neq 1 \quad (k=1,\dots,2m_i;\ i=1,\dots,n), \quad (1.32)$$

$$a_{ik}\, a_{i\,2m_i+1-k} = 1 \quad (k=1,\dots,2m_i;\ i=1,\dots,n), \quad (1.33)$$

and let the quadratic form (1.27) *be nonnegative defined. Then problem* (1.24), (1.30) *is* α*–well–posed.*

Theorem 1.14. *Let* $p_{\boldsymbol\alpha} \in C_0^{2\boldsymbol{m}}(\mathbb{R}^n)$ $(\boldsymbol\alpha \in \boldsymbol\Upsilon_{\boldsymbol{m}})$, $p_{\boldsymbol\alpha} \in C_0^{\boldsymbol\alpha}(\Omega)$ $(\boldsymbol\alpha \notin \boldsymbol\Upsilon_{\boldsymbol{m}})$, *and let inequalities* (1.29), (1.32) *and* (1.33) *hold. Then problem* (1.28), (1.31) *is* α*–well–posed.*

Consider the particular case of equation (1.30)

$$u^{(2\mathbf{m})} = \sum_{\boldsymbol\sigma \le \mathbf{1}} p_{\boldsymbol\sigma}(\mathbf{x}_{\widehat{\boldsymbol\sigma}})u^{(2\mathbf{m}_{\boldsymbol\sigma})} + q(\mathbf{x}). \quad (1.34)$$

Theorem 1.15. *Let* $p_{\boldsymbol{\sigma}} \in C(\Omega)$ $(\boldsymbol{\sigma} \le \mathbf{1})$, *let inequalities* (1.32) *and* (1.33) *hold, and let*

$$(-1)^{\|\boldsymbol{m}\|+\|\boldsymbol{\sigma}\|-1} p_{\boldsymbol{\sigma}}(\mathbf{x}_{\widehat{\boldsymbol{\sigma}}}) \ge 0 \quad (\boldsymbol{\sigma} \le \mathbf{1}). \tag{1.35}$$

The problem (1.34), (1.31) *is* α*–well–posed.*

1.6. **Initial–Boundary Value Problems.** Consider the following initial–boundary conditions:

$$\begin{gathered} h_{ik}\big(u^{(\mathbf{m}^{i-1})}(\bullet, \widehat{\mathbf{x}}_i)\big) = \varphi_{ik}^{(\mathbf{m}^{i-1})}(\widehat{\mathbf{x}}_i) \ \ (k = 1, \dots, m_i; \ \ i = 1, \dots, l), \\ u^{(\mathbf{m}^{i-1}+k\mathbf{1}_i)}\big(0, \widehat{\mathbf{x}}_i\big) = \varphi_{ik}^{(\mathbf{m}^{i-1}+k\mathbf{1}_i)}(\widehat{\mathbf{x}}_i) \ \ (k = 1, \dots, m_i; \ \ i = l+1, \dots, n) \end{gathered} \tag{1.36}$$

and

$$u^{(\mathbf{m}^{i-1}+k\mathbf{1}_i)}\big(0, \widehat{\mathbf{x}}_i\big) = \varphi_{ik}^{(\mathbf{m}^{i-1}+k\mathbf{1}_i)}(\widehat{\mathbf{x}}_i) \ \ (k = 1, \dots, m_i; \ \ i = 1, \dots, n). \tag{1.37}$$

Theorem 1.16. *Let for every* $\boldsymbol{\sigma} \le \boldsymbol{\sigma}^l$ *the* $\boldsymbol{\sigma}$*–associated problem*

$$v^{(\mathbf{m}_{\boldsymbol{\sigma}})} = \sum_{\boldsymbol{\alpha} < \mathbf{m}_{\boldsymbol{\sigma}}} p_{\boldsymbol{\alpha}+\mathbf{m}_{\widehat{\boldsymbol{\sigma}}}}(\mathbf{x}) v^{(\boldsymbol{\alpha})}, \tag{1.38}$$

$$h_{ik}\big(u^{(\mathbf{m}^{i-1})}(\bullet, \widehat{\mathbf{x}}_i)\big) = 0 \quad (k = 1, \dots, m_i); \ \ i \in \operatorname{supp} \boldsymbol{\sigma} \tag{1.39}$$

be well–posed for every $\mathbf{x}_{\widehat{\boldsymbol{\sigma}}} \in \Omega_{\widehat{\boldsymbol{\sigma}}}$. *Then problem* (1.1), (1.36) *is well–posed. Moreover, if* $p_{\boldsymbol{\alpha}} \in C^{\boldsymbol{m}}(\Omega)$ $(\boldsymbol{\alpha} \in \boldsymbol{\Upsilon}_{\boldsymbol{m}})$, *then problem* (1.1), (1.36) *is* α*–well–posed too.*

Theorem 1.17. *Let* $l = 1$ *and* $p_{\boldsymbol{\alpha}} \in C(\Omega)$ $(\boldsymbol{\alpha} < \boldsymbol{m})$. *Then problem* (1.1), (1.36) *is well–posed if and only if the problem*

$$z^{(m_1)} = \sum_{k=0}^{m_1-1} p_{k\mathbf{1}_1+\widehat{\mathbf{m}}_1}(\mathbf{x}) z^{(k)}, \tag{1.40}$$

$$h_{1k}(z) = 0 \quad (k = 1, \dots, m_1) \tag{1.41}$$

has only the trivial solution for every $\widehat{\mathbf{x}}_1 \in \widehat{\Omega}_1$.

Theorem 1.18. *Let* $p_{\boldsymbol{\alpha}} \in C(\Omega)$ $(\boldsymbol{\alpha} < \boldsymbol{m})$. *Then problem* (1.1), (1.37) *is well–posed. Moreover, if* $p_{\boldsymbol{\alpha}} \in C^{\boldsymbol{m}}(\Omega)$, *then problem* (1.1), (1.37) *is* α*–well–posed.*

2. Auxiliary Statements

2.1. **Some facts from the theory of ODE.** Consider the boundary value problem

$$z^{(m)} = \sum_{k=0}^{m-1} p_k(t) z^{(k)} + q(t), \tag{2.1}$$

$$h_k(z) = c_k \quad (k = 1, \dots, m), \tag{2.2}$$

and its corresponding homogeneous problem

$$z^{(m)} = \sum_{k=0}^{m-1} p_k(t) z^{(k)}, \tag{2.1$_0$}$$

$$h_k(z) = 0 \quad (k = 1, \dots, m), \tag{2.2$_0$}$$

where $p_k \in C([0,\omega])$ $(k = 0, \dots, m-1)$, $q \in C([0,\omega])$, $c_k \in \mathbb{R}$ $(k = 1, \dots, m)$, and $h_k : C^{m-1}([0,\omega]) \to \mathbb{R}$ $(k = 1, \dots, m)$ are bounded linear functionals.

Lemma 2.1. *The following facts are equivalent:*

(A_1) *problem* (2.1), (2.2) *is solvable for arbitrary* $q \in C(\Omega)$ *and* $c_k \in \mathbb{R}$ $(k = 1, \dots, m)$*;*

(A_2) *problem* (2.1), (2.2$_0$) *is solvable for arbitrary* $q \in C([0,\omega])$*;*

(A_3) *problem* (2.1$_0$), (2.2) *is solvable for arbitrary* $c_k \in \mathbb{R}$ $(k = 1, \dots, m)$*;*

(A_4) *problem* (2.1$_0$), (2.2$_0$) *has only the trivial solution.*

Lemma 2.1 is a well–known fact from the theory of ordinary differential equation (e.g. see Theorem 1.1 from [19]). If problem (2.1$_0$), (2.2$_0$) has only the trivial solution then a solution of problem (2.1),(2.2) admits the representation

$$z(t) = \Gamma(c_1, \dots, c_m)(t) + \mathcal{G}(q)(t), \tag{2.3}$$

where $\Gamma : \mathbb{R}^m \to C^m([0,\omega])$ and $\mathcal{G} : C([0,\omega]) \to C^m([0,\omega])$ are bounded linear operators. Moreover, the operator $\mathcal{G}$ admits the representation

$$\mathcal{G}(q)(t) = \int_0^\omega g(t,\tau) q(\tau)\, d\tau, \tag{2.4}$$

where $g : [0,\omega] \times [0,\omega] \to \mathbb{R}$ is called the **Green's function of problem** (2.1$_0$), (2.2$_0$) (for more about Green's functions see [19]).

Definition 2.1. $\mathcal{G} : C([0,\omega]) \to C^m([0,\omega])$ is called the **Green's operator** of problem $(2.1_0), (2.2_0)$.

Definition 2.2. $\Gamma : \mathbb{R}^m \to C^m([0,\omega])$ is called the **Green's boundary operator** of problem $(2.1_0), (2.2_0)$.

2.2. **Case** $n = 2$**.** Along with the associated problems $(1.1_j), (1.2_j)$ consider the problem

$$u^{(\mathbf{m}_1)} = \sum_{j=0}^{m_1-1} \widetilde{p}_{j\,m_2}(\mathbf{x}) u^{(j,0)}, \tag{2.5}$$

$$h_{1k}(u(\bullet, x_2)) = 0 \ \ (k = 1, \dots, m_1), \tag{2.6}$$

where $\widetilde{p}_{j\,m_2} \in C^{0,m_2}(\Omega)$ $(j = 0, \dots, m_1 - 1)$.

Lemma 2.2. *Let* $n = 2$*, conditions* (A_1) *and* (A_2) *of Theorem* 1.4 *hold, and let problem* $(2.5), (2.6)$ *have only the trivial solution for every* $x_2 \in [0, \omega_2]$*. Then an arbitrary solution* u *of problem* $(1.1), (1.2)$ *admits the following representations:*

$$\begin{aligned} u^{(m_1,0)}(x_1,x_2) = \int_0^{\omega_2} g_2(x_2,s_2;x_1)\Big(\sum_{j=0}^{m_1-1} p_{j\,m_2}(x_1,s_2) u^{(j,m_2)}(x_1,s_2) \\ + \sum_{j=0}^{m_1-1}\sum_{k=0}^{m_2-1} p_{jk}(x_1,s_2) u^{(j,k)}(x_1,s_2) + q(x_1,s_2)\Big)\, ds_2 \\ + \Gamma_2\big(\varphi_{21}^{(m_1)}(x_1), \dots, \varphi_{2m_2}^{(m_1)}(x_1)\big)(x_2); \end{aligned} \tag{2.7}$$

$$\begin{aligned} u^{(0,m_2)}(x_1,x_2) = \int_0^{\omega_1} g_1(x_1,s_1;x_2)\Big(\sum_{k=0}^{m_2-1} p_{m_1\,k}(\mathbf{x}) u^{(m_1,k)}(s_1,x_2) \\ + \sum_{j=0}^{m_1-1}\sum_{k=0}^{m_2-1} p_{jk}(s_1,x_2) u^{(j,k)}(s_1,x_2) + q(s_1,x_2)\Big)\, ds_1 \\ + \Gamma_1\big(\varphi_{11}^{(m_2)}(x_2), \dots, \varphi_{1m_1}^{(m_2)}(x_2)\big)(x_1); \end{aligned} \tag{2.8}$$

$$u(x_1,x_2)=\int\limits_0^{\omega_1}\widetilde g_1(x_1,s_1;x_2)\int\limits_0^{\omega_2}g_2(x_2,s_2;s_1)\biggl(\sum_{j=0}^{m_1-1}\sum_{k=0}^{m_2-1}\rho_{jk}(s_1,s_2)u^{(j,k)}(s_1,s_2)$$
$$+\sum_{j=0}^{m_1-1}\bigl(p_{j\,m_2}(s_1,s_2)-\widetilde p_{j\,m_2}(s_1,s_2)\bigr)u^{(j,m_2)}(s_1,s_2)+q(s_1,s_2)\biggr)\,ds_2\,ds_1$$
$$+\mathcal P\bigl[u;\varphi_{21}^{(m_1)},\dots,\varphi_{2m_2}^{(m_1)}\bigr](x_1,x_2)+\widetilde\Gamma_1\bigl(\varphi_{11}(x_2),\dots,\varphi_{1m_1}(x_2)\bigr)(x_1),\quad (2.9)$$

where

$$\rho_{jk}(x_1,x_2)=p_{jk}(x_1,x_2)+\sum_{i=k}^{m_2-1}\frac{k!}{i!(i-k)!}\,p_{m_1\,i}(\mathbf x)\,\widetilde p_{j\,m_2}^{(0,i-k)}(x_1,x_2)$$
$$-\frac{m_2!}{k!(m_2-k)!}\,\widetilde p_{j\,m_2}^{(0,m_2-k)}(x_1,x_2)\quad (j=0,\dots,m_1-1;\ k=0,\dots m_2-1),\quad (2.10)$$

$$\mathcal P\bigl[u;\varphi_{21}^{(m_1)},\dots,\varphi_{2m_2}^{(m_1)}\bigr](x_1,x_2)=$$
$$\int\limits_0^{\omega_1}\widetilde g_1(x_1,s_1;x_2)\Gamma_2\Biggl[\varphi_{21}^{(m_1)}(s_1)-\sum_{j=0}^{m_1-1}h_{21}\Bigl(\widetilde p_{j\,m_2}((\mathbf{s_1},\bullet))u^{(j,0)}(s_1,\bullet)\Bigr),\dots,$$
$$\varphi_{2m_2}^{(m_1)}(s_1)-\sum_{j=0}^{m_1-1}h_{2m_2}\Bigl(\widetilde p_{j\,m_2}((\mathbf{s_1},\bullet))u^{(j,0)}(s_1,\bullet)\Bigr)\Biggr]\,ds_1,\quad (2.11)$$

g_j *and* Γ_j*, respectively, are the Green's function and Green's boundary operator of problem* $(1.1_j),(1.2_j)$ $(j=1,2)$*, and* $\widetilde g_1$ *and* $\widetilde\Gamma_1$*, respectively,are the Green's function and the Green's boundary operator of problem* $(2.5),(2.6)$*.*

Proof. Let u be a solution of problem (1.1),(1.2). Set

$$v(x_1,x_2)=u^{(m_1,0)}(x_1,x_2);\quad w(x_1,x_2)=u^{(0,m_2)}(x_1,x_2);$$
$$\widetilde v(x_1,x_2)=u^{(m_1,0)}(x_1,x_2)-\sum_{j=0}^{m_1-1}\widetilde p_{jn}(x_1,x_2)u^{(j,0)}(x_1,x_2).$$

In order to prove Lemma 2.2, one needs to notice that v, w and $\widetilde{v}$, respectively, are solution of the following boundary value problems:

$$v^{(0,m_2)} = \sum_{k=0}^{m_2-1} p_{m_1k}(x_1,x_2)v^{(0,k)} + \sum_{j=0}^{m_1-1} p_{jm_2}(x_1,x_2)u^{(j,m_2)}(x_1,x_2) + \sum_{j=0}^{m_1-1}\sum_{k=0}^{m_2-1} p_{jk}(x_1,x_2)u^{(j,k)}(x_1,x_2) + q(x_1,x_2), \tag{2.12}$$

$$h_{2k}(v(x_1,\bullet)) = \varphi_{2k}^{(m_1)}(x_1) \quad (k=1,\dots,m_2); \tag{2.13}$$

$$w^{(m_1,0)} = \sum_{j=0}^{m_1-1} p_{jm_2}(x_1,x_2)(x_1,x_2)w^{(j,0)} + \sum_{k=0}^{m_2-1} p_{m_1k}(x_1,x_2)u^{(m_1,k)}(x_1,x_2) + \sum_{j=0}^{m_1-1}\sum_{k=0}^{m_2-1} p_{jk}(x_1,x_2)u^{(j,k)}(x_1,x_2) + q(x_1,x_2), \tag{2.14}$$

$$h_{1k}(w(\bullet,x_2)) = \varphi_{1k}^{(m_2)}(x_2) \quad (k=1,\dots,m_1); \tag{2.15}$$

$$\widetilde{v}^{(0,m_2)} = \sum_{k=0}^{m_2-1} p_{m_1k}(x_1,x_2)\widetilde{v}^{(0,k)} + \sum_{j=0}^{m_1-1}\big(p_{jm_2}(x_1,x_2) - \widetilde{p}_{jm_2}(x_1,x_2)\big)u^{(j,m_2)}(x_1,x_2) + \sum_{j=0}^{m_1-1}\sum_{k=0}^{m_2-1} \rho_{jk}(x_1,x_2)u^{(j,k)}(x_1,x_2) + q(x_1,x_2), \tag{2.16}$$

$$h_{2k}(v(x_1,\bullet)) = \varphi_{2k}^{(m_1)}(x_1) - h_{2k}\Big(\sum_{j=0}^{m_1-1} \widetilde{p}_{jm_2}(x_1,\bullet)u^{(j,m_2)}(x_1,\bullet)\Big) \quad (k=1,\dots,m_2). \tag{2.17}$$

□

Lemma 2.3. *Let $n=2$, and let conditions (A_1) and (A_2) of Theorem* 1.4 *hold. Then problem* (1.1), (1.2) *has the Fredholm property.*

Proof. In view of Lemma 2.2, problem (1.1), (1.2) is equivalent to the following system of integral equations

$$v(x_1,x_2) = \mathcal{F}_1(u,w)(x_1,x_2); \tag{2.18}$$

$$w(x_1,x_2) = \mathcal{F}_2(u,v)(x_1,x_2); \tag{2.19}$$

$$u(x_1,x_2) = \mathcal{F}(u,w)(x_1,x_2), \tag{2.20}$$

where

$$
\begin{aligned}
\mathcal{F}_1(u,w)(x_1,x_2) = \int_0^{\omega_2} g_2(x_2,s_2;x_1)\Bigg(\sum_{j=0}^{m_1-1} p_{j\,m_2}(x_1,s_2)w^{(j,0)}(x_1,s_2)\\
+\sum_{j=0}^{m_1-1}\sum_{k=0}^{m_2-1} p_{jk}(x_1,s_2)u^{(j,k)}(x_1,s_2)+q(x_1,s_2)\Bigg)\,ds_2\\
+\Gamma_2\big(\varphi_{21}^{(m_1)}(x_1),\dots,\varphi_{2m_2}^{(m_1)}(x_1)\big)(x_2); \quad (2.21)
\end{aligned}
$$

$$
\begin{aligned}
\mathcal{F}_2(u,v)(x,y) = \int_0^{\omega_1} g_1(x_1,s_1;x_2)\Bigg(\sum_{k=0}^{m_2-1} p_{m_1\,k}(\mathbf{x})v^{(0,k)}(s_1,x_2)\\
+\sum_{j=0}^{m_1-1}\sum_{k=0}^{m_2-1} p_{jk}(x_1,s_2)(s_1,x_2)u^{(j,k)}(s_1,x_2)+q(s_1,x_2)\Bigg)\,ds_1\\
+\Gamma_1\big(\varphi_{11}^{(m_2)}(x_2),\dots,\varphi_{1m_1}^{(m_2)}(x_2)\big)(x_1); \quad (2.22)
\end{aligned}
$$

$$
\begin{aligned}
\mathcal{F}(u,w)(x_1,x_2) = \int_0^{\omega_1} \widetilde{g}_1(x_1,s_1;x_2)\int_0^{\omega_2} g_2(x_2,s_2;s_1)\Bigg(\sum_{j=0}^{m_1-1}\sum_{k=0}^{m_2-1} \rho_{jk}(s_1,s_2)u^{(j,k)}(s_1,s_2)\\
+\sum_{j=0}^{m_1-1}\big(p_{j\,m_2}(s_1,s_2)-\widetilde{p}_{j\,m_2}(s_1,s_2)\big)w^{(j,0)}(s_1,s_2)+q(s_1,s_2)\Bigg)\,ds_2\,ds_1\\
+\mathcal{P}\big[u;\varphi_{21}^{(m_1)},\dots,\varphi_{2m_2}^{(m_1)}\big](x_1,x_2)+\widetilde{\Gamma}_1\big(\varphi_{11}(x_1),\dots,\varphi_{1m_1}(x_1)\big)(x_2), \quad (2.23)
\end{aligned}
$$

Let $\mathcal{F}_1^0(u,w)$, $\mathcal{F}_2^0(u,v)$ and $\mathcal{F}^0(u,w)$ be the homogeneous parts of the operators $\mathcal{F}_1(u,w)$, $\mathcal{F}_2(u,v)$ and $\mathcal{F}(u,w)$, respectively, and set:

$$
\mathcal{K}(u,v,w) = \Big(\mathcal{F}_1^0(u,w),\mathcal{F}_2^0(u,v),\mathcal{F}^0(u,w)\Big) \quad (2.24)
$$

It is clear that $\mathcal{K}$ is a bounded linear operator from $C^{m_1-1,m_2-1}(\Omega)\times C^{m_1-1,m_2-1}(\Omega)\times C^{m_1-1,m_2-1}(\Omega)$ into $C^{m_1-1,m_2}(\Omega)\times C^{m_1,m_2-1}(\Omega)\times C^{m_1,m_2}(\Omega)$.

Notice that $\mathcal{K}^2=\mathcal{K}\circ\mathcal{K}$ is a compact operator from $C^{m_1-1,m_2-1}(\Omega)\times C^{m_1-1,m_2-1}(\Omega)\times C^{m_1-1,m_2-1}(\Omega)$ into $C^{m_1-1,m_2-1}(\Omega)\times C^{m_1-1,m_2-1}(\Omega)\times C^{m_1-1,m_2-1}(\Omega)$. The latter fact implies that the system of operator equations (2.18)–(2.20) and, consequently, problem (1.1),(1.2) have the Fredholm property. □

Remark 2.1. Notice that if $p_{jm_2} \in C^{0,m_2}(\Omega)$ $(j = 0,\dots,m_1-1)$, then choosing $\widetilde{p}_{jm_2}(\mathbf{x}) \equiv p_{jm_2}(\mathbf{x})$ $(j = 0,\dots,m_1-1)$ in representation (2.9), one can show that problem (1.1),(1.2) is equivalent to the Fredholm equation

$$u(x_1,x_2) = \mathcal{F}_0(u)(x_1,x_2), \tag{2.25}$$

where

$$\begin{aligned}\mathcal{F}_0(u)(x_1,x_2) = \int_0^{\omega_1} g_1(x_1,s_1;x_2)\int_0^{\omega_2} g_2(x_2,s_2;s_1)\Big(\sum_{j=0}^{m_1-1}\sum_{k=0}^{m_2-1}\rho_{jk}(s_1,s_2)u^{(j,k)}(s_1,s_2)\\ + q(s_1,s_2)\Big)\,ds_2\,ds_1 + \mathcal{P}\big[u;\varphi_{21}^{(m_1)},\dots,\varphi_{2m_2}^{(m_1)}\big](x_1,x_2)\\ + \Gamma_1\big(\varphi_{11}(x_1),\dots,\varphi_{1m_1}(x_1)\big)(x_2).\end{aligned} \tag{2.26}$$

Remark 2.2. By Lemma 2.3, problem (1.1),(1.2) has the Fredholm property in $C^{\mathbf{m}}(\Omega)$, i.e. problem $(1.1_0),(1.2_0)$ has a finite dimensional space of solutions and problem $(1.1),(1.2)$ has a unique solution in $C^{\mathbf{m}}(\Omega)$ if and only if problem $(1.1_0),(1.2_0)$ has only the trivial solution in $C^{m,n}(\Omega)$.

It is not difficult to show that problem (1.1),(1.2) has the Fredholm property $AC^{\mathbf{m}-\mathbf{1}}(\Omega)$, i.e. problem $(1.1_0),(1.2_0)$ has a finite dimensional space of solutions in $AC^{\mathbf{m}-\mathbf{1}}(\Omega)$, and for any $q \in L(\Omega)$, $\varphi_{ik} \in AC^{\widehat{\mathbf{m}}_i-\widehat{\mathbf{1}}_i}(\widehat{\Omega}_i)$ $(k = 1,\dots,m_i;\ i = 1,2)$, problem $(1.1),(1.2)$ has a unique solution in $AC^{\mathbf{m}-\mathbf{1}}(\Omega)$ if and only if problem $(1.1_0),(1.2_0)$ has only the trivial solution in $AC^{\mathbf{m}-\mathbf{1}}(\Omega)$.

2.3. **Case** $n \geq 2$**.** If problem $(1.1),(1.2)$ is well–posed, then its solution u admits the representation

$$u(x) = \Gamma\big[\Phi\big](x) + \mathcal{G}(q)(x), \tag{2.27}$$

where $\Phi = \Big(\varphi_{ik}\Big)_{k=1,\dots,m_i}^{i=1,\dots,n}$, $\Gamma : C^{m_1-1}(\widehat{\Omega}_1;\mathbb{R}^{m_1}) \times \cdots \times C^{m_n-1}(\widehat{\Omega}_n;\mathbb{R}^{m_n}) \to C^{\mathbf{m}}(\Omega)$ and $\mathcal{G} : C(\Omega) \to C^{\mathbf{m}}(\Omega)$ are bounded linear operators.

Definition 2.3. $\mathcal{G} : C(\Omega) \to C^{\mathbf{m}}(\Omega)$ is called the **Green's operator** of problem $(1.1_0),(1.2_0)$.

Definition 2.4. $\Gamma : C^{m_1-1}(\widehat{\Omega}_1; \mathbb{R}^{m_1}) \times \cdots \times C^{m_n-1}(\widehat{\Omega}_n; \mathbb{R}^{m_n}) \to C^{\mathbf{m}}(\Omega)$ is called the **Green's boundary operator** of problem $(1.1_0), (1.2_0)$.

Remark 2.3. If problem (1.1),(1.2) is well–posed in $C^{\mathbf{m}}(\Omega)$, and its solution admits the estimate (3.57) (see the proof of Theorem 1.6 below), then problem (1.1),(1.2) is well–posed in $AC^{\mathbf{m-1}}(\Omega)$ too.

In other words, the Green's operator $\mathcal{G}$ is a bounded linear operator from $C(\Omega)$ to $C^{\mathbf{m}}(\Omega)$, as well as from $L(\Omega)$ to $AC^{\mathbf{m-1}}(\Omega)$. Therefore, by the Dunford-Pettis theorem (see [18], Chapter XI, § 1, Theorem 6) $\mathcal{G}$ admits the representation

$$\mathcal{G}(q)(\mathbf{x}) = \iint\limits_{\Omega} G(\mathbf{x}, \mathbf{s}) q(\mathbf{s})\, d\mathbf{s}, \tag{2.28}$$

where $G \in L^{\infty}(\Omega \times \Omega)$ is called the Green's function of problem $(1.1_0), (1.2_0)$.

Lemma 2.4. *Let* $\mathbf{m} \geq \mathbf{0}$ *and* $p, z \in C^{\mathbf{m}}(\Omega)$ *be arbitrary functions. Then there exist integers* $c_{\boldsymbol{\alpha}}$ *depending on* $\mathbf{m}$ *only and functions* $\rho_{\boldsymbol{\alpha}} \in C^{\mathbf{m}}(\Omega)$ *depending on* p *only such that*

$$p(\mathbf{x}) z^{(\mathbf{m})}(\mathbf{x}) = \sum_{\boldsymbol{\alpha} \leq \mathbf{m}} c_{\boldsymbol{\alpha}} (\rho_{\boldsymbol{\alpha}}(\mathbf{x}) z(\mathbf{x}))^{(\boldsymbol{\alpha})}. \tag{2.29}$$

The proof of the lemma is obvious. Coefficients c_α and functions ρ_α are easily calculated by applying the product rule of differentiation multiple times.

Lemma 2.5. *Let for some* $\boldsymbol{\sigma} \in \boldsymbol{\Xi}$ *the* $\boldsymbol{\sigma}$*–associated problem* $(1.1_{\boldsymbol{\sigma}}), (1.2_{\boldsymbol{\sigma}})$ *be well–posed for every* $x_{\widehat{\boldsymbol{\sigma}}} \in \Omega_{\widehat{\boldsymbol{\sigma}}}$*, and let* $p_{\mathbf{m}_{\widehat{\boldsymbol{\sigma}}}+\boldsymbol{\alpha}} \in C(\Omega)$ $(\boldsymbol{\alpha} < \mathbf{m}_{\boldsymbol{\sigma}})$*. Then the Green's operator* $\mathcal{G}_{\boldsymbol{\sigma}}$ *is a bounded linear operator from* $C(\Omega)$ *to* $C^{\mathbf{m}_{\boldsymbol{\sigma}}}(\Omega)$*. Moreover, if* $p_{\mathbf{m}_{\widehat{\boldsymbol{\sigma}}}+\boldsymbol{\alpha}} \in C^{\mathbf{m}_{\widehat{\boldsymbol{\sigma}}}}(\Omega)$ $(\boldsymbol{\alpha} < \mathbf{m}_{\boldsymbol{\sigma}})$*, then the Green's operator* $\mathcal{G}_{\boldsymbol{\sigma}}$ *is a bounded linear operator from* $C(\Omega)$ *to* $C^{\mathbf{m}}(\Omega)$*.*

Proof. Let v be a solution of the problem

$$v^{(\mathbf{m}_{\boldsymbol{\sigma}})} = \sum_{\boldsymbol{\alpha}<\mathbf{m}_{\boldsymbol{\sigma}}} p_{\boldsymbol{\alpha}+\widehat{\mathbf{m}}_{\boldsymbol{\sigma}}}(\mathbf{x})v^{(\boldsymbol{\alpha})} + q(\mathbf{x}), \tag{2.30}$$

$$h_{ik}\big(v^{(\mathbf{m}_{\boldsymbol{\sigma}}^{i-1})}(\bullet\,\widehat{\mathbf{x}}_i)\big) = 0 \qquad (k = 1,\dots,m_i;\ \ i \in \operatorname{supp}\boldsymbol{\sigma}). \tag{2.31}$$

Set:

$$w(\mathbf{x}) = v(\mathbf{x}_{\boldsymbol{\sigma}}, \mathbf{x}_{\widehat{\boldsymbol{\sigma}}}) - v(\mathbf{x}_{\boldsymbol{\sigma}}, \widetilde{\mathbf{x}}_{\widehat{\boldsymbol{\sigma}}}). \tag{2.32}$$

Then w is a solution of the problem

$$\begin{aligned} w^{(\mathbf{m}_{\boldsymbol{\sigma}})} = \sum_{\boldsymbol{\alpha}<\mathbf{m}_{\boldsymbol{\sigma}}} & p_{\boldsymbol{\alpha}+\widehat{\mathbf{m}}_{\boldsymbol{\sigma}}}(\mathbf{x})w^{(\boldsymbol{\alpha})} \\ &+ \sum_{\boldsymbol{\alpha}<\mathbf{m}_{\boldsymbol{\sigma}}} \Big(p_{\boldsymbol{\alpha}+\widehat{\mathbf{m}}_{\boldsymbol{\sigma}}}(\mathbf{x}_{\boldsymbol{\sigma}}, \mathbf{x}_{\widehat{\boldsymbol{\sigma}}}) - p_{\boldsymbol{\alpha}+\widehat{\mathbf{m}}_{\boldsymbol{\sigma}}}(\mathbf{x}_{\boldsymbol{\sigma}}, \widetilde{\mathbf{x}}_{\widehat{\boldsymbol{\sigma}}})\Big)v^{(\boldsymbol{\alpha})}(\mathbf{x}_{\boldsymbol{\sigma}}, \widetilde{\mathbf{x}}_{\widehat{\boldsymbol{\sigma}}}) \\ &\qquad + q(\mathbf{x}_{\boldsymbol{\sigma}}, \mathbf{x}_{\widehat{\boldsymbol{\sigma}}}) - q(\mathbf{x}_{\boldsymbol{\sigma}}, \widetilde{\mathbf{x}}_{\widehat{\boldsymbol{\sigma}}}), \end{aligned} \tag{2.33}$$

$$h_{ik}\big(w^{(\mathbf{m}_{\boldsymbol{\sigma}}^{i-1})}(\bullet\,\widehat{\mathbf{x}}_i)\big) = 0 \qquad (k = 1,\dots,m_i;\ \ i \in \operatorname{supp}\boldsymbol{\sigma}). \tag{2.34}$$

Hence we get the estimate

$$\begin{aligned} \|w(\bullet\,\mathbf{x}_{\widehat{\boldsymbol{\sigma}}})\|_{C^{\mathbf{m}_{\boldsymbol{\sigma}}}(\Omega_{\boldsymbol{\sigma}})} \le M^2(\mathbf{x}_{\widehat{\boldsymbol{\sigma}}})\|q\|_{C(\Omega)} \sum_{\boldsymbol{\alpha}<\mathbf{m}_{\boldsymbol{\sigma}}} & \big\|p_{\boldsymbol{\alpha}+\widehat{\mathbf{m}}_{\boldsymbol{\sigma}}}(\bullet, \mathbf{x}_{\widehat{\boldsymbol{\sigma}}}) - p_{\boldsymbol{\alpha}+\widehat{\mathbf{m}}_{\boldsymbol{\sigma}}}(\bullet, \widetilde{\mathbf{x}}_{\widehat{\boldsymbol{\sigma}}})\big\|_{C(\Omega_{\boldsymbol{\sigma}})} \\ &+ M(\mathbf{x}_{\widehat{\boldsymbol{\sigma}}})\big\|q(\bullet, \mathbf{x}_{\widehat{\boldsymbol{\sigma}}}) - q(\bullet, \widetilde{\mathbf{x}}_{\widehat{\boldsymbol{\sigma}}})\big\|_{C(\Omega_{\boldsymbol{\sigma}})}, \end{aligned} \tag{2.35}$$

where $M(\mathbf{x}_{\widehat{\boldsymbol{\sigma}}})$ is the norm of the Green's operator $\mathcal{G}_{\boldsymbol{\sigma}}$ depending on the parameter $\mathbf{x}_{\widehat{\boldsymbol{\sigma}}}$.

The latter estimate implies the continuity of $\mathcal{G}_{\boldsymbol{\sigma}}$ as an operator from $C(\Omega)$ to $C^{\mathbf{m}_{\boldsymbol{\sigma}}}(\Omega)$. The second part of the lemma can be proved similarly. $\square$

Lemma 2.6. *Let problem* (1.1), (1.2) *be well–posed, and let for every* $\boldsymbol{\sigma} \in \boldsymbol{\Xi}$ *the* $\boldsymbol{\sigma}$*–associated problem* $(1.1_{\boldsymbol{\sigma}}), (1.2_{\boldsymbol{\sigma}})$ *be well–posed for every* $\mathbf{x}_{\widehat{\boldsymbol{\sigma}}} \in \Omega_{\widehat{\boldsymbol{\sigma}}}$. *Moreover, let* $p_{\boldsymbol{\alpha}} \in C^{\mathbf{m}}(\Omega)$ $(\boldsymbol{\alpha} \in \boldsymbol{\Upsilon}_{\mathbf{m}})$. *Then problem* (1.1), (1.2) *is* α*–well–posed.*

Lemma 2.7. *Let problem* (1.1), (1.2) *be well–posed, and let for every* $\boldsymbol{\sigma} \in \boldsymbol{\Xi}$ *the* $\boldsymbol{\sigma}$*–associated problem* $(1.1_{\boldsymbol{\sigma}}), (1.2_{\boldsymbol{\sigma}})$ *be well–posed for every* $\mathbf{x}_{\widehat{\boldsymbol{\sigma}}} \in \Omega_{\widehat{\boldsymbol{\sigma}}}$. *Moreover, let*

$p_{\boldsymbol\alpha} \in C^{\mathbf{m}}(\Omega)$ $(\boldsymbol\alpha < \mathbf{m})$. *Then*

$$\mathcal{G}(q^{(\mathbf{m})})(\mathbf{x}) = q(\mathbf{x}) + \mathcal{A}(q)(\mathbf{x}), \tag{2.36}$$

where $\mathcal{G}$ *is the Green's operator of problem* $(1.1_0), (1.2_0)$, *and* $\mathcal{A} : C^{(\mathbf{m}-\mathbf{1})}(\Omega) \to C^{(\mathbf{m}-\mathbf{1})}(\Omega)$ *is a bounded linear operator.*

Proofs of Lemmas 2.6 *and* 2.7. Let problem (1.1),(1.2) be well–posed, and let u be a solution of problem $(1.5), (1.2_0)$. Set

$$v_{\mathbf{m}_{\boldsymbol\sigma}}(\mathbf{x}) = u^{(\mathbf{m}_{\boldsymbol\sigma})}(\mathbf{x}) \qquad (\boldsymbol\sigma \in \boldsymbol\Xi). \tag{2.37}$$

Then $v_{\mathbf{m}_{\boldsymbol\sigma}}(\mathbf{x})$ is a solution of the problem

$$v^{(\mathbf{m}_{\widehat{\boldsymbol\sigma}})} = \sum_{\boldsymbol\alpha < \mathbf{m}_{\widehat{\boldsymbol\sigma}}} p_{\mathbf{m}_{\boldsymbol\sigma}+\alpha}(\mathbf{x}) v^{(\boldsymbol\alpha)} + \sum_{\boldsymbol\alpha < \mathbf{m}_{\boldsymbol\sigma}} \sum_{\beta \le \mathbf{m}_{\widehat{\boldsymbol\sigma}}} p_{\boldsymbol\alpha+\boldsymbol\beta}(\mathbf{x}) u^{(\boldsymbol\alpha+\boldsymbol\beta)} + q^{(\boldsymbol m)}(\mathbf{x}), \tag{2.38}$$

$$h_{ik}\big(v^{(\mathbf{m}_{\boldsymbol\sigma}^{i-1})}(\bullet\, \widehat{\mathbf{x}}_i)\big) = 0 \qquad (k = 1, \dots, m_i;\;\; i \in \operatorname{supp} \widehat{\boldsymbol\sigma}). \tag{2.39}$$

We prove the lemmas by induction. The validity of Lemmas 2.6 and 2.7 $n = 2$ follows from Theorem 1.4 (see the proof of Theorem 1.4 below). Let $n \ge 3$, and let us assume that the theorem is true for $n-1$ dimensional problem. Then each $\boldsymbol\sigma$–associated problem $(1.1_{\boldsymbol\sigma}), (1.2_{\boldsymbol\sigma})$ $(\boldsymbol\sigma \in \boldsymbol\Xi)$ is α–well–posed. Consequently, u admits the following representations:

$$u^{(\mathbf{m}_{\boldsymbol\sigma})}(\mathbf{x}) - q^{(\mathbf{m}_{\boldsymbol\sigma})}(\mathbf{x}) = \mathcal{G}_{\widehat{\boldsymbol\sigma}}\Big(\sum_{\boldsymbol\alpha < \mathbf{m}_{\boldsymbol\sigma}} \sum_{\beta \le \mathbf{m}_{\widehat{\boldsymbol\sigma}}} p_{\boldsymbol\alpha+\boldsymbol\beta}(\mathbf{x}) u^{(\boldsymbol\alpha+\boldsymbol\beta)} \Big) + \mathcal{B}_{\circ\, \widehat{\boldsymbol\sigma}}(q)(\mathbf{x}), \tag{2.40}$$

where $\mathcal{G}_{\widehat{\boldsymbol\sigma}}$ is the Green's operator of problem $(1.1_{\widehat{\boldsymbol\sigma}}), (1.2_{\widehat{\boldsymbol\sigma}})$, and $\mathcal{B}_{\circ\, \widehat{\boldsymbol\sigma}}$ is a bounded linear operator from $C^{\mathbf{m}-\mathbf{1}}(\Omega)$ into itself.

Set

$$w_{\boldsymbol\alpha}(\mathbf{x}) = u^{(\boldsymbol\alpha)}(\mathbf{x}) - q^{(\boldsymbol\alpha)}(\mathbf{x}). \tag{2.41}$$

Then

$$\begin{aligned} w_{\mathbf{m}_{\boldsymbol\sigma}}(\mathbf{x}) = \mathcal{G}_{\widehat{\boldsymbol\sigma}}\Big(&\sum_{\boldsymbol\alpha < \mathbf{m}_{\boldsymbol\sigma}} \sum_{\beta \le \mathbf{m}_{\widehat{\boldsymbol\sigma}}} p_{\boldsymbol\alpha+\boldsymbol\beta}(\mathbf{x}) w_{\boldsymbol\alpha+\boldsymbol\beta}(\mathbf{x}) \Big) \\ &+ \mathcal{G}_{\widehat{\boldsymbol\sigma}}\Big(\sum_{\boldsymbol\alpha < \mathbf{m}_{\boldsymbol\sigma}} \sum_{\beta \le \mathbf{m}_{\widehat{\boldsymbol\sigma}}} p_{\boldsymbol\alpha+\boldsymbol\beta}(\mathbf{x}) q^{(\boldsymbol\alpha+\boldsymbol\beta)} \Big) + \mathcal{B}^{\circ}_{\widehat{\boldsymbol\sigma}}(q) \qquad (\boldsymbol\sigma \in \boldsymbol\Xi). \end{aligned} \tag{2.42}$$

It is clear that problem $(1.5), (1.2_0)$ is equivalent to the system of operator equations

$$w_{\boldsymbol{\gamma}}(\mathbf{x}) = \mathcal{A}_{\boldsymbol{\gamma}}\big(w\big)(\mathbf{x}) + \mathcal{B}_{\boldsymbol{\gamma}}(q) \qquad (\boldsymbol{\gamma} < \mathbf{m}), \tag{2.43}$$

where

$$\mathcal{A}_{\boldsymbol{\gamma}}\big(w\big)(\mathbf{x}) = \mathcal{G}_{\widehat{\boldsymbol{\sigma}}}^{(\boldsymbol{\gamma}-\widehat{\boldsymbol{\sigma}})}\Big(\sum_{\boldsymbol{\alpha}<\mathbf{m}_{\boldsymbol{\sigma}}}\sum_{\beta\leq\mathbf{m}_{\widehat{\boldsymbol{\sigma}}}} p_{\boldsymbol{\alpha}+\boldsymbol{\beta}}(\mathbf{x}) w_{\boldsymbol{\alpha}+\boldsymbol{\beta}}(\mathbf{x})\Big) \quad \text{for} \quad \mathbf{m}_{\widehat{\boldsymbol{\sigma}}} \leq \boldsymbol{\gamma} \leq \mathbf{m} - \boldsymbol{\sigma}, \tag{2.44}$$

and

$$\mathcal{B}_{\boldsymbol{\gamma}}\big(w\big)(\mathbf{x}) = \mathcal{A}_{\boldsymbol{\gamma}}\big(q\big)(\mathbf{x}) + \mathcal{B}_{\circ\,\widehat{\boldsymbol{\sigma}}}^{(\boldsymbol{\gamma}-\widehat{\boldsymbol{\sigma}})}(q)(\mathbf{x}) \quad \text{for} \quad \mathbf{m}_{\widehat{\boldsymbol{\sigma}}} \leq \boldsymbol{\gamma} \leq \mathbf{m} - \boldsymbol{\sigma}. \tag{2.45}$$

By Lemma 2.4, $\mathcal{B}_{\boldsymbol{\gamma}}$ is a bounded linear operator from $C^{\mathbf{m}-\mathbf{1}}(\Omega)$ into itself. Due to well-posedness of problem (1.1),(1.2), system (2.43) is uniquely solvable, and the operator $\mathcal{I} - \mathcal{K}$ is invertible, where $\mathcal{K} = \big(\mathcal{A}_{\boldsymbol{\gamma}}\big)_{\boldsymbol{\gamma}<\mathbf{m}}$. Consequently, problem (1.1),(1.2) is α–well–posed and equality (2.36) holds. $\square$

3. Proofs of the Main Results

Proof of Theorem 1.1. Let $\varphi_{ik}(\widehat{\mathbf{x}}_i) \equiv 0$ $(k = 1, \dots, m_i;\ i = 2, \dots, n)$, and let

$$\varphi_{1k}(\widehat{\mathbf{x}}_1) = c_k \varphi(x_2) \quad (k = 1, \dots, m_1), \tag{3.1}$$

where $c_1, \dots, c_{m_1}$ are arbitrary real numbers and

$$h_{21}(\varphi(\bullet)) = 1 \tag{3.2}$$

(the latter equality is possible, since h_{21} is not a zero functional).

Let u be an arbitrary solution of problem (1.1),(1.2). Set $z = h_{21}(u(\bullet, \widehat{\mathbf{x}}_2))$. Then z is a solution of the problem

$$z^{(m_1)} = 0, \tag{3.3}$$

$$h_{1k}(z) = c_k \quad (k = 1, \dots, m_1). \tag{3.4}$$

Consequently, problem (3.3),(3.4) is solvable for arbitrary boundary values c_1, ..., c_{m_1}. By Lemma 2.1, this is equaivalent to the fact that the homogeneous problem

$$z^{(m_1)} = 0, \tag{3.3_0}$$

$$h_{1k}(z) = 0 \quad (k = 1, \dots, m_1). \tag{3.4_0}$$

has only the trivial solution.

In order to complete the proof of the theorem one needs to consider

$$z = h_{j+1\,1}\big(u^{(\mathbf{m}^j}(\bullet, \widehat{\mathbf{x}}_{j+1})\big)$$

and choose $\varphi_{ik}(\widehat{\mathbf{x}}_i) \equiv 0$ $(k = 1, \dots, m_i;\ i \neq j)$,

$$\varphi_{jk}(\widehat{\mathbf{x}}_j) = c_k \varphi(x_{j+1}) \quad (k = 1, \dots, m_j),$$

where $c_1, \dots, c_{m_j}$ are arbitrary real numbers and

$$h_{j+1,1}(\varphi(\bullet)) = 1. \quad \square$$

Proof of Theorem 1.2. Let u be a solution of problem (1.1),(1.2). Set:

$$w(\mathbf{x}) = u^{(\mathbf{m}_{\widehat{\sigma}})}(\mathbf{x}) - \sum_{\boldsymbol{\beta}<\mathbf{m}_{\widehat{\sigma}}} p_{\boldsymbol{\beta}+\mathbf{m}_{\boldsymbol{\sigma}}} u^{(\boldsymbol{\beta})}(\mathbf{x}), \tag{3.5}$$

$$v_{jl}(\widehat{\mathbf{x}}_j) = h_{jl}\big(w(\bullet, \widehat{\mathbf{x}}_{\mathbf{j}})\big) \quad (l = 1, \ldots, m_j), \quad j \in \operatorname{supp} \widehat{\boldsymbol{\sigma}}. \tag{3.6}$$

Then, in view of conditions (1.8), we get that w is a solution of the problem

$$w^{(\mathbf{m}_{\boldsymbol{\sigma}})} = \sum_{\boldsymbol{\alpha}<\mathbf{m}_{\boldsymbol{\sigma}}} p_{\boldsymbol{\alpha}+\mathbf{m}_{\widehat{\sigma}}} w^{(\boldsymbol{\alpha})} + (p_{\mathbf{0}} + p_{\mathbf{m}_{\widehat{\sigma}}} p_{\mathbf{m}_{\boldsymbol{\sigma}}}) u(\mathbf{x}) + q(\mathbf{x}), \tag{3.7}$$

$$\begin{aligned} h_{ik}\big(w^{(\mathbf{m}^{i-1})}(\bullet,\, \widehat{\mathbf{x}}_i)\big) &= \Big[h_{ik}\Big(u^{(\mathbf{m}_{\widehat{\sigma}})}(\bullet,\, \widehat{\mathbf{x}}_i) - \sum_{\boldsymbol{\beta}<\mathbf{m}_{\widehat{\sigma}}} p_{\boldsymbol{\beta}+\mathbf{m}_{\boldsymbol{\sigma}}} u^{(\boldsymbol{\beta})}(\bullet,\, \widehat{\mathbf{x}}_i)\Big)\Big]^{(\mathbf{m}^{i-1})} \\ &= \Big[h_{ik}\Big(u^{(\mathbf{m}^{i-1})}(\bullet,\, \widehat{\mathbf{x}}_i)\Big)\Big]^{(\mathbf{m}_{\widehat{\sigma}})} - \sum_{\boldsymbol{\beta}<\mathbf{m}_{\widehat{\sigma}}} p_{\boldsymbol{\beta}+\mathbf{m}_{\boldsymbol{\sigma}}} \Big[h_{ik}\Big(u^{(\mathbf{m}^{i-1})}(\mathbf{x})\Big)\Big]^{(\boldsymbol{\beta})} \\ &= \varphi_{ik}^{(\mathbf{m}^{i-1}+\mathbf{m}_{\widehat{\sigma}})}(\widehat{\mathbf{x}}_i) - \sum_{\boldsymbol{\beta}<\mathbf{m}_{\widehat{\sigma}}} p_{\boldsymbol{\beta}+\mathbf{m}_{\boldsymbol{\sigma}}} \varphi_{ik}^{(\mathbf{m}^{i-1}+\boldsymbol{\beta})}(\widehat{\mathbf{x}}_i) \quad (k = 1, \ldots, m_i); \;\; i \in \operatorname{supp} \boldsymbol{\sigma}. \end{aligned} \tag{3.8}$$

Applying the operator h_{jl} to (3.7) and (3.8), we immediately get that v_{jl} is a solution of problem (1.9),(1.10). $\square$

Proof of Theorem 1.3. Let problem (1.1),(1.2) be well–posed. Assume the contrary: let problem $(1.1_2), (1.2_2)$ have a nontrivial solution $\xi_0(x_1)$ for some $x_2^* \in [0, \omega_2]$. Then due to the well–posedness of problem (1.1),(1.2) there exist $\delta > 0$ and $\widetilde{p}_{j\, m_2} \in C^{(0,m_2)}(\Omega)$ $(j = 0, \ldots, m_1 - 1)$ such that

$$\widetilde{p}_{j\, m_2}(x_1, x_2) = p_{j\, m_2}(x_1, x_2^*) \quad \text{for} \quad x_2 \in [x_2^* - \delta, x_2^* + \delta] \cap [0, \omega_2] \quad (j = 0, \ldots, m_1 - 1),$$

and the problem

$$u^{(m_1,m_2)} = \sum_{k=0}^{m_2-1} p_{m_1\, k}(\mathbf{x}) u^{(m_1,k)} + \sum_{j=0}^{m_1-1} \widetilde{p}_{j\, m_2}(\mathbf{x}) u^{(j,m_2)} + \sum_{\boldsymbol{\alpha}\leq\mathbf{m}-\mathbf{1}} p_{\boldsymbol{\alpha}}(\mathbf{x}) u^{(\boldsymbol{\alpha})}, \tag{3.9}$$

$$h_{1k}(u(\bullet, x_2) = 0 \;\; (k = 1, \ldots, m_1), \quad h_{2k}(u^{(\mathbf{m}_1)}(x_1, \bullet)) = 0 \;\; (k = 1, \ldots, m_2) \tag{3.10}$$

is well–posed. From Remark 2.1 it follows that the problem

$$u^{(m_1,m_2)} = \sum_{k=0}^{m_2-1} p_{m_1\,k}(\mathbf{x})u^{(m_1,k)} + \sum_{j=0}^{m_1-1} \widetilde{p}_{j\,m_2}(\mathbf{x})u^{(j,m_2)} + \sum_{\boldsymbol{\alpha}\le\mathbf{m-1}} \widetilde{p}_{\boldsymbol{\alpha}}(\mathbf{x})u^{(\boldsymbol{\alpha})}, \tag{3.11}$$

$$h_{1k}(u(\bullet,x_2) = 0 \ \ (k=1,\dots,m_1),$$
$$h_{2k}\Big(u^{(\mathbf{m}_1)}(x_1,\bullet) - \sum_{j=0}^{m_1-1} \widetilde{p}_{j\,m_2}(\mathbf{x})u^{(j,0)}(x_1,\bullet)\Big) = 0 \ \ (k=1,\dots,m_2), \tag{3.12}$$

where

$$\widetilde{p}_{j\,k}(\mathbf{x}) = -\sum_{i=k}^{m_2-1} \frac{k!}{i!(i-k)!}\, p_{m_1\,i}(\mathbf{x})\, \widetilde{p}_{j\,m_2}^{(0,i-k)}(\mathbf{x})$$
$$+\frac{m_2!}{k!(m_2-k)!}\, \widetilde{p}_{j\,m_2}^{(0,m_2-k)}(\mathbf{x}) \quad (j=0,\dots,m_1-1;\ k=0,\dots m_2-1),$$

has the Fredholm property. On the other hand, an arbitrary solution u of problem (3.11), (3.12) is a solution of the problem

$$\Big(u^{(\mathbf{m}_1)} - \sum_{j=0}^{m_1-1} \widetilde{p}_{j\,m_2}(\mathbf{x})u^{(j,0)}\Big)^{(0,m_2)}$$
$$= \sum_{k=0}^{m_2-1} p_{m_1\,k}(\mathbf{x})\Big(u^{(\mathbf{m}_1)} - \sum_{j=0}^{m_1-1} \widetilde{p}_{j\,m_2}(\mathbf{x})u^{(j,0)}\Big)^{(0,k)},$$
$$h_{2k}\Big(u^{(\mathbf{m}_1)}(x_1,\bullet) - \sum_{j=0}^{m_1-1} \widetilde{p}_{j\,m_2}(\mathbf{x})u^{(j,0)}(x_1,\bullet)\Big) = 0 \ \ (k=1,\dots,m_2).$$

Hence, every solution $u\in C^{\mathbf{m}}(\Omega)$ of the problem

$$u^{(\mathbf{m}_1)} = \sum_{j=0}^{m_1-1} \widetilde{p}_{j\,m_2}(\mathbf{x})u^{(j,0)}, \tag{3.13}$$

$$h_{1k}(u(\bullet,x_2)) = 0 \ \ (k=1,\dots,m_1), \tag{3.14}$$

is a solution of problem (3.11), (3.12).

Let $\gamma\in C^\infty([0,\omega_2])$, $\operatorname{supp}\gamma\subset[x_2^*-\delta,x_2^*+\delta]\cap[0,\omega_2]$ be an arbitrary function. Then

$$u(\mathbf{x}) = \xi_0(x_1)\gamma(x_2)$$

is a solution of problem (3.13),(3.14), and, consequently, is a solution of the problem (3.11), (3.12). Thus problem (3.11), (3.12) has an infinite dimensional space of solutions, which contradicts to the fact that is has the Fredholm property. The obtained contradiction completes the proof of the theorem. □

Proof of Theorem 1.4. Theorem 1.4 follows from Lemma 2.3 and Remark 2.1, in particular, representations (2.9) and (2.25).

Proof of Theorem 1.5. First notice that it is sufficient to prove the theorem for problem (1.1), (1.2_0) only. Indeed, choose $\rho_k \in C([0,\omega_n])$ such that the problem

$$z^{(m_n)} = \sum_{k=0}^{m_n-1} \rho_k(x_n) z^{(k)}, \qquad h_{nk}(z) = 0 \quad (k = 1, \dots, m_n) \tag{3.15}$$

has only the trivial solution, and consider the auxiliary problem

$$u^{(\mathbf{m})} = \sum_{k=0}^{m_n-1} \rho_k(x_n) u^{(\widehat{\mathbf{m}}_n + k\mathbf{1}_n)}, \tag{3.16}$$

$$h_{ik}\big(u^{(\mathbf{m}^{i-1})}(\bullet, \widehat{\mathbf{x}}_i)\big) = \varphi_{ik}^{(\mathbf{m}^{i-1})}(\widehat{\mathbf{x}}_i) \ \text{ for } \ \widehat{\mathbf{x}}_i \in \widehat{\Omega}_i \ \ (k = 1, \dots, m_i; \ \ i = 1, \dots, n). \tag{3.17}$$

By Lemma 2.1, we have

$$u^{(\widehat{\mathbf{m}}_n)}(\mathbf{x}) = \Gamma_n\big(\varphi_{n\,1}(\bullet, \widehat{\mathbf{x}}_n), \dots, \varphi_{n\,m_n}(\bullet, \widehat{\mathbf{x}}_n)\big)(x_n), \tag{3.18}$$

where Γ_n is the Green's boundary operator of problem (3.15). Hence it is clear that the function

$$\begin{aligned} u(\mathbf{x}) = \Gamma_1\big(\varphi_{1\,1}(\bullet, \widehat{\mathbf{x}}_1), \dots, \varphi_{1\,m_1}(\bullet, \widehat{\mathbf{x}}_1)\big)(x_1) \\ + \mathcal{G}_1\Big(\Gamma_2\big(\varphi_{2\,1}, \dots, \varphi_{2\,m_2}\big)\Big)(\mathbf{x}) \\ + \dots + \mathcal{G}_1 \circ \dots \circ \mathcal{G}_{n-1}\Big(\Gamma_n\big(\varphi_{n\,1}, \dots, \varphi_{n\,m_n}\big)\Big)(\mathbf{x}) \end{aligned} \tag{3.19}$$

is the unique solution of problem (3.16),(3.17).

It is rather obvious that by means of function (3.19) one can always reduce problem (1.1),(1.2) to the problem (1.1).(1.2_0).

Let u be a solution of problem $(1.1), (1.2_0)$. Set

$$w(\mathbf{x}) = \mathcal{L}_{\widehat{\boldsymbol{\sigma}}}(u)(\mathbf{x}). \tag{3.20}$$

Then w is a solution of the problem

$$w^{(\mathbf{m}_{\boldsymbol{\sigma}})} = \sum_{\boldsymbol{\alpha}<\mathbf{m}_{\boldsymbol{\sigma}}} p_{\boldsymbol{\alpha}+\mathbf{m}_{\widehat{\boldsymbol{\sigma}}}} w^{(\boldsymbol{\alpha})} + \sum_{\boldsymbol{\alpha}\le\mathbf{m}-\mathbf{1}} p_{\boldsymbol{\alpha}}(\mathbf{x}) u^{(\boldsymbol{\alpha})}(\mathbf{x}) + q(\mathbf{x}), \tag{3.21}$$

$$h_{ik}\big(w^{(\mathbf{m}^{i-1})}(\bullet, \widehat{\mathbf{x}}_i)\big) = 0 \quad (k = 1, \dots, m_i; \ \ i \in \operatorname{supp}\boldsymbol{\sigma}). \tag{3.22}$$

Due to the well–posedness of $\boldsymbol{\sigma}$–associated problem, we have the following representation

$$w(x) = \mathcal{G}_{\boldsymbol{\sigma}}\bigg(\sum_{\boldsymbol{\alpha}\le\mathbf{m}-\mathbf{1}} p_{\boldsymbol{\alpha}} u^{(\boldsymbol{\alpha})} + q\bigg)(\mathbf{x}), \tag{3.23}$$

where $\mathcal{G}_{\boldsymbol{\sigma}}$ is the Green's operator of the $\boldsymbol{\sigma}$–associated problem. Consequently, we arrive at the problem

$$u^{(\mathbf{m}_{\widehat{\boldsymbol{\sigma}}})} = \sum_{\boldsymbol{\alpha}<\mathbf{m}_{\widehat{\boldsymbol{\sigma}}}} p_{\boldsymbol{\alpha}+\mathbf{m}_{\boldsymbol{\sigma}}} u^{(\boldsymbol{\alpha})} + \mathcal{G}_{\boldsymbol{\sigma}}\bigg(\sum_{\boldsymbol{\alpha}\le\mathbf{m}-\mathbf{1}} p_{\boldsymbol{\alpha}} u^{(\boldsymbol{\alpha})} + q\bigg)(\mathbf{x}), \tag{3.24}$$

$$h_{ik}\big(u^{(\mathbf{m}^{i-1})}(\bullet, \widehat{\mathbf{x}}_i)\big) = 0 \quad (k = 1, \dots, m_i; \ \ i \in \operatorname{supp}\widehat{\boldsymbol{\sigma}}). \tag{3.25}$$

Hence, due to the well–posedness of $\widehat{\boldsymbol{\sigma}}$–associated problem, we get

$$u(x) = \mathcal{G}_{\widehat{\boldsymbol{\sigma}}} \circ \mathcal{G}_{\boldsymbol{\sigma}}\bigg(\sum_{\boldsymbol{\alpha}\le\mathbf{m}-\mathbf{1}} p_{\boldsymbol{\alpha}} u^{(\boldsymbol{\alpha})} + q\bigg)(\mathbf{x}), \tag{3.26}$$

where $\mathcal{G}_{\widehat{\boldsymbol{\sigma}}}$ is the Green's operator of the $\widehat{\boldsymbol{\sigma}}$–associated problem.

It is clear that the operator equation (3.26) is equivalent to problem $(1.1), (1.2_0)$. On the other hand, the operator

$$\mathcal{A}(z) = \mathcal{G}_{\widehat{\boldsymbol{\sigma}}} \circ \mathcal{G}_{\boldsymbol{\sigma}}\bigg(\sum_{\boldsymbol{\alpha}\le\mathbf{m}-\mathbf{1}} p_{\boldsymbol{\alpha}} z^{(\boldsymbol{\alpha})}\bigg) \tag{3.27}$$

is a bounded linear operator from $C^{\mathbf{m}-\mathbf{1}}(\Omega)$ to $C^{\mathbf{m}}(\Omega)$. Consequently, $\mathcal{A}$ is a compact operator from $C^{\mathbf{m}-\mathbf{1}}(\Omega)$ to $C^{\mathbf{m}-\mathbf{1}}(\Omega)$. The latter fact immediately implies the Fredholm property of problem $(1.1), (1.2)$. $\square$

Proof of Theorem 1.6. As we have already noticed in the proof of Theorem 1.5, it is sufficient to prove the theorem for problem $(1.1), (1.2_0)$ only.

First let us prove solvability of problem (1.1), (1.2_0) under the assumption of the homogeneous problem (1.1_0), (1.2_0) having only the trivial solution.

Let $\lambda \in [0,1]$, and u be a solution of problem (1.23), (1.2_0). Choose $\rho_k \in C([0,\omega_n])$ $(k = 0, \dots, m_n - 1)$ so that the boundary value problem

$$z^{(m_n)} = \sum_{k=0}^{m_n-1} \rho_k(x_n) z^{(k)}, \qquad h_{nk}(z) = 0 \quad (k = 1, \dots, m_n) \tag{3.28}$$

has only the trivial solution, and set:

$$v_j(\mathbf{x}) = u^{(\mathbf{m_j})}(\mathbf{x}) \qquad (j = 1, \dots, n-1), \tag{3.29}$$

$$v_n(\mathbf{x}) = u^{(\mathbf{m_n})}(\mathbf{x}) - \sum_{k=0}^{m_n-1} \rho_k(x_n) u^{(k\mathbf{1}_n)}(\mathbf{x}). \tag{3.30}$$

Then, v_j is a solution of $n-1$ dimensional problem depending on the parameter $x_j \in [0, \omega_j]$ $\;(j = 1, \dots, n)$:

$$v^{(\widehat{\mathbf{m}}_j)} = \sum_{\boldsymbol{\alpha} < \widehat{\mathbf{m}}_j} p_{\lambda\, \mathbf{m}_j + \boldsymbol{\alpha}}(\mathbf{x}) v^{(\boldsymbol{\alpha})} + Q_j[u](\mathbf{x}), \tag{3.31}$$

$$h_{ik}\big(v^{(\mathbf{m}^{i-1})}(\bullet, \widehat{\mathbf{x}}_i)\big) = 0 \quad \text{for} \quad \widehat{\mathbf{x}}_i \in \widehat{\Omega_i} \quad (k = 1, \dots, m_i,\ i \neq j), \tag{3.32}$$

where

$$Q_j[u](\mathbf{x}) = \sum_{\boldsymbol{\alpha} < \mathbf{m}_j} \sum_{\boldsymbol{\beta} \leq \widehat{\mathbf{m}}_j} p_{\lambda\, \boldsymbol{\alpha} + \boldsymbol{\beta}}(\mathbf{x}) u^{(\boldsymbol{\alpha} + \boldsymbol{\beta})} + q^{(\mathbf{m})}(\mathbf{x}) \quad (j = 1, \dots, n-1), \tag{3.33}$$

$$\begin{aligned} Q_n[u](\mathbf{x}) = & \sum_{\boldsymbol{\alpha} < \mathbf{m}_n} \sum_{\boldsymbol{\beta} \leq \widehat{\mathbf{m}}_n} p_{\lambda\, \boldsymbol{\alpha} + \boldsymbol{\beta}}(\mathbf{x}) u^{(\boldsymbol{\alpha} + \boldsymbol{\beta})} \\ & + \sum_{\boldsymbol{\alpha} < \widehat{\mathbf{m}}_n} \sum_{k=0}^{m_n-1} p_{\lambda\, \boldsymbol{m}_n + \boldsymbol{\alpha}}\, \rho_k(x_n) u^{(\boldsymbol{\alpha} + k\mathbf{1}_n)} + q^{(\mathbf{m})}(\mathbf{x}). \end{aligned} \tag{3.34}$$

We prove the theorem by induction. The validity of Theorem 1.6 for $n = 2$ was already proved (see Theorem 1.4). Let $n \geq 3$, and let us assume that the theorem is true for $n-1$ dimensional problem. Then each problem (3.31)(3.32) is α–well–posed. Consequently, u admits the following representations:

$$u^{(\mathbf{m_j})}(\mathbf{x}) = \mathcal{G}_j(Q_j[u])(\mathbf{x}) \qquad (j = 1, \dots, n-1), \tag{3.35}$$

$$u^{(\mathbf{m_n})}(\mathbf{x}) = \sum_{k=0}^{m_n-1} \rho_k(x_n) u^{(k\mathbf{1}_n)}(\mathbf{x}) + \mathcal{G}_n(Q_n[u])(\mathbf{x}), \tag{3.36}$$

where $\mathcal{G}_j$ is the Green's operator of the associated problem

$$v^{(\widehat{\mathbf{m}}_j)} = \sum_{\boldsymbol{\alpha}<\widehat{\mathbf{m}}_j} p_{\lambda\,\mathbf{m}_j+\boldsymbol{\alpha}}(\mathbf{x})v^{(\boldsymbol{\alpha})}, \tag{3.37}$$

$$h_{ik}\big(v^{(\mathbf{m}^{i-1})}(\bullet, \widehat{\mathbf{x}}_i)\big) = 0 \quad \text{for } \widehat{\mathbf{x}}_i \in \widehat{\Omega}_i \quad (k = 1, \dots, m_i,\ i \neq j). \tag{3.38}$$

In view of differentiability of coefficients p_α and Lemmas 2.4 and 2.6, from (3.33), (3.34), (3.35) and (3.36) we get

$$u^{(\mathbf{m_j})}(\mathbf{x}) = q^{(\mathbf{m_j})}(\mathbf{x}) + \mathcal{A}_j^\circ(u)(\mathbf{x}) + \mathcal{B}_j^\circ(q)(\mathbf{x}) \quad (j = 1, \dots, n-1), \tag{3.39}$$

$$u^{(\mathbf{m_n})}(\mathbf{x}) = q^{(\mathbf{m_n})}(\mathbf{x}) + \sum_{k=0}^{m_n-1} \rho_k(x_n)u^{(k\mathbf{1}_n)}(\mathbf{x}) + \mathcal{A}_n^\circ(u)(\mathbf{x}) + \mathcal{B}_n^\circ(q)(\mathbf{x}), \tag{3.40}$$

where $\mathcal{A}_j^\circ$ and $\mathcal{B}_j^\circ : C^{\mathbf{m-1}}(\Omega) \to C^{\mathbf{m-1}}(\Omega)$ are bounded linear operators. Hence

$$u(\mathbf{x}) = \mathcal{A}_j(u)(\mathbf{x}) + \mathcal{B}_j(q)(\mathbf{x}) \quad (j = 1, \dots, n), \tag{3.41}$$

where

$$\mathcal{A}_j(u)(\mathbf{x}) = \int_0^{\omega_j} g_j(x_j, s_j)\mathcal{A}_j^\circ(u)(s_j, \hat{x}_j)ds_j \quad (j = 1, \dots, n), \tag{3.42}$$

$$\mathcal{B}_j(q)(\mathbf{x}) = \int_0^{\omega_j} g_j(x_j, s_j)\big(q^{(\mathbf{m}_j)}(s_j, \widehat{\mathbf{x}}_j) + \mathcal{B}_j^\circ(q)(s_j, \widehat{\mathbf{x}}_j)\big)\, ds_j \quad (j = 1, \dots, n), \tag{3.43}$$

and g_j is the Green's function of problem (1.7) if $j \in \{1, \dots, n-1\}$, and of problem (3.28) if $j = n$. Notice, that by Lemma 2.7, $\mathcal{B}_j$ is a bounded linear operator from $C^{\mathbf{m-1}}(\Omega) \to C^{\mathbf{m-1}}(\Omega)$ $(j = 1, \dots, n)$.

(3.41) implies the representation

$$u(\mathbf{x}) = \mathcal{A}(u)(\mathbf{x}) + \mathcal{B}(q)(\mathbf{x}), \tag{3.44}$$

where

$$\mathcal{A} = \mathcal{A}_1 \circ \mathcal{A}_2 \circ \dots \circ \mathcal{A}_n, \tag{3.45}$$

$$\mathcal{B} = \mathcal{B}_1 + \mathcal{A}_1 \circ \mathcal{B}_2 + \mathcal{A}_1 \circ \mathcal{A}_2 \circ \mathcal{B}_3 + \dots + \mathcal{A}_1 \circ \mathcal{A}_2 \dots \circ \mathcal{A}_{n-1} \circ \mathcal{B}_n. \tag{3.46}$$

From the construction of the operator $\mathcal{A}$ it is clear that $\mathcal{A} : C^{\mathbf{m-1}}(\Omega) \to C^{\mathbf{m}}(\Omega)$ is a bounded linear operator. Consequently, $\mathcal{A} : C^{\mathbf{m-1}}(\Omega) \to C^{\mathbf{m-1}}(\Omega)$ is a compact operator.

Furthermore, (3.39) and (3.40) yield the estimates

$$\|u^{(\mathbf{m}_j+\boldsymbol{\beta})}\|_{C(\Omega)} \leq M\Big(\|q^{(\mathbf{m}_j+\boldsymbol{\beta})}\|_{C(\Omega)} + \|q\|_{C^{\mathbf{m-1}}(\Omega)} + \|u\|_{C^{\mathbf{m-1}}(\Omega)}\Big) \qquad \boldsymbol{\beta} \leq \widehat{\mathbf{m}}_{\boldsymbol{j}} - \widehat{\mathbf{1}}_j \quad (j = 1, \ldots, n), \quad (3.47)$$

where M is a positive number independent of u and q.

Now set

$$w_{ij}(\mathbf{x}) = u^{(\mathbf{m_{ij}})}(\mathbf{x}) \quad (\mathbf{m_{ij}} = \mathbf{m_i} + \mathbf{m_j}). \quad (3.48)$$

Then w_{ij} is a solution of the $n-2$ dimensional problem

$$w^{(\widehat{\mathbf{m}}_{ij})} = \sum_{\boldsymbol{\alpha}<\widehat{\mathbf{m}}_{ij}} p_{\lambda \mathbf{m}_{ij}+\boldsymbol{\alpha}}(\mathbf{x}) w^{(\boldsymbol{\alpha})} + Q_{ij}[u](\mathbf{x}), \quad (3.49)$$

$$h_{lk}\big(w^{(\mathbf{m}^{l-1})}(\bullet, \widehat{\mathbf{x}}_l)\big) = 0 \quad \text{for } \widehat{\mathbf{x}}_l \in \widehat{\Omega}_l \quad (k = 1, \ldots, m_l;\ l \neq i, j), \quad (3.50)$$

where

$$Q_{ij}[u](\mathbf{x}) = \sum_{\boldsymbol{\alpha}<\mathbf{m}_{ij}} \sum_{\beta \leq \widehat{\mathbf{m}}_{ij}} p_{\lambda\, \boldsymbol{\alpha}+\boldsymbol{\beta}}(\mathbf{x}) u^{(\boldsymbol{\alpha}+\boldsymbol{\beta})} + q^{(\mathbf{m})}(\mathbf{x}).$$

Problem (3.49),(3.50) satisfies all of the conditions of Theorem 1.6. Since (3.49),(3.50) is an $n-2$ dimensional problem, by our assumption, it is α–well–posed. Therefore we have

$$u^{(\mathbf{m_{ij}})} = \mathcal{G}_{ij}\big(Q_{ij}[u]\big)(\mathbf{x}), \quad (3.51)$$

where $\mathcal{G}_{ij}$ is the Green's operator of the associated problem

$$w^{(\widehat{\mathbf{m}}_{ij})} = \sum_{\boldsymbol{\alpha}<\widehat{\mathbf{m}}_{ij}} p_{\lambda \mathbf{m}_{ij}+\alpha}(\mathbf{x}) w^{(\boldsymbol{\alpha})}, \quad (3.52)$$

$$h_{lk}\big(w^{(\mathbf{m}^{l-1})}(\bullet, \widehat{\mathbf{x}}_l)\big) = 0 \quad \text{for } \widehat{\mathbf{x}}_l \in \widehat{\Omega}_l \quad (k = 1, \ldots, m_l;\ l \neq i, j). \quad (3.53)$$

By Lemma 2.7, we get:

$$u^{(\mathbf{m_{ij}})}(\mathbf{x}) = q^{(\mathbf{m_{ij}})}(\mathbf{x}) + \sum_{\boldsymbol{\alpha}<\mathbf{m}_{ij}} p_{\lambda \widehat{\mathbf{m}}_{ij}+\boldsymbol{\alpha}}(\mathbf{x}) u^{(\boldsymbol{\alpha})} + \mathcal{A}^{\circ}_{ij}(u)(\mathbf{x}) + \mathcal{B}^{\circ}_{ij}(q)(\mathbf{x}), \quad (3.54)$$

where $\mathcal{A}^{\circ}_{ij}$ and $\mathcal{B}^{\circ}_{ij} : C^{\mathbf{m-1}}(\Omega) \to C^{\mathbf{m-1}}(\Omega)$ are bounded linear operators.

Continuing this process step-by-step, one can obtain similar representation

$$u^{(\mathbf{m}_{\boldsymbol{\sigma}})}(\mathbf{x}) = q^{(\mathbf{m}_{\boldsymbol{\sigma}})}(\mathbf{x}) + \sum_{\boldsymbol{\alpha}<\mathbf{m}_{\boldsymbol{\sigma}}} p_{\lambda \mathbf{m}_{\widehat{\boldsymbol{\sigma}}}+\boldsymbol{\alpha}}(\mathbf{x}) u^{(\boldsymbol{\alpha})} + \mathcal{A}^{\circ}_{\boldsymbol{\sigma}}(u)(\mathbf{x}) + \mathcal{B}^{\circ}_{\boldsymbol{\sigma}}(q)(\mathbf{x}) \quad (\boldsymbol{\sigma} \in \boldsymbol{\Xi}), \quad (3.55)$$

where $\mathcal{A}^{\circ}_{\boldsymbol{\sigma}}$ and $\mathcal{B}^{\circ}_{\boldsymbol{\sigma}} : C^{\mathbf{m-1}}(\Omega) \to C^{\mathbf{m-1}}(\Omega)$ $(\boldsymbol{\sigma} \in \boldsymbol{\Xi})$ are bounded linear operators.

(3.47), (3.54) and (3.55) yield the estimates

$$\|u^{(\mathbf{m}_{\boldsymbol{\sigma}}+\boldsymbol{\beta})}\|_{C(\Omega)} \le M\Big(\|q^{(\mathbf{m}_{\boldsymbol{\sigma}}+\boldsymbol{\beta})}\|_{C(\Omega)} + \|q\|_{C^{\mathbf{m-1}}(\Omega)} + \|u\|_{C^{\mathbf{m-1}}(\Omega)}\Big) \qquad \boldsymbol{\beta} \le \mathbf{m}_{\widehat{\boldsymbol{\sigma}}} - \widehat{\boldsymbol{\sigma}} \quad (\boldsymbol{\sigma} \in \boldsymbol{\Xi}). \quad (3.56)$$

and, hence,

$$\|u\|_{C^{\mathbf{m}}(\Omega)} \le M\Big(\|q\|_{C^{(\mathbf{m})}(\Omega)} + \|u\|_{C^{\mathbf{m-1}}(\Omega)}\Big), \qquad (3.57)$$

where M is a positive number independent of u and q.

Now we are ready to prove solvability of problem $(1.1), (1.2_0)$.

Problem $(1.23), (1.2_0)$ is well–posed for $\lambda = 0$. Therefore it is well–posed for $\lambda \in [0, \delta]$ for some $\delta \le 1$. Our goal is to prove that $\delta = 1$. Assume the contrary. Then there exists $\lambda_0 \le 1$ and a sequence $\lambda_k \nearrow \lambda_0$ such that

$$\lim_{k \to +\infty} \|u_k\|_{C^{\mathbf{m}}(\Omega)} = +\infty, \qquad (3.58)$$

where u_k is a solution of problem $(1.23), (1.2_0)$ for $\lambda = \lambda_k$. Set:

$$\eta_k = \|u_k\|_{C^{\mathbf{m}}(\Omega)}, \qquad \widetilde{u}_k(\mathbf{x}) = \frac{u_k(\mathbf{x})}{\eta_k}. \qquad (3.59)$$

Then

$$\|\widetilde{u}_k\|_{C^{\mathbf{m}}(\Omega)} = 1 \quad (k = 1, 2, \dots) \qquad (3.60)$$

and $\widetilde{u_k}$ is a solution of the problem

$$u^{(\mathbf{m})} = \sum_{\boldsymbol{\alpha}<\mathbf{m}} p_{\lambda\,\boldsymbol{\alpha}}(\mathbf{x}) u^{(\boldsymbol{\alpha})} + \frac{q(\mathbf{x})}{\eta_k}, \qquad (3.61)$$

$$h_{ik}\big(u^{(\mathbf{m}^{i-1})}(\bullet, \widehat{\mathbf{x}}_i)\big) = \frac{\varphi_{ik}^{(\mathbf{m}^{i-1})}(\widehat{\mathbf{x}}_i)}{\eta_k} \quad \text{for } \widehat{\mathbf{x}}_i \in \widehat{\Omega}_i \ \ (k = 1, \dots, m_i; \ \ i = 1, \dots, n). \quad (3.62)$$

By virtue of (3.57),(3.60) and Arzela–Ascoli lemma, without loss of generality, one may assume that there exists $\widetilde{u} \in C^{\mathbf{m}}(\omega)$ such that

$$\lim_{k\to+\infty} \|\widetilde{u} - \widetilde{u}_k\|_{C^{\mathbf{m}}(\Omega)} = 0 \tag{3.63}$$

and

$$\|\widetilde{u}\|_{C^{\mathbf{m}}(\Omega)} = 1. \tag{3.64}$$

On the other hand, it is clear that $\widetilde{u}$ is a nonzero solution of problem $(1.23_0), (1.2_0)$ for $\lambda = \lambda_0$, which contradicts to the assumption that the homogenous problem $(1.23_0), (1.2_0)$ has only the trivial solution for every $\lambda \in [0, 1]$. The obtained contradiction proves the solvability of problem (1.1),(1.2) under the assumption that the homogeneous problem $(1.1_0), (1.2_0)$ has only the trivial solution.

At the same time, along with solvability of problem $(1.1), (1.2_0)$ we have also proved its α–well–posedness, by establishing the estimates (3.56) and (3.57).

In order to complete the proof, it remains to show that problem $(1.1_0), (1.2_0)$ has a finite dimensional space of solution. This follows from the fact, that every solution of problem $(1.1_0), (1.2_0)$ is also a solution of the equation

$$u(\mathbf{x}) = \mathcal{A}(u)(\mathbf{x}), \tag{3.65}$$

where $\mathcal{A} : C^{\mathbf{m-1}}(\Omega) \to C^{\mathbf{m-1}}(\Omega)$ is the compact operator defined by (3.14). $\square$

Proof of Theorem 1.7. First let us show that the homogeneous problem

$$u^{(2\mathbf{m})} = \sum_{\boldsymbol{\alpha}+\boldsymbol{\beta}<2\mathbf{m}} \big(p_{\boldsymbol{\alpha}+\boldsymbol{\beta}}(\mathbf{x}) u^{(\boldsymbol{\alpha})}\big)^{(\boldsymbol{\beta})}, \tag{3.66}$$

$$\begin{aligned} &u^{(2\mathbf{m}^{i-1}+(k-1)\mathbf{1}_i)}(0, \widehat{\mathbf{x}}_{\mathbf{i}}) = 0 \quad (k = 1, \dots, m_i;\ i = 1, \dots, n),\\ &u^{(2\mathbf{m}^{i-1}+(k-1)\mathbf{1}_i)}(\omega_i, \widehat{\mathbf{x}}_{\mathbf{i}}) = 0 \quad (k = 1, \dots, m_i;\ i = 1, \dots, n), \end{aligned} \tag{3.67}$$

has only the trivial solution. Indeed, let u be an arbitrary solution of problem (3.66),(3.67). Multiply equation (3.66) by u and integrate over Ω. After integrating by parts multiple times and taking into account conditions (3.67), we get:

$$(-1)^{\|\mathbf{m}\|} \iint\limits_{\Omega} \big|u^{(\mathbf{m})}(\mathbf{x})\big|^2 \, d\mathbf{x} = \iint\limits_{\Omega} \sum_{\boldsymbol{\alpha}+\boldsymbol{\beta}<2\mathbf{m}} (-1)^{\|\boldsymbol{\beta}\|} p_{\boldsymbol{\alpha}+\boldsymbol{\beta}}(\mathbf{x}) u^{(\boldsymbol{\alpha})}(\mathbf{x}) u^{(\boldsymbol{\alpha})}(\mathbf{x}) \, d\mathbf{x}. \tag{3.68}$$

From (1.27) and (3.68) we get

$$u^{(\mathbf{m})}(\mathbf{x}) = 0. \tag{3.69}$$

(3.69), along with (3.67), immediately implies $u(\mathbf{x}) \equiv 0$.

Now consider the equation

$$u^{(2\mathbf{m})} = \sum_{\boldsymbol{\alpha}+\boldsymbol{\beta}<2\mathbf{m}} \lambda\big(p_{\boldsymbol{\alpha}+\boldsymbol{\beta}}(\mathbf{x})u^{(\boldsymbol{\alpha})}\big)^{(\boldsymbol{\beta})} + q(\mathbf{x}) \tag{3.70}$$

By Theorem 1.5 (or the remark in the beginning of the proof of Theorem 1.5), problem (3.70),(3.67) is well–posed for $\lambda = 0$. Also, it is obvious, that for every $\lambda \in [0, 1]$ the homogeneous equation

$$u^{(2\mathbf{m})} = \sum_{\boldsymbol{\alpha}+\boldsymbol{\beta}<2\mathbf{m}} \lambda\big(p_{\boldsymbol{\alpha}+\boldsymbol{\beta}}(\mathbf{x})u^{(\boldsymbol{\alpha})}\big)^{(\boldsymbol{\beta})} \tag{3.71}$$

has only the trivial solution satisfying conditions (3.67). Consequently, by Theorem 1.6, problem (1.24),(1.25) is α–well–posed. □

Proof of Theorem 1.8. The proof is almost identical to the proof of Theorem 1.7.

Proof of Theorem 1.9. The proof is similar to the proof of Theorem 1.7. One needs to notice that according to (1.28), the coefficient $p_{\mathbf{m}}(\mathbf{x})$ is nonpositive. Therefore equation (1.28) can be written in the form of equation (1.1) since the leading coefficient, i.e. the coefficient of the term $u^{(2\mathbf{m})}$ is

$$1 - p_{\mathbf{m}}(\mathbf{x}) \geq 1. \quad \square$$

Proof of Theorem 1.10. The proof is almost identical to the to the proof of Theorem 1.9.

Proof of Theorem 1.11. The proof is almost identical to the to the proof of Theorem 1.9.

Proof of Theorem 1.12. The proof is almost identical to the to the proof of Theorem 1.9.

Proof of Theorem 1.13. First let us show that the homogeneous problem

$$u^{(2\mathbf{m})} = \sum_{\boldsymbol{\alpha}+\boldsymbol{\beta}<2\mathbf{m}} \big(p_{\boldsymbol{\alpha}+\boldsymbol{\beta}}(\mathbf{x})u^{(\boldsymbol{\alpha})}\big)^{(\boldsymbol{\beta})}, \tag{3.72}$$

$$u^{(2\mathbf{m}^{i-1}+(k-1)\mathbf{1}_i)}(0,\widehat{\mathbf{x}}_\mathbf{i}) - a_{ik}u^{(2\mathbf{m}^{i-1}+(k-1)\mathbf{1}_i)}(\omega_i,\widehat{\mathbf{x}}_\mathbf{i})$$
$$= 0 \quad (k=1,\dots,2m_i;\ i=1,\dots,n), \tag{3.73}$$

has only the trivial solution. Indeed, let u be an arbitrary solution of problem (3.72), (3.73). Multiply equation (3.72) by u and integrate over Ω. After integrating by parts multiple times and taking into account conditions (3.73) and (1.33), we get:

$$(-1)^{\|\mathbf{m}\|} \iint\limits_{\Omega} \big|u^{(\mathbf{m})}(\mathbf{x})\big|^2\, d\mathbf{x} = \iint\limits_{\Omega} \sum_{\boldsymbol{\alpha}+\boldsymbol{\beta}<2\mathbf{m}} (-1)^{\|\boldsymbol{\beta}\|} p_{\boldsymbol{\alpha}+\boldsymbol{\beta}}(\mathbf{x})u^{(\boldsymbol{\alpha})}(\mathbf{x})u^{(\boldsymbol{\alpha})}(\mathbf{x})\, d\mathbf{x}. \tag{3.74}$$

From (1.27) and (3.74) we get

$$u^{(\mathbf{m})}(\mathbf{x}) = 0. \tag{3.75}$$

(3.75), along with (3.73) and (1.32), implies $u(\mathbf{x}) \equiv 0$.

Now consider the equation

$$u^{(2\mathbf{m})} = \sum_{\boldsymbol{\alpha}+\boldsymbol{\beta}<2\mathbf{m}} \lambda\big(p_{\boldsymbol{\alpha}+\boldsymbol{\beta}}(\mathbf{x})u^{(\boldsymbol{\alpha})}\big)^{(\boldsymbol{\beta})} + q(\mathbf{x}) \tag{3.76}$$

In view of conditions (1.32) and (1.33), by Theorem 1.5 (or the remark in the beginning of the proof of Theorem 1.5), problem (3.76),(3.73) is well–posed for $\lambda = 0$. Also, it is obvious, that for every $\lambda \in [0,1]$ the homogeneous equation

$$u^{(2\mathbf{m})} = \sum_{\boldsymbol{\alpha}+\boldsymbol{\beta}<2\mathbf{m}} \lambda\big(p_{\boldsymbol{\alpha}+\boldsymbol{\beta}}(\mathbf{x})u^{(\boldsymbol{\alpha})}\big)^{(\boldsymbol{\beta})} \tag{3.77}$$

has only the trivial solution satisfying conditions (3.73). Consequently, by Theorem 1.6, problem (1.24),(1.31) is α–well–posed. □

Proof of Theorem 1.14. The proof of the theorem is similar to the proof of Theorem 1.9.

Proof of Theorem 1.15. The proof of the theorem is similar to the proof of Theorem 1.9.

Proof of Theorem 1.16. It is clear that it is enough to prove the theorem in the particular case of initial–boundary conditions (1.36) with $l = n-1$ and homogeneous initial–boundary values, i.e. the initial boundary conditions:

$$\begin{gathered} h_{ik}\big(u^{(\mathbf{m}^{i-1})}(\bullet, \widehat{\mathbf{x}}_i)\big) = 0) \;\; (k = 1, \ldots, m_i; \;\; i = 1, \ldots, n-1), \\ u^{(\mathbf{m}^{n-1} + k\mathbf{1}_n)}\big(\widehat{\mathbf{x}}_n, 0\big) = 0 \;\; (k = 1, \ldots, m_n). \end{gathered} \tag{3.78}$$

We prove the theorem by induction. The validity of Theorem 1.16 for $n = 2$ was already proved (see Theorem 1.4). Let $n \geq 3$, and let us assume that the theorem is true for $n-1$ dimensional problem. Then every level $l \leq n-1$ associated problem of problem (1.1),(3.78) is well–posed.

Let u be a solution of problem (1.1),(3.78). Then u admits the following representations:

$$u^{(\mathbf{m_j})}(\mathbf{x}) = \mathcal{G}_j(Q_j[u])(\mathbf{x}) \qquad (j = 1, \ldots, n), \tag{3.79}$$

where

$$Q_j[u](\mathbf{x}) \;=\; \sum_{\boldsymbol{\alpha} < \mathbf{m}_j} \sum_{\boldsymbol{\beta} \leq \widehat{\mathbf{m}}_j} p_{\boldsymbol{\alpha}+\boldsymbol{\beta}}(\mathbf{x}) u^{(\boldsymbol{\alpha}+\boldsymbol{\beta})} \;+\; q(\mathbf{x}) \quad (j \;=\; 1, \ldots, n \;-\; 1), \tag{3.80}$$

$\mathcal{G}_j$ $(j = 1, \ldots, n-1)$ is the Green's operator of the associated problem

$$v^{(\widehat{\mathbf{m}}_j)} = \sum_{\boldsymbol{\alpha} < \widehat{\mathbf{m}}_j} p_{\mathbf{m}_j + \boldsymbol{\alpha}}(\mathbf{x}) v^{(\boldsymbol{\alpha})}, \tag{3.81}$$

$$\begin{gathered} h_{ik}\big(v^{(\mathbf{m}^{i-1})}(\bullet, \widehat{\mathbf{x}}_i)\big) = 0) \;\; (k = 1, \ldots, m_i, \; i \neq j), \\ u^{(\mathbf{m}^{n-1} + k\mathbf{1}_n)}\big(\widehat{\mathbf{x}}_n, 0\big) = 0 \;\; (k = 1, \ldots, m_n), \end{gathered} \tag{3.82}$$

and $\mathcal{G}_n$ is the Green's operator of the associated problem

$$v^{(\widehat{\mathbf{m}}_n)} = \sum_{\boldsymbol{\alpha} < \widehat{\mathbf{m}}_n} p_{\mathbf{m}_n + \boldsymbol{\alpha}}(\mathbf{x}) v^{(\boldsymbol{\alpha})}, \tag{3.83}$$

$$h_{ik}\big(v^{(\mathbf{m}^{i-1})}(\bullet, \widehat{\mathbf{x}}_i)\big) = 0 \;\; (k = 1, \ldots, m_i, \; i = 1, \ldots, n-1) \tag{3.84}$$

Notice that the Green's operator $\mathcal{G}_j$ $(j = 1, \ldots, n-1)$ are of Volterra type with respect to x_n variable, i.e.

$$\mathcal{G}_j(z)(\mathbf{x}) = \int_0^{\omega_1} \ldots \int_0^{\omega_{j-1}} \int_0^{\omega_{j+1}} \ldots \int_0^{x_n} G(\widehat{\mathbf{x}}_{\mathbf{j}}, \widehat{\mathbf{s}}_j; x_j)\, z(x_j, \widehat{\mathbf{s}}_j)\, d\widehat{\mathbf{s}}_j \quad (j = 1, \ldots, n-1), \tag{3.85}$$

and that

$$Q_n[u](\mathbf{x}) = \sum_{\boldsymbol{\alpha}<\mathbf{m}_n} \sum_{\boldsymbol{\beta}\le\widehat{\mathbf{m}}_n} p_{\boldsymbol{\alpha}+\boldsymbol{\beta}}(\mathbf{x})u^{(\boldsymbol{\alpha}+\boldsymbol{\beta})} + q(\mathbf{x})$$

contains terms $u^{(\boldsymbol{\alpha})}$ such that $\alpha_n < m_n$.

Continuing this process step-by-step, one can reduce problem (1.1),(3.78) to the equivalent system of integral equations that are Volterra type with respect to x_n variable.

Unique solvability of such systems can be proved by means of Picard's successive approximations method. □

Proof of Theorem 1.17. Sufficiency was already proved in Theorem 1.16. Let us prove the necessity. Let problem (1.1),(1.36) be well–posed. Then for arbitrary $c_k \in \mathbb{R}$ $(k = 1, \dots, m_1)$ the problem

$$u^{(\mathbf{m})} = \sum_{\boldsymbol{\alpha}<\mathbf{m}} p_{\boldsymbol{\alpha}}(\mathbf{x})u^{(\boldsymbol{\alpha})}, \tag{3.86}$$

$$\begin{gathered} h_{1k}\big(u(\bullet, \widehat{\mathbf{x}}_1)\big) = c_k \frac{x_2^{m_2-1} \dots x_n^{m_n-1}}{(m_2-1)! \cdots (m_n-1)!} \ \text{ for } \widehat{\mathbf{x}}_1 \in \Omega_1 \ (k = 1, \dots, m_1), \\ u^{(\mathbf{m}^{i-1}+k\mathbf{1}_i)}\big(0, \widehat{\mathbf{x}}_i\big) = 0 \ \ (k = 1, \dots, m_i; \ \ i = 2, \dots, n) \end{gathered} \tag{3.87}$$

has a unique solution u. It is clear that the function $z(x_1) = u^{\widehat{\mathbf{m}}_1}(x_1, 0, \dots, 0)$ is a solution of problem

$$z^{(m_1)} = \sum_{k=0}^{m_1-1} p_{k\mathbf{1}_1+\widehat{\mathbf{m}}_1}(\mathbf{x})z^{(k)}, \tag{3.88}$$

$$h_{1k}(z) = c_k \quad (k = 1, \dots, m_1) \tag{3.89}$$

for $\widehat{\mathbf{x}}_1 = \mathbf{0}$. hence, by Lemma 2.1, it follows that problem (1.40),(1.41) has only the trivial solution for $\widehat{\mathbf{x}}_1 = \mathbf{0}$.

Now let $\widehat{\mathbf{x}}_1^0 = (0, x_2^0, \dots, x_n^0) \in \widehat{\Omega}_1$ be an arbitrary point such that $x_i^0 > 0$ $(i = 2, \dots, n)$. Set:

$$\delta_l(x_i) = \begin{cases} l^2\left(\frac{x_i}{x_i^0} - \frac{l-1}{l}\right)\left(\frac{l+1}{l} - \frac{x_i}{x_i^0}\right) & \text{for } \frac{x_i}{x_i^0} \in \left[\frac{l-1}{l}, \frac{l+1}{l}\right] \\ 0 \quad \text{for } \frac{x_i}{x_i^0} \notin \left[\frac{l-1}{l}, \frac{l+1}{l}\right] \end{cases}, \tag{3.90}$$

$$\delta(\widehat{\mathbf{x}}_1) = \int_0^{x_2} \cdots \int_0^{x_n} \frac{(x_2 - s_2)^{m_2-1} \dots (x_n - s_n)^{m_n-1}}{(m_2-1)! \cdots (m_n-1)!} \delta_l(s_2) \cdots \delta_l(s_n)\, ds_n \dots ds_2. \tag{3.91}$$

For the equation (3.78) consider the initial–boundary conditions

$$h_{1k}\big(u(\bullet, \widehat{\mathbf{x}}_1)\big) = c_k \delta(\widehat{\mathbf{x}}_1) \ \text{ for } \widehat{\mathbf{x}}_1 \in \Omega_1 \ (k = 1, \dots, m_1),$$
$$u^{(\mathbf{m}^{i-1} + k\mathbf{1}_i)}\big(0, \widehat{\mathbf{x}}_i\big) = 0 \ \ (k = 1, \dots, m_i; \ \ i = 2, \dots, n). \tag{3.92}$$

In view of wellposedness of problem (1.1),(1.36), problem (3.86),(3.92) has a unique solution u_l. Furthermore

$$u_l(x) = 0 \quad \text{for} \quad x_i \le \frac{l-1}{l} x_i^0 \ \ (i = 2, \dots, n). \tag{3.93}$$

Let $z_l(x_1) = u^{(\widehat{\mathbf{m}}_1)}(x_1, \widehat{\mathbf{x}}_1^0)$. Then it is clear that z_l is a solution of the problem

$$z^{(m_1)} = \sum_{k=0}^{m_1-1} p_{k\mathbf{1}_1 + \widehat{\mathbf{m}}_1}(x_1, \widehat{\mathbf{x}}_1^0) z^{(k)} + q_l(x_1), \tag{3.94}$$

$$h_{1k}(z) = c_k \quad (k = 1, \dots, m_1), \tag{3.95}$$

where

$$q_l(x_1) = \sum_{\boldsymbol{\alpha} < \mathbf{m}_1} \sum_{\boldsymbol{\beta} \le \widehat{\mathbf{m}}_1} p_{\boldsymbol{\alpha}+\boldsymbol{\beta}}(\mathbf{x}) u^{(\boldsymbol{\alpha}+\boldsymbol{\beta})}(x_1, \widehat{\mathbf{x}}_1^0). \tag{3.96}$$

By ArzelaAscoli lemma, without loss of generality, one can assume that there

$$\lim_{l \to +\infty} \|q_l\|_{C([0,\omega_1])} = 0, \tag{3.97}$$

$$\lim_{l \to +\infty} \|z_l - z\|_{C([0,\omega_1])} = 0 \tag{3.98}$$

for some $z \in C([0, \omega_1])$. It is easy to show that in fact $z \in C^{m_1}([0, \omega_1])$ and it is a solution of the problem

$$z^{(m_1)} = \sum_{k=0}^{m_1-1} p_{k\mathbf{1}_1 + \widehat{\mathbf{m}}_1}(x_1, \widehat{\mathbf{x}}_1^0) z^{(k)} \tag{3.99}$$

$$h_{1k}(z) = c_k \quad (k = 1, \dots, m_1). \tag{3.100}$$

Hence, by Lemma 2.1, problem (1.40),(1.41) has only the trivial solution for arbitrary $\widehat{\mathbf{x}}_1^0 \in \widehat{\Omega}_1$. $\square$

Proof of Theorem 1.18. The proof of Theorem 1.18 is similar to the proof of Theorem 1.16.

CHAPTER II
Problems on Periodic Solutions

4. Formulation of the Main Results

As we have already noted, the periodic boundary conditions

$$u^{(2\mathbf{m}^{i-1}+(k-1)\mathbf{1}_i)}(0,\widehat{\mathbf{x}}_\mathbf{i}) - u^{(2\mathbf{m}^{i-1}+(k-1)\mathbf{1}_i)}(\omega_i,\widehat{\mathbf{x}}_\mathbf{i})$$
$$= \varphi_{ik}^{(2\mathbf{m}^{i-1})}(\widehat{\mathbf{x}}_\mathbf{i}) \quad (k=1,\dots,2m_i;\ i=1,\dots,n)$$

do not satisfy the conditions of Theorem 1.1, since problem (1.7) with periodic boundary conditions has nontrivial solutions for every $i \in \{1,\dots,n-1\}$. Consequently, problem (1.1),(1.2) nonhomogeneous periodic conditions is *not* well–posed in the sense of Definition 1.2.

Therefore it makes sense to study periodic problem with the homogeneous boundary conditions only. Furthermore, instead of studying periodic problems in the domain Ω, it is more convenient to study problems on periodic solutions in $\mathbb{R}^n$:

$$u^{(\mathbf{m})} = \sum_{\boldsymbol{\alpha}<\mathbf{m}} p_{\boldsymbol{\alpha}}(\mathbf{x}) u^{(\boldsymbol{\alpha})} + q(\mathbf{x}), \tag{4.1}$$

$$u(\mathbf{x}+\boldsymbol{\omega}_i) = u(\mathbf{x}) \quad (i=1,\dots,n), \tag{4.2}$$

where $\boldsymbol{\omega}_\mathbf{i} = (0,\dots,\omega_i,\dots,0)$, $\boldsymbol{\omega} = (\omega_1,\dots,\omega_n)$, $p_{\boldsymbol{\alpha}} \in C_{\boldsymbol{\omega}}(\mathbb{R}^n)$ $(\boldsymbol{\alpha} < \mathbf{m})$, $q \in C_{\boldsymbol{\omega}}(\mathbb{R}^n)$, and $C_{\boldsymbol{\omega}}(\mathbb{R}^n)$ is the space of $\boldsymbol{\omega}$-periodic continuous functions, i.e. continuous functions that are ω_i-periodic with respect to the variable x_i $(i=1,\dots,n)$.

Along with equation (4.1) consider its corresponding homogeneous equation

$$u^{(\mathbf{m})} = \sum_{\boldsymbol{\alpha}<\mathbf{m}} p_{\boldsymbol{\alpha}}(x) u^{(\boldsymbol{\alpha})}, \tag{4.1$_0$}$$

the $\|\boldsymbol{\sigma}\|$–dimensional homogeneous $\boldsymbol{\omega}_{\boldsymbol{\sigma}}$–periodic problem depending on the parameter $\mathbf{x}_{\widehat{\boldsymbol{\sigma}}} \in \Omega_{\widehat{\boldsymbol{\sigma}}}$

$$v^{(\mathbf{m}_{\boldsymbol{\sigma}})} = \sum_{\boldsymbol{\alpha}<\mathbf{m}_{\boldsymbol{\sigma}}} p_{\boldsymbol{\alpha}+\widehat{\mathbf{m}}_{\boldsymbol{\sigma}}}(\mathbf{x}) v^{(\boldsymbol{\alpha})}, \tag{4.1$_{\boldsymbol{\sigma}}$}$$

$$v(\mathbf{x}+\boldsymbol{\omega}_i) = v(\mathbf{x}) \quad (i \in \operatorname{supp}\boldsymbol{\sigma}). \tag{4.2$_{\boldsymbol{\sigma}}$}$$

Definition 4.1. Let $\boldsymbol{\sigma} \in \Xi \cup \{\mathbf{1}\}$. Then problem $(4.1_{\boldsymbol{\sigma}}), (4.2_{\boldsymbol{\sigma}})$ is called $\boldsymbol{\sigma}$–associated problem, or associate problem of level $l = \|\sigma\|$.

Associated problems of the level $n-1$ can be written in the relatively simpler form

$$v^{(\widehat{\mathbf{m}}_j)} = \sum_{\alpha < \widehat{\mathbf{m}}_j} p_{\mathbf{m}_j + \alpha}(\mathbf{x}) v^{(\alpha)}, \tag{4.1$_j$}$$

$$v(\mathbf{x} + \boldsymbol{\omega}_i) = v(\mathbf{x}) \quad (i \neq j). \tag{4.2$_j$}$$

We have modified the definitions of well–posedness and α–well–posedness, adapting them to problem (4.1),(4.2).

Definition 4.2. The periodic problem $(4.1), (4.2)$ is called well-posed, if it is uniquely solvable for arbitrary $q \in C_{\boldsymbol{\omega}}(\Omega)$, and its solution u admits the estimate

$$\|u\|_{C^{\mathbf{m}}_{\boldsymbol{\omega}}} \leq M \|q\|_{C_{\boldsymbol{\omega}}},$$

where M is a positive constant independent of q.

Consider the equation

$$u^{(\mathbf{m})} = \sum_{\boldsymbol{\alpha} < \mathbf{m}} p_{\boldsymbol{\alpha}}(\mathbf{x}) u^{(\boldsymbol{\alpha})} + q^{(\mathbf{m})}(\mathbf{x}). \tag{4.3}$$

Definition 4.3. We say that problem $(4.1), (4.2)$ is called α–well–posed, if it is well-posed and the solution u of problem $(4.3), (4.2)$ admits the estimate

$$\|u^{(\boldsymbol{\alpha})} - q^{(\boldsymbol{\alpha})}\|_{C_{\boldsymbol{\omega}}} \leq M \|q\|_{C_{\boldsymbol{\omega}}} \quad (\boldsymbol{\alpha} \leq \mathbf{m}), \tag{4.4}$$

where M is a positive constant independent of q.

4.1. **case** $n = 2$**.** .

Theorem 4.1. *Let $n = 2$, and let problem $(4.1), (4.2)$ be well–posed. Then each associated problem $(4.1_j), (4.2_j)$ has only the trivial solution for every $x_j \in [0, \omega_j]$ $(j = 1, 2)$.*

Theorem 4.2. *Let $n = 2$, and let each associated problem $(4.1_j), (4.2_j)$ have only the trivial solution for every $x_j \in [0, \omega_j]$ $(j = 1, 2)$. Then problem $(4.1), (4.2)$ has the Fredholm property. Moreover, if problem $(4.1_0), (4.2)$ has only the trivial solution, then problem $(4.1), (4.2)$ is well–posed. Furthermore, if $p_{\boldsymbol{\alpha}} \in C^{\boldsymbol{\alpha}}_{\boldsymbol{\omega}}(\mathbb{R}^n)$ $(\boldsymbol{\alpha} < \boldsymbol{m})$, then problem $(4.1), (4.2)$ is α–well–posed.*

4.2. **case $n \geq 2$.** Set:

$$\mathcal{L}_{\mathbf{m}} v = v^{(\mathbf{m})} - \sum_{\boldsymbol{\alpha}<\mathbf{m}} p_{\boldsymbol{\alpha}}(\mathbf{x}) v^{(\boldsymbol{\alpha})}, \tag{4.5}$$

$$\mathcal{L}_{\boldsymbol{\sigma}} v = v^{(\mathbf{m}_{\boldsymbol{\sigma}})} - \sum_{\boldsymbol{\alpha}<\mathbf{m}_{\boldsymbol{\sigma}}} p_{\boldsymbol{\alpha}+\widehat{\mathbf{m}}_{\boldsymbol{\sigma}}}(\mathbf{x}) v^{(\boldsymbol{\alpha})}. \tag{4.6}$$

Theorem 4.3. *Let for some $\boldsymbol{\sigma} \in \boldsymbol{\Xi}$ $\boldsymbol{\sigma}$ and $\widehat{\boldsymbol{\sigma}}$–associated problems be well–posed. Moreover, let $p_{\boldsymbol{\alpha}+\mathbf{m}_{\boldsymbol{\sigma}}}(\mathbf{x}) = p_{\boldsymbol{\alpha}+\mathbf{m}_{\boldsymbol{\sigma}}}(\mathbf{x}_{\widehat{\boldsymbol{\sigma}}})$ $(\boldsymbol{\alpha} < \mathbf{m}_{\widehat{\boldsymbol{\sigma}}})$, and*

$$\mathcal{L}_{\mathbf{m}}\, v = \mathcal{L}_{\boldsymbol{\sigma}} \circ \mathcal{L}_{\widehat{\boldsymbol{\sigma}}}\, v + \sum_{\boldsymbol{\alpha}\leq \boldsymbol{m}-\mathbf{1}} p_{\boldsymbol{\alpha}}(\mathbf{x}) v^{(\boldsymbol{\alpha})}. \tag{4.7}$$

Then problem $(4.1), (4.2)$ has the Fredholm property.

Along with the equation (4.1_0) consider the equation

$$u^{(\mathbf{m})} = \sum_{\boldsymbol{\alpha}<\mathbf{m}} p_{\lambda\,\boldsymbol{\alpha}}(\mathbf{x}) u^{(\boldsymbol{\alpha})} + q(\mathbf{x}), \tag{4.8}$$

where $\lambda \in [0, 1]$, $p_{\lambda\,\boldsymbol{\alpha}}(\mathbf{x}) = (1-\lambda)p_{0\,\boldsymbol{\alpha}}(\mathbf{x}) + \lambda p_{\boldsymbol{\alpha}}(\mathbf{x})$, $p_{0\,\boldsymbol{\alpha}}(\mathbf{x})$ $C(\Omega)$ $(\boldsymbol{\alpha} < \boldsymbol{m})$, and also its corresponding homogeneous and associated equations

$$u^{(\mathbf{m})} = \sum_{\boldsymbol{\alpha}<\mathbf{m}} p_{\lambda\,\boldsymbol{\alpha}}(\mathbf{x}) u^{(\boldsymbol{\alpha})} \tag{4.8_0}$$

and

$$v^{(\mathbf{m}_{\boldsymbol{\sigma}})} = \sum_{\boldsymbol{\alpha}<\mathbf{m}_{\boldsymbol{\sigma}}} p_{\lambda\,\boldsymbol{\alpha}+\mathbf{m}_{\widehat{\boldsymbol{\sigma}}}}(\mathbf{x}) v^{(\boldsymbol{\alpha})},. \tag{$4.8_{\boldsymbol{\sigma}}$}$$

Theorem 4.4. *Let $p_{0\boldsymbol{\alpha}}$, $p_{\boldsymbol{\alpha}} \in C^{\mathbf{1}}_{\boldsymbol{\omega}}(\Omega)$ $(\boldsymbol{\alpha} \in \boldsymbol{\Upsilon}_{\mathbf{m}})$, and let:*

(A_1) *each $\boldsymbol{\sigma}$–associated problem $(4.8_{\boldsymbol{\sigma}}), (4.2_{\boldsymbol{\sigma}})$ be well–posed for $\lambda = 0$;*

(A_2) *problem $(4.8), (4.2)$ be well–posed for $\lambda = 0$;*

(A_3) *each* $\boldsymbol{\sigma}$*–associated problem* $(4.8_{\boldsymbol{\sigma}}), (4.2_{\boldsymbol{\sigma}})$ *have only the trivial solution for every* $\mathbf{x}_{\widehat{\boldsymbol{\sigma}}} \in \Omega_{\widehat{\boldsymbol{\sigma}}}$ $(\boldsymbol{\sigma} \in \boldsymbol{\Xi})$ *and every* $\lambda \in [0,1]$*;*

(A_4) *problem* $(4.8_0), (4.2)$ *have only the trivial solution for every* $\lambda \in [0,1)$.

Then problem $(4.1), (4.2)$ *has the Fredholm property. Moreover, if problem* $(1.1_0), (1.2_0)$ *has only the trivial solution, then problem* $(1.1), (1.2)$ *is well–posed, and if* $p_{\boldsymbol{\alpha}} \in C^{\boldsymbol{\alpha}}_{\boldsymbol{\omega}}(\Omega)$ $(\boldsymbol{\alpha} < \boldsymbol{m})$*, then problem* $(4.1), (4.2)$ *is* α*–well–posed.*

Consider the equations of even and odd order:

$$u^{(2\mathbf{m})} = \sum_{\boldsymbol{\alpha}+\boldsymbol{\beta}<2\mathbf{m}} \big(p_{\boldsymbol{\alpha}+\boldsymbol{\beta}}(\mathbf{x}) u^{(\boldsymbol{\alpha})}\big)^{(\boldsymbol{\beta})} + q(\mathbf{x}), \tag{4.9}$$

$$u^{(2\mathbf{m})} = \sum_{\boldsymbol{\alpha}\le\mathbf{m}} \big(p_{\boldsymbol{\alpha}}(\mathbf{x}) u^{(\boldsymbol{\alpha})}\big)^{(\boldsymbol{\alpha})} + q(\mathbf{x}), \tag{4.10}$$

and

$$u^{(2\mathbf{m}+\mathbf{1}_n)} = \sum_{\boldsymbol{\alpha}+\boldsymbol{\beta}<2\mathbf{m}} \big(p_{\boldsymbol{\alpha}+\boldsymbol{\beta}+\mathbf{1}_n}(\mathbf{x}) u^{(\boldsymbol{\alpha}+1_n)}\big)^{(\boldsymbol{\beta})} + \sum_{\boldsymbol{\alpha}\le\mathbf{m}} p_{2\boldsymbol{\alpha}}(\widehat{\boldsymbol{x}}_{\boldsymbol{\alpha}}) u^{(2\boldsymbol{\alpha})} + q(\mathbf{x}) \tag{4.11}$$

Notice that due to the term $\big(p_{\boldsymbol{m}}(\mathbf{x})u^{(\boldsymbol{m})}\big)^{(\boldsymbol{m})}$, equation (4.10) is not a particular case of equation (4.9).

Theorem 4.5. *Let* $p_{\boldsymbol{\alpha}+\boldsymbol{\beta}} \in C^{\boldsymbol{\beta}}_{\boldsymbol{\omega}}(\mathbb{R}^n)$ $(\boldsymbol{\alpha}+\boldsymbol{\beta} < 2\boldsymbol{m})$*, and let there exist* $\delta > 0$ *such that*

$$\sum_{\boldsymbol{\alpha}+\boldsymbol{\beta}<2\mathbf{m}} (-1)^{\|\boldsymbol{m}\|+\|\boldsymbol{\beta}\|-1} p_{\boldsymbol{\alpha}+\boldsymbol{\beta}}(\mathbf{x}) z_{\boldsymbol{\alpha}} z_{\boldsymbol{\beta}} \ge \delta \sum_{\boldsymbol{\alpha}<2\mathbf{m}} z^2_{\boldsymbol{\alpha}} > 0 \quad \text{for} \quad \mathbf{x} \in \Omega. \tag{4.12}$$

Then problem $(4.9), (4.2)$ *is well–posed. Moreover, if* $p_{\boldsymbol{\alpha}+\boldsymbol{\beta}} \in C^{\boldsymbol{\alpha}+\boldsymbol{\beta}}_{\boldsymbol{\omega}}(\mathbb{R}^n)$ $(\boldsymbol{\alpha}+\boldsymbol{\beta} < 2\boldsymbol{m})$*, then problem* $(4.9), (4.2)$ *is* α*–well–posed.*

Theorem 4.6. *Let* $p_{\boldsymbol{\alpha}} \in C^{\boldsymbol{\alpha}}_{\boldsymbol{\omega}}(\mathbb{R}^n)$ $(\boldsymbol{\alpha} \le \boldsymbol{m})$*, and let the inequalities hold:*

$$(-1)^{\|\boldsymbol{m}\|+\|\boldsymbol{\alpha}\|-1} p_{\boldsymbol{\alpha}}(\mathbf{x}) \ge \delta > 0 \quad \text{for} \quad \mathbf{x} \in \Omega \quad (\boldsymbol{\alpha} \le \boldsymbol{m}). \tag{4.13}$$

Then problem $(4.10), (4.2)$ *is well–posed. Moreover, if* $p_{\boldsymbol{\alpha}} \in C^{2\boldsymbol{\alpha}}_{\boldsymbol{\omega}}(\mathbb{R}^n)$ $(\boldsymbol{\alpha} \le \boldsymbol{m})$*, then problem* $(4.10), (4.2)$ *is* α*–well–posed.*

Theorem 4.7. *Let* $p_{\boldsymbol{\alpha}+\boldsymbol{\beta}+\mathbf{1}_n} \in C^{\boldsymbol{\beta}}_{\boldsymbol{\omega}}(\mathbb{R}^n)$ $(\boldsymbol{\alpha}+\boldsymbol{\beta} < 2\boldsymbol{m})$, *and let there exist* $\delta > 0$ *and* $\sigma \in \{-1, 1\}$ *such that*

$$\sum_{\boldsymbol{\alpha}+\boldsymbol{\beta}<2\mathbf{m}} (-1)^{\|\boldsymbol{m}\|+\|\boldsymbol{\beta}\|-1} p_{\boldsymbol{\alpha}+\boldsymbol{\beta}+\mathbf{1}_n}(\mathbf{x})\, z_{\boldsymbol{\alpha}}\, z_{\boldsymbol{\beta}} \;\geq\; \delta \sum_{\boldsymbol{\alpha}<2\mathbf{m}} z^2_{\boldsymbol{\alpha}} > 0 \quad \textit{for} \quad \mathbf{x} \in \Omega \tag{4.14}$$

and

$$(-1)^{\|\boldsymbol{\alpha}\|}\, \sigma\, p_{2\boldsymbol{\alpha}}(\widehat{\boldsymbol{x}}_{\boldsymbol{\alpha}}) \;\geq\; \delta > 0 \quad \textit{for} \quad \mathbf{x} \in \Omega. \tag{4.15}$$

Then problem (4.11), (4.2) *is well–posed. Moreover, if* $p_{\boldsymbol{\alpha}+\boldsymbol{\beta}+\mathbf{1}_n} \in C^{\boldsymbol{\alpha}+\boldsymbol{\beta}+\mathbf{1}_n}_{\boldsymbol{\omega}}(\mathbb{R}^n)$ $(\boldsymbol{\alpha} + \boldsymbol{\beta} < 2\boldsymbol{m})$, *then problem* (4.11), (4.2) *is* α*–well–posed.*

Remark 4.1. In Theorems 4.5 and 4.6 the condition $\delta > 0$ cannot be replaced by the weaker condition $\delta \geq 0$. Indeed, consider the equation

$$u^{(2,\dots,2)} = (-1)^n \sum_{i=1}^{n} u^{(2\mathbf{1}_i)} + (-1)^{n-1} u + q(x_1, \dots, x_{n-1}). \tag{4.16}$$

The coefficients of the equation satisfy the conditions of Theorems 4.5 and 4.6 with $\delta = 0$. Let us show that problem (4.16),(4.2) has no (classical) solution.

Assume that problem (4.16),(4.2) has a solution u. One can easily verify that u is the unique solution of problem (4.16),(4.2) and thus is independent of x_n. Therefore, u satisfies the equation

$$\sum_{i=1}^{n-1} u_{x_i x_i} - u = q(x_1, \dots, x_{n-1}).$$

From the theory of elliptic equations it is well-known, that if $q \in C(\widehat{\Omega}_n)$, then, generally speaking, u is not a classical solution, i.e., it does not belong $C^2(\widehat{\Omega}_n)$, and thus does not belong to $C^{2,\dots,2}(\widehat{\Omega}_n)$.

4.3. **Ill–posed case.** Let $n = 2$ and $\mathbf{m} = (m_1, m_2)$. Consider the equations

$$u^{(2\mathbf{m})} = p_0(\mathbf{x})u + p_1(x_2)u^{(2\mathbf{m}_1)} + q(\mathbf{x}) \tag{4.17}$$

and

$$u^{(2\mathbf{m})} = p_0(\mathbf{x})u + q(\mathbf{x}). \tag{4.18}$$

According to Theorems 4.1 and 4.2, problems (4.17),(4.2) and (4.18),(4.2) are not well–posed and do not have the Fredholm property. Nevertheless, under certain conditions they still may be uniquely solvable.

Along with equations (4.17) and (4.18) consider the equations

$$u^{(\mathbf{2m})} = p_0(\mathbf{x})u + p_1(x_2)u^{(\mathbf{2m}_1)} + \mathcal{E}_2^{\varepsilon}[u] + q(\mathbf{x}), \tag{4.17_ε}$$

and

$$u^{(\mathbf{2m})} = p_0(\mathbf{x})u + \mathcal{E}_1^{\varepsilon}[u] + \mathcal{E}_2^{\varepsilon}[u] + q(\mathbf{x}), \tag{4.18_ε}$$

where

$$\mathcal{E}_k^{\varepsilon}[u] = (-1)^{m_k}\left(\left(\varepsilon^2 \frac{\partial^2}{\partial x_k^2} - I\right)^{m_k} - (-1)^{m_k} I\right)[u] \quad (k = 1, 2). \tag{4.19}$$

By Theorems 4.2 and 4.6, for every $\varepsilon > 0$ problems $(4.17_\varepsilon), (4.2)$ and $(4.18_\varepsilon), (4.2)$ are uniquely solvable.

Theorem 4.8. *Let p_0, $q \in C_{\boldsymbol{\omega}}^{\mathbf{2m}_2}(\mathbb{R}^2)$ and let there exist $\delta > 0$ such that*

$$(-1)^{m_2-1} p_1(x_2) \ge \delta > 0 \quad \textit{for} \quad x_2 \in [0, \omega_2], \tag{4.20}$$

$$(-1)^{m_1+m_2-1} p_0(\mathbf{x}) \ge \delta > 0 \quad \textit{for} \quad \mathbf{x} \in \Omega. \tag{4.21}$$

Then problem $(4.17), (4.2)$ *has a unique classical solution u and*

$$\lim_{\varepsilon \to 0} \|u_\varepsilon - u\|_{C_{\boldsymbol{\omega}}^{\mathbf{2m}}} = 0, \tag{4.22}$$

where u_ε is a solution of problem $(4.17_\varepsilon), (4.2)$.

Theorem 4.9. *Let $p_0, q \in C_{\boldsymbol{\omega}}^{\mathbf{2m}}(\mathbb{R}^2)$ and let there exist $\delta > 0$ such that condition* (4.21) *holds. Then problem* $(4.18), (4.2)$ *has a unique classical solution u and*

$$\lim_{\varepsilon \to 0} \|u_\varepsilon - u\|_{C_{\boldsymbol{\omega}}^{\mathbf{2m}}} = 0, \tag{4.23}$$

where u_ε is a solution of problem $(4.18_\varepsilon), (4.2)$.

5. Auxiliary Statements

5.1. **Case $n = 2$.** Along with the associated problems $(4.1_j), (4.2_j)$ consider the problem

$$u^{(\mathbf{m}_1)} = \sum_{j=0}^{m_1-1} \widetilde{p}_{j\,m_2}(\mathbf{x}) u^{(j,0)}, \tag{5.1}$$

$$u(x_1 + \omega_1, x_2) = u(x_1, x_2) \tag{5.2}$$

where $\widetilde{p}_{j\,m_2} \in C^{0,m_2}_{\boldsymbol{\omega}}(\mathbb{R}^2)$ $(j = 0, \dots, m_1 - 1)$.

Lemma 5.1. *Let $n = 2$, condition (A_1) of Theorem 4.2 hold, and let problem $(5.1), (5.2)$ have only the trivial solution for every $x_2 \in [0, \omega_2]$. Then an arbitrary solution u of problem $(4.1), (4.2)$ admits the following representations:*

$$u^{(m_1,0)}(x_1, x_2) = \int_0^{\omega_2} g_2(x_2, s_2; x_1) \Bigg(\sum_{j=0}^{m_1-1} p_{j\,m_2}(x_1, s_2) u^{(j,m_2)}(x_1, s_2) + \sum_{j=0}^{m_1-1} \sum_{k=0}^{m_2-1} p_{jk}(x_1, s_2) u^{(j,k)}(x_1, s_2) + q(x_1, s_2) \Bigg) ds_2; \tag{5.3}$$

$$u^{(0,m_2)}(x_1, x_2) = \int_0^{\omega_1} g_1(x_1, s_1; x_2) \Bigg(\sum_{k=0}^{m_2-1} p_{m_1\,k}(\mathbf{x}) u^{(m_1,k)}(s_1, x_2) + \sum_{j=0}^{m_1-1} \sum_{k=0}^{m_2-1} p_{jk}(s_1, x_2) u^{(j,k)}(s_1, x_2) + q(s_1, x_2) \Bigg) ds_1; \tag{5.4}$$

$$u(x_1, x_2) = \int_0^{\omega_1} \widetilde{g}_1(x_1, s_1; x_2) \int_0^{\omega_2} g_2(x_2, s_2; s_1) \Bigg(\sum_{j=0}^{m_1-1} \sum_{k=0}^{m_2-1} \rho_{jk}(s_1, s_2) u^{(j,k)}(s_1, s_2) + \sum_{j=0}^{m_1-1} \big(p_{j\,m_2}(s_1, s_2) - \widetilde{p}_{j\,m_2}(s_1, s_2)\big) u^{(j,m_2)}(s_1, s_2) + q(s_1, s_2) \Bigg) ds_2\, ds_1, \tag{5.5}$$

where

$$\rho_{jk}(x_1,x_2) = p_{jk}(x_1,x_2) + \sum_{i=k}^{m_2-1} \frac{i!}{k!(i-k)!}\, p_{m_1\, i}(\mathbf{x})\, \widetilde{p}_{j\, m_2}^{(0,i-k)}(x_1,x_2)$$
$$- \frac{m_2!}{k!(m_2-k)!}\, \widetilde{p}_{j\, m_2}^{(0,m_2-k)}(x_1,x_2) \quad (j=0,\dots,m_1-1;\ k=0,\dots m_2-1), \quad (5.6)$$

g_j *is the Green's function of problem* $(4.1_j),(4.2_j)$ $(j=1,2)$*, and* $\widetilde{g}_1$ *is the Green's function of problem* $(5.1),(5.2)$*.*

The proof of Lemma 5.1 is similar to the proof of the Lemma 2.2.

Lemma 5.2. *Let* $n=2$*, and let condition* (A_1) *of Theorem* 4.2 *hold. Then problem* $(4.1),(4.2)$ *has the Fredholm property.*

Proof. In view of Lemma 5.1, problem $(4.1),(4.2)$ is equivalent to the following system of integral equations

$$v(x_1,x_2) = \mathcal{F}_1(u,w)(x_1,x_2); \quad (5.7)$$

$$w(x_1,x_2) = \mathcal{F}_2(u,v)(x_1,x_2); \quad (5.8)$$

$$u(x_1,x_2) = \mathcal{F}(u,w)(x_1,x_2), \quad (5.9)$$

where

$$\mathcal{F}_1(u,w)(x_1,x_2) = \int_0^{\omega_2} g_2(x_2,s_2;x_1)\Bigg(\sum_{j=0}^{m_1-1} p_{j\, m_2}(x_1,s_2) w^{(j,0)}(x_1,s_2)$$
$$+ \sum_{j=0}^{m_1-1}\sum_{k=0}^{m_2-1} p_{jk}(x_1,s_2) u^{(j,k)}(x_1,s_2) + q(x_1,s_2)\Bigg)\, ds_2; \quad (5.10)$$

$$\mathcal{F}_2(u,v)(x_1,x_2) = \int_0^{\omega_1} g_1(x_1,s_1;x_2)\Bigg(\sum_{k=0}^{m_2-1} p_{m_1\, k}(\mathbf{x}) v^{(0,k)}(s_1,x_2)$$
$$+ \sum_{j=0}^{m_1-1}\sum_{k=0}^{m_2-1} p_{jk}(s_1,x_2) u^{(j,k)}(s_1,x_2) + q(s_1,x_2)\Bigg)\, ds_1; \quad (5.11)$$

$$\mathcal{F}(u,w)(x_1,x_2) = \int_0^{\omega_1} \widetilde{g}_1(x_1,s_1;x_2) \int_0^{\omega_2} g_2(x_2,s_2;s_1) \Bigg(\sum_{j=0}^{m_1-1} \sum_{k=0}^{m_2-1} \rho_{jk}(s_1,s_2) u^{(j,k)}(s_1,s_2)$$
$$+ \sum_{j=0}^{m_1-1} \big(p_{j\,m_2}(s_1,s_2) - \widetilde{p}_{j\,m_2}(s_1,s_2)\big) w^{(j,0)}(s_1,s_2) + q(s_1,s_2) \Bigg) \, ds_2 \, ds_1. \quad (5.12)$$

Let $\mathcal{F}_1^0(u,w)$, $\mathcal{F}_2^0(u,v)$ and $\mathcal{F}^0(u,w)$ be the homogeneous parts of the operators $\mathcal{F}_1(u,w)$, $\mathcal{F}_2(u,v)$ and $\mathcal{F}(u,w)$, respectively, and set:

$$\mathcal{K}(u,v,w) = \Big(\mathcal{F}_1^0(u,w), \mathcal{F}_2^0(u,v), \mathcal{F}^0(u,w)\Big) \quad (5.13)$$

It is clear that $\mathcal{K}$ ia s bounded linear operator from $C_{\boldsymbol{\omega}}^{m_1-1,m_2-1}(\mathbb{R}^2) \times C_{\boldsymbol{\omega}}^{m_1-1,m_2-1}(\mathbb{R}^2) \times C_{\boldsymbol{\omega}}^{m_1-1,m_2-1}(\mathbb{R}^2)$ into $C_{\boldsymbol{\omega}}^{m_1-1,m_2}(\mathbb{R}^2) \times C_{\boldsymbol{\omega}}^{m_1,m_2-1}(\mathbb{R}^2) \times C_{\boldsymbol{\omega}}^{m_1,m_2}(\mathbb{R}^2)$.

Notice that $\mathcal{K}^2 = \mathcal{K} \circ \mathcal{K}$ is a compact operator from $C_{\boldsymbol{\omega}}^{m_1-1,m_2-1}(\mathbb{R}^2) \times C_{\boldsymbol{\omega}}^{m_1-1,m_2-1}(\mathbb{R}^2) \times C_{\boldsymbol{\omega}}^{m_1-1,m_2-1}(\mathbb{R}^2)$ into $C_{\boldsymbol{\omega}}^{m_1-1,m_2-1}(\mathbb{R}^2) \times C_{\boldsymbol{\omega}}^{m_1-1,m_2-1}(\mathbb{R}^2) \times C_{\boldsymbol{\omega}}^{m_1-1,m_2-1}(\mathbb{R}^2)$. The latter fact implies that the system of operator equations (5.7)–(5.9) and, consequently, problem (4.1),(4.2) have the Fredholm property. □

Remark 5.1. Notice that if $p_{jm_2} \in C_{\boldsymbol{\omega}}^{0,m_2}(\mathbb{R}^2)$ $(j = 0, \dots, m_1 - 1)$, then choosing $\widetilde{p}_{jm_2}(\mathbf{x}) \equiv p_{jm_2}(\mathbf{x})$ $(j = 0, \dots, m_1 - 1)$ in representation (5.5), one can show that problem (4.1),(4.2) is equivalent to the Fredholm integral equation

$$u(x_1,x_2) = \mathcal{F}_0(u)(x_1,x_2), \quad (5.14)$$

where

$$\mathcal{F}_0(u,v)(x_1,x_2) = \int_0^{\omega_1} g_1(x_1,s_1;x_2) \int_0^{\omega_2} g_2(x_2,s_2;s_1) \Bigg(\sum_{j=0}^{m_1-1} \sum_{k=0}^{m_2-1} \rho_{jk}(s_1,s_2) u^{(j,k)}(s_1,s_2)$$
$$+ q(s_1,s_2) \Bigg) \, ds_2 \, ds_1. \quad (5.15)$$

5.2. **Case** $n \geq 2$**.** If problem (4.1), (4.2) is well–posed, then its solution u admits the representation

$$u(x) = \mathcal{G}(q)(x), \quad (5.16)$$

where $\mathcal{G} : C_{\boldsymbol{\omega}}(\mathbb{R}^n) \to C_{\boldsymbol{\omega}}^{\mathbf{m}}(\mathbb{R}^n)$ is a bounded linear operator.

Definition 5.1. $\mathcal{G} : C_{\boldsymbol{\omega}}(\mathbb{R}^n) \to C^{\mathbf{m}}_{\boldsymbol{\omega}}(\mathbb{R}^n)$ is called the **Green's operator** of problem $(4.1_0), (4.2)$.

Remark 5.2. If problem (4.1),(4.2) is well–posed in $C^{\mathbf{m}}_{\boldsymbol{\omega}}(\mathbb{R}^n)$, and its solution admits the estimate (6.34) (see the proof of Theorem 4.4 below), then problem (1.1),(1.2) is well–posed in $AC^{\mathbf{m-1}}_{\boldsymbol{\omega}}(\mathbb{R}^n)$ too.

In other words, the Green's operator $\mathcal{G}$ is a bounded linear operator from $C_{\boldsymbol{\omega}}(\mathbb{R}^n)$ to $C^{\mathbf{m}}_{\boldsymbol{\omega}}(\mathbb{R}^n)$, as well as from $L_{\boldsymbol{\omega}}(\mathbb{R}^n)$ to $AC^{\mathbf{m-1}}_{\boldsymbol{\omega}}(\mathbb{R}^n)$. Therefore, by the Dunford-Pettis theorem (see [18], Chapter XI, § 1, Theorem 6) $\mathcal{G}$ admits the representation

$$\mathcal{G}(q)(\mathbf{x}) = \iint\limits_{\Omega} G(\mathbf{x}, \mathbf{s}) q(\mathbf{s})\, d\mathbf{s} \qquad \mathbf{x} \in \Omega, \tag{5.17}$$

where $G \in L^{\infty}(\Omega \times \Omega)$ is called the Green's function of problem $(4.1_0), (4.2)$.

Lemma 5.3. *Let for some* $\boldsymbol{\sigma} \in \boldsymbol{\Xi}$ *the* $\boldsymbol{\sigma}$*–associated problem* $(4.1_{\boldsymbol{\sigma}}), (4.2_{\boldsymbol{\sigma}})$ *be well–posed for every* $x_{\widehat{\boldsymbol{\sigma}}} \in \Omega_{\widehat{\boldsymbol{\sigma}}}$*, and let* $p_{\mathbf{m}_{\widehat{\sigma}}+\boldsymbol{\alpha}} \in C_{\boldsymbol{\omega}}(\mathbb{R}^n)$ $(\boldsymbol{\alpha} < \mathbf{m}_{\boldsymbol{\sigma}})$*. Then the Green's operator* $\mathcal{G}_{\boldsymbol{\sigma}}$ *is a bounded linear operator from* $C_{\boldsymbol{\omega}}(\mathbb{R}^n)$ *to* $C^{\mathbf{m}_{\sigma}}_{\boldsymbol{\omega}}(\mathbb{R}^n)$*. Moreover, if* $p_{\mathbf{m}_{\widehat{\sigma}}+\boldsymbol{\alpha}} \in C^{\mathbf{m}_{\widehat{\sigma}}}_{\boldsymbol{\omega}}(\mathbb{R}^n)$ $(\boldsymbol{\alpha} < \mathbf{m}_{\boldsymbol{\sigma}})$*, then the Green's operator* $\mathcal{G}_{\boldsymbol{\sigma}}$ *is a bounded linear operator from* $C_{\boldsymbol{\omega}}(\mathbb{R}^n)$ *to* $C^{\mathbf{m}}_{\boldsymbol{\omega}}(\mathbb{R}^n)$*.*

Lemma 5.3 can be proved similarly to Lemma 2.5

Lemma 5.4. *Let problem* $(4.1), (4.2)$ *be well–posed, and let for every* $\boldsymbol{\sigma} \in \boldsymbol{\Xi}$ *the* $\boldsymbol{\sigma}$*–associated problem* $(4.1_{\boldsymbol{\sigma}}), (4.2_{\boldsymbol{\sigma}})$ *be well–posed for every* $\mathbf{x}_{\widehat{\boldsymbol{\sigma}}} \in \Omega_{\widehat{\boldsymbol{\sigma}}}$*. Moreover, let* $p_{\boldsymbol{\alpha}} \in C^{\boldsymbol{\alpha}}_{\boldsymbol{\omega}}(\mathbb{R}^n)$ $(\boldsymbol{\alpha} < \mathbf{m})$*. Then problem* $(4.1), (4.2)$ *is* α*–well–posed.*

Lemma 5.5. *Let problem* $(4.1), (4.2)$ *be well–posed, and let for every* $\boldsymbol{\sigma} \in \boldsymbol{\Xi}$ *the* $\boldsymbol{\sigma}$*–associated problem* $(4.1_{\boldsymbol{\sigma}}), (4.2_{\boldsymbol{\sigma}})$ *be well–posed for every* $\mathbf{x}_{\widehat{\boldsymbol{\sigma}}} \in \Omega_{\widehat{\boldsymbol{\sigma}}}$*. Moreover, let* $p_{\boldsymbol{\alpha}} \in C^{\boldsymbol{\alpha}}_{\boldsymbol{\omega}}(\mathbb{R}^n)$*. Then*

$$\mathcal{G}(q^{(\mathbf{m})})(\mathbf{x}) = q(\mathbf{x}) + \mathcal{A}(q)(\mathbf{x}), \tag{5.18}$$

where $\mathcal{G}$ is the Green's operator of problem $(4.1_0), (4.2)$, *and* $\mathcal{A} : C_{\boldsymbol{\omega}}(\mathbb{R}^n) \to C_{\boldsymbol{\omega}}(\mathbb{R}^n)$ *is a bounded linear operator.*

Lemmas 5.4 and 5.5 can be proved similarly to Lemmas 2.6 and 2.7.

5.3. **Ill–posed case.** Consider the problems

$$z'' = p(t)z + q(t), \qquad z(t+\omega) = z(t), \tag{5.19}$$

$$z'' = p(t)z, \qquad z(t+\omega) = z(t), \tag{5.19_0}$$

where p and $q \in C_\omega(\mathbb{R})$.

Lemma 5.6. *Let*

$$p(t) \geq 0 \quad for \quad t \in \mathbb{R}, \qquad \int_0^\omega p(t)\,dt \geq \delta > 0. \tag{5.20}$$

Then problem (5.19_0) *has only a trivial solution. Moreover the Green's function of problem* (5.19_0) *satisfies the inequalities*

$$g(t,\tau) < 0 \quad for \quad (t,\tau) \in \mathbb{R}^2, \quad t \leq \tau \leq t+\omega, \tag{5.21}$$

$$\int_t^{t+\omega} g^2(t,\tau)\,d\tau \leq \omega(\delta^{-\frac{1}{2}} + \omega^{\frac{1}{2}})^4, \tag{5.22}$$

$$\int_t^{t+\omega} \Big(\frac{\partial g(t,\tau)}{\partial t}\Big)^2 d\tau \leq \big((\delta^{-\frac{1}{2}} + \omega^{\frac{1}{2}})^2\omega\|p\|_{L^2_\omega} + \omega^{\frac{1}{2}}\big)^2 \quad for \quad t \in \mathbb{R}. \tag{5.23}$$

Proof. Let z be a solution of problem (5.19). Integrating the equality

$$z''(t)z(t) = p(t)z^2(t) + q(t)z(t)$$

from 0 to ω, we get

$$\int_0^\omega p(t)z^2(t)\,dt + \int_0^\omega z'^2(t)\,dt = \int_0^\omega q(t)z(t)\,dt. \tag{5.24}$$

By Schwartz's inequality and condition (5.20) we obtain

$$z^2(t_0)\delta + \|z'\|^2_{L^2_\omega} \leq \omega^{\frac{1}{2}}\|q\|_{L^2_\omega}\|z\|_{C_\omega},$$

where $|z(t_0)| = \min\{|z(t)| : t \in \mathbb{R}\}$. On the other hand

$$\|z\|_{C_\omega} \le |z(t_0)| + \omega^{\frac{1}{2}} \|z'\|_{L^2_\omega}.$$

The latter two inequalities imply

$$\|z\|_{C_\omega} \le \big(\delta^{-\frac{1}{2}} + \omega^{\frac{1}{2}}\big)^2 \omega^{\frac{1}{2}} \|q\|_{L^2_\omega}. \tag{5.25}$$

On the other hand

$$\begin{aligned} \|z'\|_{C_\omega} &\le \int\limits_0^\omega |z''(t)|\, dt \le \omega^{\frac{1}{2}} \big(\|p\|_{L^2_\omega} \|z\|_{C_\omega} + \|q\|_{L^2_\omega}\big) \\ &\le \Big(\big(\delta^{-\frac{1}{2}} + \omega^{\frac{1}{2}}\big)^2 \|p\|_{L^2_\omega} + \omega^{\frac{1}{2}}\Big) \|q\|_{L^2_\omega}. \end{aligned} \tag{5.26}$$

If $q(t) \equiv 0$, then (5.25) yields $z(t) \equiv 0$. Therefore problem (5.19_0) has only a trivial solution. By this fact and Lemma 2.1, for every q problem (5.19) has a unique solution z.

Estimates (5.22) and (5.23) follow from (5.25) and (5.26). Inequality (5.21) follows from the fact that every nonzero solution of the equation

$$z'' = p(t) z \tag{5.28}$$

has at most one zero. $\square$

Lemma 5.7. *Let*

$$p \in C^k_\omega(\mathbb{R}), \quad q \in C^k_\omega(\mathbb{R}) \tag{5.29}$$

and there exist a positive constant γ such that

$$p(t) \ge \gamma \quad \textit{for} \quad t \in \mathbb{R}. \tag{5.30}$$

Then a solution z of problem (5.19) *for $k = 0$ admits the estimate*

$$\|z\|_{C_\omega} \le \gamma^{-1} \|q\|_{C_\omega}, \tag{5.31}$$

and for $k \ge 1$ the estimate

$$\|z\|_{C^k_\omega} \le \frac{r_k}{\gamma} \Big(\|\gamma^{-1} p'\|_{C^{k-1}_\omega} + 1 \Big) \|q\|_{C^k_\omega}, \tag{5.32}$$

where r_k is a positive constant depending on k only.

Proof. By condition (5.30), from (5.24) we have

$$\gamma\|z\|^2_{L^2_\omega} \le \|q\|_{L^2_\omega}\ \|z\|_{L^2_\omega},$$

and therefore

$$\|z\|_{L^2_\omega} \le \gamma^{-1}\|q\|_{L^2_\omega}.$$

Now assume that $q \in C_\omega(\mathbb{R})$. Then

$$z(t) = \int\limits_0^\omega g(t,\tau)q(\tau)\,d\tau, \tag{5.33}$$

where g is a Green's function of problem (5.19_0) and satisfies inequality (5.21). If $q(t) \equiv p(t)$, then (5.19) has a unique solution $z(t) \equiv -1$. Therefore

$$\int\limits_t^{t+\omega} |g(t,\tau)|p(\tau)\,d\tau = 1 \quad \text{for} \quad t \in [0,\omega].$$

By this identity and condition (5.30), from (5.33) we get

$$|z(t)| \le \|q\|_{C_\omega}\int\limits_0^\omega |g(t,\tau)|\,d\tau \le \gamma^{-1}\|q\|_{C_\omega}\int\limits_0^\omega |g(t,\tau)|p(\tau)\,d\tau = \gamma^{-1}\|q\|_{C_\omega} \quad \text{for} \quad t \in [0,\omega].$$

Therefore

$$\|z\|_{C_\omega} \le \gamma^{-1}\|q\|_{C_\omega}.$$

Thus for $k = 0$ the validity of estimate (5.31) is proved.

Now let k=1. Then

$$z''' = p(t)z' + p'(t)z + q'(t), \quad z'(t+\omega) = z'(t).$$

The above proved estimate implies

$$\|z'\|_{L^2_\omega} \le \gamma^{-1}\|p'z + q'\|_{L^2_\omega}$$
$$\le \gamma^{-1}(\|p'\|_{L^2_\omega}\|z\|_{L^2_\omega} + \|q\|_{L^2_\omega}) \le \gamma^{-1}(\gamma^{-1}\|p'\|_{L^2_\omega}\|q\|_{L^2_\omega} + \|q\|_{L^2_\omega})$$
$$\big(\|z'\|_{C_\omega} \le \gamma^{-1}(\gamma^{-1}\|p'\|_{C_\omega}\|q\|_{C_\omega} + \|q\|_{C_\omega})\big).$$

Therefore estimate (5.32) is valid, where $r_1 = 1$.

Now applying the method of induction we can easily show that for every natural k estimate (5.32) is valid, where r_k is a positive number depending on k only. □

Consider now the problem

$$\varepsilon^2 z'' = z - q(t), \qquad z(t+\omega) = z(t). \tag{5.34}$$

Lemma 5.8. *Let $q \in C^k_\omega(\mathbb{R})$. Then for every $\varepsilon > 0$ problem* (5.34) *has a unique solution z_ε and*

$$\lim_{\varepsilon \to 0} \|z_\varepsilon - q\|_{C^k_\omega} = 0. \tag{5.35}$$

Proof. By Lemma 5.6, problem (5.34) has a unique solution

$$z_\varepsilon(t) = \int_0^\omega g_\varepsilon(t,\tau) q(\tau)\, d\tau, \tag{5.36}$$

where

$$g_\varepsilon(t,\tau) = \begin{cases} \frac{\cosh \varepsilon^{-1}(\tau - t + \omega/2)}{2\sinh \varepsilon^{-1}\omega/2} & \text{for} \quad 0 \le \tau \le t \le \omega \\ \frac{\cosh \varepsilon^{-1}(t - \tau + \omega/2)}{2\sinh \varepsilon^{-1}\omega/2} & \text{for} \quad 0 \le t < \tau \le \omega \end{cases}. \tag{5.37}$$

Notice that for every $\varepsilon > 0$

$$g_\varepsilon(t,\tau) > 0 \quad \text{for} \quad (t,\tau) \in [0,\omega] \times [0,\omega], \qquad \int_0^\omega g_\varepsilon(t,\tau)\, d\tau \equiv 1. \tag{5.38}$$

Moreover, for every $\delta \in (0, \omega/2)$ we have

$$\int_0^{t-\delta} g_\varepsilon(t,\tau)\, d\tau + \int_{t+\delta}^\omega g_\varepsilon(t,\tau)\, d\tau = \frac{\sinh \varepsilon^{-1}(\omega/2 - \delta)}{\sinh \varepsilon^{-1}\omega/2} \to 0 \ \text{ as } \ \varepsilon \to 0. \tag{5.39}$$

If $q \in C_\omega$, then (5.38) and (5.39) imply that

$$\lim_{\varepsilon \to 0} \|z_\varepsilon - q\|_{L^2_\omega} = 0.$$

As for equalities (5.35), to prove them it is sufficient to notice that

$$z^{(m)}_\varepsilon(t) = \int_0^\omega g_\varepsilon(t,\tau) q^{(k)}(\tau)\, d\tau. \quad \square$$

Lemma 5.9. *Let $z \in C^2_\omega(\mathbb{R})$. Then*

$$z(t) = \frac{1}{\omega} \int_0^\omega z(\tau)\, d\tau + \int_0^\omega K_\omega(t,\tau) z''(\tau)\, d\tau, \tag{5.40}$$

where

$$K_\omega(t,\tau) = \begin{cases} \frac{1}{2\omega}(\tau - t)(\tau - t - \omega) & \text{for} \quad 0 \le \tau \le t \le \omega \\ \frac{1}{2\omega}(t - \tau)(t - \tau - \omega) & \text{for} \quad 0 \le t < \tau \le \omega \end{cases} \tag{5.41}$$

The proof of Lemma 5.9 is trivial.

Lemma 5.10. *Let $u \in C^{2,2}_{\omega_1\omega_2}(\mathbb{R}^2)$. Then*

$$u(x_1,x_2) = \varphi(x_2) + \psi(x_1) + w(x_1,x_2), \tag{5.42}$$

where

$$\varphi(x_2) = \frac{1}{\omega_1}\int_0^{\omega_1} u(s_1,x_2)\,ds_1, \quad \psi(x_1) = \frac{1}{\omega_2}\int_0^{\omega_2} u(x_1,s_2)\,ds_2, \tag{5.43}$$

$$w(x_1,x_2) = \int_0^{\omega_1}\int_0^{\omega_2} K_{\omega_1}(x_1,s_1)K_{\omega_2}(x_2,s_2)u^{(2,2)}(s_1,s_2)\,ds_2\,ds_1 - \frac{1}{\omega_1\omega_2}\int_0^{\omega_1}\int_0^{\omega_2} u(s_1,s_2)\,ds_1\,ds_2. \tag{5.44}$$

Lemma 5.10 immediately follows from Lemma 5.9.

5.4. **Lemmas on properties of solutions of problems** $(4.17_\varepsilon), (4.2)$ **and** (4.18_ε)**,** (4.2)**.** If conditions (4.20) and (4.21) (condition (4.21)) hold, then by Theorem 4.2, problem $(4.17_\varepsilon), (4.2)$ $\big($problem $(4.18_\varepsilon), (4.2)\big)$ has a unique solution u_ε. Consider the case, where $m_1 = m_2 = 1$. Set

$$\varphi_\varepsilon(x_2) = \frac{1}{\omega_1}\int_0^{\omega_1} u_\varepsilon(s_1,x_2)\,ds_1, \tag{5.45}$$

$$\psi_\varepsilon(x_1) = \frac{1}{\omega_2}\int_0^{\omega_2} u_\varepsilon(x_1,s_2)\,ds_2, \tag{5.46}$$

$$w_\varepsilon(x_1,x_2)=\int_0^{\omega_1}\int_0^{\omega_2}K_{\omega_1}(x_1,s_1)K_{\omega_2}(x_2,s_2)\big(p_0(s_1,s_2)u_\varepsilon(s_1,s_2)+q(s_1,s_2)\big)\,ds_2\,ds_1$$
$$+\,\varepsilon^2\int_0^{\omega_1}K_{\omega_1}(x_1,s_1)u_\varepsilon(s_1,x_2)\,ds_1+\varepsilon^2\int_0^{\omega_2}K_{\omega_2}(x_2,s_2)u_\varepsilon(x_1,s_2)\,ds_2$$
$$-\frac{1}{\omega_1\omega_2}\int_0^{\omega_1}\int_0^{\omega_2}u_\varepsilon(s_1,s_2)\,ds_2\,ds_1, \quad (5.47)$$

$$v_\varepsilon(x_1,x_2)=\int_0^{\omega_1}\int_0^{\omega_2}K_{\omega_1}(x_1,s_1)g_2(x_2,s_2)\big((p_0(s_1,s_2)+\varepsilon^2p_1(s_2))u_\varepsilon(s_1,s_2)+q(s_1,s_2)\big)\,ds_2\,ds_1$$
$$+\,\varepsilon^2\int_0^{\omega_1}K_{\omega_1}(x_1,s_1)u_\varepsilon(s_1,x_2)\,ds_1, \quad (5.48)$$

where g_2 is the Green's function of problem

$$z''=p_1(x_2)z, \qquad z(x_2+\omega_2)=z(x_2).$$

Lemma 5.11. *Let $m_1=m_2=1$, and conditions (4.20) and (4.21) hold. Then for every $\varepsilon>0$ a solution u_ε of problem (4.17_ε), (4.2) admits the representation*

$$u_\varepsilon(x,y)=\varphi_\varepsilon(y)+v_\varepsilon(x,y), \quad (5.49)$$

where φ_ε and v_ε are the functions given by (5.45) and (5.48). Moreover, if

$$p_0,\,q\in C^{0,2}_{\omega_1\omega_2}(\mathbb{R}^2), \quad (5.50)$$

then $\varphi_\varepsilon\in C^4_{\omega_2}(\mathbb{R})$, $v_\varepsilon\in C^{2,4}_{\omega_1\omega_2}(\mathbb{R}^2)$,

$$\begin{gathered}\|\varphi_\varepsilon\|_{C^2_{\omega_2}}\le r\big(\|p_0\|_{C^{0,2}_{\omega_1\omega_2}}+\|p_1\|_{C^2_{\omega_2}}+1\big)\|q\|_{C^{0,2}_{\omega_1\omega_2}},\\ \|v_\varepsilon\|_{C^{2,4}_{\omega_1\omega_2}}\le r\big(\|p_0\|_{C^{0,2}_{\omega_1\omega_2}}+\|p_1\|_{C^2_{\omega_2}}+1\big)\|q\|_{C^{0,2}_{\omega_1\omega_2}}\quad \textit{for}\quad 0<\varepsilon\le\varepsilon_0,\end{gathered} \quad (5.51)$$

where $r>0$ and $\varepsilon_0>0$ are constants depending on ω_1 and ω_2 only.

Proof. By (4.17_ε) we have

$$\Big(u_\varepsilon^{(2,0)}(x_1,x_2)-\varepsilon^2u_\varepsilon(x_1,x_2)\Big)^{(0,2)}=p_1(y)\Big(u_\varepsilon^{(2,0)}(x_1,x_2)-\varepsilon^2u_\varepsilon(x_1,x_2)\Big)$$
$$+\big(p_0(x_1,x_2)+\varepsilon^2p_1(x_2)\big)u_\varepsilon(x_1,x_2)+q(x_1,x_2).$$

Hence by Lemma 5.6, it follows that

$$u_\varepsilon^{(0,2)}(x_1,x_2) = \varepsilon^2 u_\varepsilon(x_1,x_2) + \int\limits_{x_2}^{x_2+\omega_2} g_1(x_2,s_2)\big((p_0(x_1,s_2)+\varepsilon^2 p_1(s_2))u_\varepsilon(x_1,s_2)+q(x_1,s_2)\big)\,ds_2. \tag{5.52}$$

By Lemma 5.9, we get representation (5.49), where φ_ε and v_ε are the functions defined by (5.45) and (5.48).

Now let us prove the second part of the lemma. Let conditions (5.50) hold.

Multiplying (4.17_ε) by u_ε and integrating over $[0,\omega_1]\times[0,\omega_2]$ we get

$$\int\limits_0^{\omega_1}\int\limits_0^{\omega_2}\Big(\Big|u_\varepsilon^{(1,1)}(s_1,s_2)\Big|^2 + p_1(s_2)\Big|u_\varepsilon^{(1,0)}(s_1,s_2)\Big|^2 + \varepsilon^2\Big|u_\varepsilon^{(0,1)}(s_1,s_2)\Big|^2$$

$$-p_0(s_1,s_2)u_\varepsilon^2(s_1,s_2)\Big)\,ds_2\,ds_1 = \int\limits_0^{\omega_1}\int\limits_0^{\omega_2} q(s_1,s_2)u_\varepsilon(s_1,s_2)\,ds_2\,ds_1.$$

Hence by (4.20) and (4.21) we have

$$\int\limits_0^{\omega_1}\int\limits_0^{\omega_2}\Big(\Big|u_\varepsilon^{(1,1)}(s_1,s_2)\Big|^2 + \delta u_\varepsilon^2(s_1,s_2)\Big)\,ds_2\,ds_1 \le \|q\|_{L^2_{\omega_1\omega_2}}\|u_\varepsilon\|_{L^2_{\omega_1\omega_2}}.$$

Therefore

$$\|u_\varepsilon\|_{L^2_{\omega_1\omega_2}} \le \delta^{-1}\|q\|_{L^2_{\omega_1\omega_2}}, \tag{5.53}$$

and

$$\int\limits_0^{\omega_1}\int\limits_0^{\omega_2}\Big|u_\varepsilon^{(1,1)}(s_1,s_2)\Big|^2\,ds_2\,ds_1 \le \delta^{-1}\|q\|^2_{L^2_{\omega_1\omega_2}}. \tag{5.54}$$

Consequently

$$\|\varphi_\varepsilon\|_{L^2_{\omega_2}} \le \delta^{-1}\omega_1^{\frac{1}{2}}\,\|q\|_{L^2_{\omega_1\omega_2}}, \quad \|v_\varepsilon\|_{L^2_{\omega_1\omega_2}} \le \delta^{-1}(1+\omega_1^{\frac{1}{2}})\,\|q\|_{L^2_{\omega_1\omega_2}}. \tag{5.55}$$

Integrating (4.17_ε) with respect to x_1 from 0 to ω_1, by (4.2) we get

$$\int\limits_0^{\omega_1} p_0(s_1,x_2)u_\varepsilon(s_1,x_2)\,ds_1 + \varepsilon^2\int\limits_0^{\omega_1} u_\varepsilon^{(0,2)}(s_1,x_2)\,ds_1 + \int\limits_0^{\omega_1} q(s_1,x_2)\,ds_1 = 0.$$

Hence by (5.45) and (5.49), we have

$$\varepsilon^2\varphi_\varepsilon''(x_2) = \widetilde{p}(x_2)\varphi_\varepsilon(x_2) + \widetilde{q}_\varepsilon(x_2), \qquad \varphi_\varepsilon(x_2+\omega_2) = \varphi_\varepsilon(x_2),$$

where

$$\widetilde{p}(x_2) = \frac{1}{\omega_1}\int\limits_0^{\omega_1} p_0(s_1,x_2)\,ds_1, \quad \widetilde{q}_\varepsilon(x_2) = \frac{1}{\omega_1}\int\limits_0^{\omega_1} \big(p_0(s_1,x_2)v_\varepsilon(s_1,x_2) + q(s_1,x_2)\big)\,ds_1.$$

By Lemma 5.7 and condition (4.21), for every $\varepsilon > 0$ and $l \in \{0,1,2\}$ we obtain

$$\|\varphi_\varepsilon\|_{C^l_{\omega_2}} \le r_l\big(\|p_0\|_{C^{0,l}_{\omega_1\omega_2}} + 1\big)\big(\|v_\varepsilon\|_{C^{0,l}_{\omega_1\omega_2}} + \|q\|_{C^{0,l}_{\omega_1\omega_2}}\big) \tag{5.56}$$

and

$$\varepsilon^2\|\varphi_\varepsilon\|_{C^{l+2}_{\omega_2}} \le r_l\big(\|p_0\|_{C^{0,l}_{\omega_1\omega_2}} + 1\big)\big(\|v_\varepsilon\|_{C^{0,l}_{\omega_1\omega_2}} + \|q\|_{C^{0,l}_{\omega_1\omega_2}}\big), \tag{5.57}$$

where r_l is a positive number depending on l and δ only. On the other hand by representation (5.48) and (5.49), we have

$$v_\varepsilon(x_1,x_2) = \int\limits_0^{\omega_1}\int\limits_0^{\omega_2} K_{\omega_1}(x_1,s_1)g_2(x_2,s_2)\Big(\big(p_0(s_1,s_2) + \varepsilon^2 p_1(s_2)\big)\big(v_\varepsilon(s_1,s_2) + \varphi_\varepsilon(s_2)\big)$$
$$+q(s_1,s_2)\Big)\,ds_2\,ds_1 + \varepsilon^2\int\limits_0^{\omega_1} K_{\omega_1}(x_1-s_1)\big(v_\varepsilon(s_1,x_2) + \varphi_\varepsilon(x_2)\big)\,ds_1.$$

Hence it follows immediately that for every $l \in \{0,\dots,2\}$ the inequality

$$\|v_\varepsilon\|_{C^{2,l+2}_{\omega_1\omega_2}} \le r_l(\|p_0\|_{C^{0,l}_{\omega_1\omega_2}} + \|p_1\|_{C^l_{\omega_2}} + 1)(\|v_\varepsilon\|_{C^{0,l}_{\omega_1\omega_2}} + \|q\|_{C^{0,l}_{\omega_1\omega_2}} + \varepsilon^2\|\varphi_\varepsilon\|_{C^l_{\omega_2}})$$
$$+\frac{1}{2}\|v_\varepsilon\|_{C^{2,l+2}_{\omega_1\omega_2}} \quad \text{for} \quad 0 < \varepsilon \le \varepsilon_0$$

holds, where r_l is a constant depending on l, ω_1 and ω_2 only, and

$$\varepsilon_0 = \frac{1}{\sqrt{2}}\Big(\frac{\omega_1^{\frac{3}{2}}}{8} + \frac{\omega_1^{\frac{1}{2}}}{\sqrt{3}} + 1\Big)^{-\frac{1}{2}}.$$

Therefore

$$\|v_\varepsilon\|_{C^{2,l+2}_{\omega_1\omega_2}} \le 2r_l\big(\|p_0\|_{C^{0,l}_{\omega_1\omega_2}} + \|p_1\|_{C^l_{\omega_2}} + 1\big)$$
$$\times\big(\|v_\varepsilon\|_{C^{0,l}_{\omega_1\omega_2}} + \|q\|_{C^{0,l}_{\omega_1\omega_2}} + \varepsilon^2\|\varphi_\varepsilon\|_{C^l_{\omega_2}}\big) \quad \text{for} \quad 0 \le \varepsilon \le \varepsilon_0. \tag{5.58}$$

By induction, estimates (2.55)–(2.58) imply estimates (5.51), where r is a positive constant depending on ω_1, ω_2 and δ only. □

Similarly to Lemma 5.11 one can prove

Lemma 5.12. *Let condition* (4.21) *hold. Then for every* $\varepsilon > 0$ *a solution* u_ε *of problem* $(4.18_\varepsilon), (4.2)$ *admits the representation*

$$u_\varepsilon(x_1, x_2) = \varphi_\varepsilon(x_2) + \psi_\varepsilon(x_1) + w_\varepsilon(x_1, x_2), \tag{5.59}$$

where φ_ε, ψ_ε *and* w_ε *are the functions given by* (5.45)–(5.47). *Moreover, if*

$$p_0 \in C^{2,2}_{\omega_1\omega_2}(\mathbb{R}^2), \quad q \in C^{2,2}_{\omega_1\omega_2}(\mathbb{R}^2),$$

then $\varphi_\varepsilon \in C^4_{\omega_2}(\mathbb{R})$, $\psi_\varepsilon \in C^4_{\omega_1}$, $w_\varepsilon \in C^{4,4}_{\omega_1\omega_2}(\mathbb{R}^2)$,

$$\|\varphi_\varepsilon\|_{C^2_{\omega_2}} \le r(\|p_0\|_{C^{0,2}_{\omega_1\omega_2}} + 1)\|q\|_{C^{0,2}_{\omega_1\omega_2}}, \quad \|\psi_\varepsilon\|_{C^2_{\omega_1}} \le r(\|p_0\|_{C^{2,0}_{\omega_1\omega_2}} + 1)\|q\|_{C^{2,0}_{\omega_1\omega_2}},$$

$$\|v_\varepsilon\|_{C^{4,4}_{\omega_1\omega_2}} \le r(\|p_0\|_{C^{2,2}_{\omega_1\omega_2}} + 1)\|q\|_{C^{2,2}_{\omega_1\omega_2}} \quad \textit{for} \quad 0 < \varepsilon \le \varepsilon_0,$$

where $r > 0$ *and* $\varepsilon_0 > 0$ *are constants depending on* ω_1 *and* ω_2 *only.*

Consider the equations

$$u^{(2,2)} = p_0(x_2)u + p_1(x_2)u^{(2,0)} + \varepsilon^2 u^{(0,2)} + q(x_1, x_2), \tag{5.60}$$

$$u^{(2,2)} = -\delta u + \varepsilon^2 u^{(2,0)} + \varepsilon^2 u^{(0,2)} + q(x_1, x_2), \tag{5.61}$$

where $p_i \in C^2_{\omega_2}(\mathbb{R})$ $(i = 0, 1, 2)$ and $q \in C^2_{\omega_1\omega_2}(\mathbb{R}^2)$.

For every $k \in \mathbb{Z}$ set

$$\mu_k = \frac{2\pi}{\omega_1}k, \quad \nu_k = \frac{2\pi}{\omega_2}k, \quad \rho_{\varepsilon k}(x_2) = \frac{-p_0(x_2) + \mu_k^2 p_1(x_2)}{\varepsilon^2 + \mu_k^2}.$$

Consider the problem

$$z'' = \rho_{\varepsilon k}(x_2)z, \qquad z(x_2 + \omega_2) = z(x_2). \tag{5.62}$$

If

$$p_0(x_2) < 0, \quad p_1(x_2) > 0 \quad \text{for} \quad x_2 \in [0, \omega_2], \tag{5.63}$$

then

$$\rho_{\varepsilon k}(x_2) > 0.$$

By Lemma 5.6, problem (5.62) has only a trivial solution. By $g_{\varepsilon k}$ denote its Green's function.

Lemma 5.13. *Let* (5.63) *hold,*

$$q(x_1,x_2) \sim \sum_{k\in\mathbb{Z}} q_k(x_2)\exp(i\mu_k x_1)$$

be the Fourier expansion of q and $u_\varepsilon \in C^{2,2}_{\omega_1\omega_2}(\mathbb{R}^2)$ be a solution of problem (5.60), (4.2). *Then the following representation is valid*

$$u_\varepsilon(x_1,x_2) = -\sum_{k\in\mathbb{Z}}\left(\int_0^{\omega_2} g_{\varepsilon k}(x_2,s_2)\frac{q_k(s_2)}{\varepsilon^2+\mu_k^2}\,ds_2\right)\exp(i\mu_k x_1). \tag{5.64}$$

Proof. Since $u_\varepsilon \in C^{2,2}_{\omega_1\omega_2}(\mathbb{R}^2)$, it can be represented as a uniformly convergent series

$$u_\varepsilon(x_1,x_2) = \sum_{k\in\mathbb{Z}} u_{\varepsilon k}(x_2)\exp(i\mu_k x_1), \tag{5.65}$$

where $u_{\varepsilon k} \in C^2_{\omega_2}(\mathbb{R})$ $(k \in \mathbb{Z})$. Moreover

$$u_\varepsilon^{(0,2)}(x_1,x_2) = \sum_{k\in\mathbb{Z}} u''_{\varepsilon k}(x_2)\exp(i\mu_k x_1), \quad u_\varepsilon^{(2,0)}(x_1,x_2) \sim -\sum_{k\in\mathbb{Z}} \mu_k^2 u_{\varepsilon k}(x_2)\exp(i\mu_k x_1),$$

$$u_\varepsilon^{(2,2)}(x_1,x_2) \sim -\sum_{k\in\mathbb{Z}} \mu_k^2 u''_{\varepsilon k}(x_2)\exp(i\mu_k x_1).$$

Therefore, by virtue of uniqueness of Fourier series, for every $k \in \mathbb{Z}$ we get that u_k is a solution of the periodic problem

$$z'' = \rho_{\varepsilon k}(x_2)z - \frac{q_k(x_2)}{p_2(x_2)+\mu_k^2}, \qquad z(x_2+\omega_2) = z(x_2).$$

But this means that representation (5.64) is valid. $\square$

Lemma 5.14. *Let $\varepsilon > 0$, $\delta > 0$,*

$$q(x_1,x_2) \sim \sum_{k,l\in\mathbb{Z}} q_{kl}\exp(i\mu_k x_1 + i\nu_l x_2)$$

be the Fourier expansion of q, and let $u_\varepsilon \in C^{2,2}_{\omega_1\omega_2}(\mathbb{R}^2)$ be a solution of problem (5.61), (4.2). *Then u_ε admits the following representation*

$$u_\varepsilon(x_1,x_2) = \sum_{k,l\in\mathbb{Z}} \frac{q_{kl}}{\delta+\mu_k^2\nu_l^2+\varepsilon^2(\mu_k^2+\nu_l^2)}\exp(i\mu_k x_1 + i\nu_l x_2). \tag{5.66}$$

Lemma 5.14 can be proved similarly to the proof of Lemma 5.13.

Lemma 5.15. *Let for any $\varepsilon > 0$ u_ε be a solution of problem* (5.60), (4.2)*, where $p_0(x_2) \equiv -\delta < 0$, $p_1 \in C^2_{\omega_2}(\mathbb{R})$, $q \in C^{0,2}_{\omega_1\omega_2}(\mathbb{R}^2)$ and let p_1 satisfy condition* (4.20)*. Then there exists a function $u_0 \in C^{2,2}_{\omega_1\omega_2}(\mathbb{R}^2)$ such that*

$$\lim_{\varepsilon\to 0} \|u_\varepsilon - u_0\|_{C^{2,2}_{\omega_1\omega_2}} = 0. \tag{5.67}$$

Proof. By Lemma 5.13, u_ε admits representation (5.63), where $p_2(x_2) \equiv 0$. On the other hand, by Lemma 2.1, for every $k \in \mathbb{Z}$ and $\varepsilon > 0$ the function

$$u_{\varepsilon k}(x_2) = \int\limits_0^{\omega_2} g_{\varepsilon k}(x_2, s_2) \frac{q_k(s_2)}{\varepsilon^2 + \mu_k^2}\, ds_2$$

is a solution of the problem

$$z'' = \rho_{\varepsilon k}(x_2) z - \frac{q_k(x_2)}{\varepsilon^2 + \mu_k^2}, \qquad z(x_2 + \omega_2) = z(x_2),$$

where

$$\rho_{\varepsilon k}(x_2) = \frac{\delta + \mu_k^2 p_1(x_2)}{\varepsilon^2 + \mu_k^2}.$$

Therefore by Lemma 5.8, for $k = 0$ we have

$$\lim_{\varepsilon\to 0} \Big\| u_{\varepsilon 0} - \frac{q_0}{\delta} \Big\|_{C^2_{\omega_2}} = 0.$$

If $k \neq 0$, then

$$\lim_{\varepsilon\to 0} \Big\| \rho_{\varepsilon k} - \frac{\delta + \mu_k^2 p_1(x_2)}{\mu_k^2} \Big\|_{C^2_{\omega_2}} = 0,$$

and $\rho_{\varepsilon k}(x_2) \geq p_1(x_2)/2$ for sufficiently small $\varepsilon > 0$. Therefore it is easy to see that under these conditions for any $(j = 0, 1)$ and $(k \neq 0)$ we have

$$\max\Big\{ \big| g^{(j,0)}_{\varepsilon k}(x_2, s_2) - g^{(j,0)}_{0k}(x_2, s_2) \big| : (x_2, s_2) \in [0, \omega_2] \times [0, \omega_2] \Big\} \to 0 \ \text{ as } \ \varepsilon \to 0.$$

Hence (5.67) obviously follows from (5.64), where

$$u_0(x_1, x_2) = \frac{q_0(x_2)}{\delta} - \sum_{k\neq 0} \int\limits_0^{\omega_2} g_{0k}(x_2, s_2) \frac{q_k(s_2)}{\mu_k^2}\, ds_2 \, \exp(i\mu_k x_1). \quad \square$$

Lemma 5.16. *Let $q \in C^{2,2}_{\omega_1\omega_2}(\mathbb{R}^2)$ and let u_ε be a solution of problem* (5.61), (4.2) *for any $\varepsilon > 0$. Then equality* (5.67) *holds, where*

$$u_\varepsilon(x_1, x_2) = \sum_{k,l\in\mathbb{Z}} \frac{q_{kl}}{\delta + \mu_k^2 \nu_l^2} \exp(i\mu_k x_1 + i\nu_l x_2). \tag{5.68}$$

Proof. By Lemma 5.14, u_ε admits representation (5.66). On the other hand, it is clear that the series

$$\sum_{k,l\neq 0} \frac{q_{kl}}{\delta + \mu_k^2\nu_l^2 + \varepsilon^2(\mu_k^2 + \nu_l^2)} \exp(i\mu_k x_1 + i\nu_l x_2)$$

converges absolutely and uniformly with respect to $(x_1, x_2) \in \mathbb{R}^2$ and $\varepsilon \in \mathbb{R}$. Therefore in order to prove Lemma 5.16, we have to verify that the series

$$\sum_{k\in\mathbb{Z}} \frac{q_{k0}}{\delta + \varepsilon^2\mu_k^2} \exp(i\mu_k x_1) \quad \text{and} \quad \sum_{l\in\mathbb{Z}} \frac{q_{0l}}{\delta + \varepsilon^2\nu_l^2} \exp(i\nu_l x_2) \tag{5.69}$$

converge uniformly to series

$$\sum_{k\in\mathbb{Z}} \frac{q_{k0}}{\delta} \exp(i\mu_k x_1) \quad \text{and} \quad \sum_{l\in\mathbb{Z}} \frac{q_{0l}}{\delta} \exp(i\nu_l x_2),$$

respectively. But this follows from Lemma 5.16, since for $\varepsilon > 0$ series (5.69) are solutions of following problems

$$\varepsilon^2 z'' = \delta z - \sum_{k\in\mathbb{Z}} q_{k0} \exp(i\mu_k x_1), \qquad z(x_1 + \omega_1) = z(x_1),$$

$$\varepsilon^2 z'' = \delta z - \sum_{l\in\mathbb{Z}} q_{0l} \exp(i\nu_l x_2), \qquad z(x_2 + \omega_2) = z(x_2),$$

respectively. □

Finally let us prove a simple lemma concerning a special type two–parametric family of linear bounded operators in a Banach space.

Lemma 5.17. *Let* $\boldsymbol{X}$ *be a Banach space,* $\mathcal{A}_\varepsilon : \mathbf{X} \to \mathbf{X}$ $(0 \le \varepsilon \le \varepsilon_0)$ *and* $\mathcal{P} : \mathbf{X} \to \mathbf{X}$ *be linear bounded operators such that*

$$\operatorname*{s-lim}_{\varepsilon\to 0} \mathcal{A}_\varepsilon = \mathcal{A}_0 \tag{5.70}$$

and the operator $\mathcal{I} - \lambda\mathcal{A}_\varepsilon\mathcal{P}$ *has a bounded inverse for every* $\lambda \in [0, 1]$ *and* $\varepsilon \in (0, \varepsilon_0]$. *Moreover, let there exist a number* $r_0 > 1$ *such that*

$$\|\mathcal{A}_{\varepsilon\lambda}\| \le r_0 \quad \textit{for} \quad \lambda \in [0, 1], \quad \varepsilon \in (0, \varepsilon_0], \tag{5.71}$$

where $\mathcal{A}_{\varepsilon\lambda} = (\mathcal{I} - \lambda\mathcal{A}_\varepsilon\mathcal{P})^{-1}\mathcal{A}_\varepsilon$. *Then there exists a bounded linear operator* $\mathcal{A} : \boldsymbol{X} \to \boldsymbol{X}$ *such that*

$$\operatorname*{s-lim}_{\varepsilon\to 0} \mathcal{A}_{\varepsilon 1} = \mathcal{A}. \tag{5.72}$$

Proof. By Banach-Steinhaus theorem and condition (5.70), without loss of generality one may assume that

$$\|\mathcal{A}_\varepsilon\| \le r_0, \quad \|\mathcal{A}_\varepsilon \mathcal{P}\| \le r_0 \quad \text{for} \quad \varepsilon \in [0, \varepsilon_0]. \tag{5.73}$$

Choose a natural number $l > 2r_0^2$. Put

$$\lambda_j = \frac{j}{l} \quad (j = 0, \dots, l).$$

Inequalities (5.71) and (5.73) imply

$$\|(\lambda - \lambda_j)^k (\mathcal{A}_\varepsilon \mathcal{P})^k \mathcal{A}_{\varepsilon\lambda_j}^{k+1}\| \le r_0 2^{-k}, \quad \mathcal{A}_{\varepsilon\lambda} = \sum_{k=0}^{+\infty} (\lambda - \lambda_j)^k (\mathcal{A}_\varepsilon \mathcal{P})^k \mathcal{A}_{\varepsilon\lambda_j}^{k+1} \mathcal{A}_\varepsilon$$
$$\text{for} \quad \lambda_{j-1} \le \lambda \le \lambda_j \quad j \in \{1, \dots, l\}, \quad \varepsilon \in [0, \varepsilon_0]. \tag{5.74}$$

Taking into the account condition (5.70), by the principle of mathematical induction from (5.74) we get (5.72), where $\mathcal{A} : \mathbf{X} \to \mathbf{X}$ is a bounded linear operator. □

6. Proof of the Main Results

Proof of Theorem 4.1. Let problem (4.1),(4.2) be well–posed. Assume the contrary: let problem $(4.1_2), (4.2_2)$ have a nontrivial solution $\xi_0(x_1)$ for some $x_2^* \in [0, \omega_2]$. Then due to the well–posedness of problem (1.1),(1.2) there exist $\delta > 0$ and $\widetilde{p}_{j\,m_2} \in C^{(0,m_2)}(\Omega)$ $(j = 0, \dots, m_1 - 1)$ such that

$$\widetilde{p}_{j\,m_2}(x_1, x_2) = p_{j\,m_2}(x_1, x_2^*) \quad \text{for} \quad x_2 \in [x_2^* - \delta, x_2^* + \delta] \cap [0, \omega_2] \quad (j = 0, \dots, m_1 - 1),$$

and the problem

$$u^{(m_1,m_2)} = \sum_{k=0}^{m_2-1} p_{m_1\,k}(\mathbf{x}) u^{(m_1,k)} + \sum_{j=0}^{m_1-1} \widetilde{p}_{j\,m_2}(\mathbf{x}) u^{(j,m_2)} + \sum_{\boldsymbol{\alpha} \le \mathbf{m-1}} p_{\boldsymbol{\alpha}}(\mathbf{x}) u^{(\boldsymbol{\alpha})}, \tag{6.1}$$

$$u(x_1 + \omega_1, x_2) = u(x_1, x_2), \qquad u(x_1, x_2 + \omega_2) = u(x_1, x_2) \tag{6.2}$$

is well–posed. From Remark 5.1 it follows that the problem

$$u^{(m_1,m_2)} = \sum_{k=0}^{m_2-1} p_{m_1\,k}(\mathbf{x}) u^{(m_1,k)} + \sum_{j=0}^{m_1-1} \widetilde{p}_{j\,m_2}(\mathbf{x}) u^{(j,m_2)} + \sum_{\boldsymbol{\alpha} \le \mathbf{m-1}} \widetilde{p}_{\boldsymbol{\alpha}}(\mathbf{x}) u^{(\boldsymbol{\alpha})}, \tag{6.3}$$

$$u(x_1 + \omega_1, x_2) = u(x_1, x_2), \tag{6.4}$$

where

$$\widetilde{p}_{j\,k}(\mathbf{x}) = - \sum_{i=k}^{m_2-1} \frac{k!}{i!(i-k)!} p_{m_1\,i}(\mathbf{x}) \, \widetilde{p}_{j\,m_2}^{(0,i-k)}(\mathbf{x})$$
$$+ \frac{m_2!}{k!(m_2-k)!} \widetilde{p}_{j\,m_2}^{(0,m_2-k)}(\mathbf{x}) \quad (j = 0, \dots, m_1 - 1; \ k = 0, \dots m_2 - 1),$$

has the Fredholm property. On the other hand, an arbitrary solution u of problem $(6.3), (6.4)$ is a solution of the problem

$$\Big(u^{(\mathbf{m}_1)} - \sum_{j=0}^{m_1-1} \widetilde{p}_{j\,m_2}(\mathbf{x}) u^{(j,0)}\Big)^{(0,m_2)} = \sum_{k=0}^{m_2-1} p_{m_1\,k}(\mathbf{x}) \Big(u^{(\mathbf{m}_1)} - \sum_{j=0}^{m_1-1} \widetilde{p}_{j\,m_2}(\mathbf{x}) u^{(j,0)}\Big)^{(0,k)},$$

$$u^{(\mathbf{m}_1)}(x_1, x_2 + \omega_2) - \sum_{j=0}^{m_1-1} \widetilde{p}_{j\,m_2}(x_1, x_2 + \omega_2) u^{(j,0)}(x_1, x_2 + \omega_2)$$
$$= u^{(\mathbf{m}_1)}(x_1, x_2) - \sum_{j=0}^{m_1-1} \widetilde{p}_{j\,m_2}(x_1, x_2) u^{(j,0)}(x_1, x_2).$$

Hence, every solution $u \in C^{\mathbf{m}}_{\boldsymbol{\omega}}(\mathbb{R}^2)$ of the problem

$$u^{(\mathbf{m}_1)} = \sum_{j=0}^{m_1-1} \widetilde{p}_{j\,m_2}(\mathbf{x})u^{(j,0)}, \tag{6.5}$$

$$u(x_1+\omega_1, x_2) = u(x_1, x_2) \tag{6.6}$$

is a solution of problem (6.3), (6.4).

Let $\gamma \in C^{\infty}_{\omega_2}(\mathbb{R})$, $\operatorname{supp}\gamma \cap [0, \omega_2] \subset [x_2^* - \delta, x_2^* + \delta]$ be an arbitrary function. Then

$$u(\mathbf{x}) = \xi_0(x_1)\gamma(x_2) \tag{6.7}$$

is a solution of problem (6.5),(6.6), and, consequently, is a solution of the problem (6.3), (6.4). Thus problem (6.3), (6.4) has an infinite dimensional space of solutions, which contradicts to the fact that is has the Fredholm property. The obtained contradiction completes the proof of the theorem. □

Proof of Theorem 4.2. Theorem 4.2 follows from Lemma 5.2 and Remark 5.1, in particular, representations (5.5) and (5.14).

Proof of Theorem 4.3. The proof of Theorem 4.3 is similar to the proof of Theorem 1.5.

Proof of Theorem 4.4. First let us prove solvability of problem (4.1), (4.2) under the assumption of the homogeneous problem (4.1_0), (4.2) having only the trivial solution.

Let $\lambda \in [0, 1]$, and u be a solution of problem (4.8), (4.2). Set:

$$\gamma_i = (-1)^{[\frac{m_i}{2}]-1} \quad ([t] \text{ is the integer part of } t), \tag{6.8}$$

$$v_j(\mathbf{x}) = u^{(\mathbf{m_j})}(\mathbf{x}) - \gamma_j\, u(\mathbf{x}) \qquad (j = 1, \dots, n). \tag{6.9}$$

Then, v_j is a solution of $n-1$ dimensional problem depending on the parameter $x_j \in [0, \omega_j]$ $(j = 1, \dots, n):$

$$v^{(\widehat{\mathbf{m}}_j)} = \sum_{\boldsymbol{\alpha}<\widehat{\mathbf{m}}_j} p_{\lambda\, \mathbf{m}_j+\boldsymbol{\alpha}}(\mathbf{x})v^{(\boldsymbol{\alpha})} + Q_j[u](\mathbf{x}), \tag{6.10}$$

$$v(\mathbf{x}+\boldsymbol{\omega}_i) = v(\mathbf{x}) \quad (i \neq j). \tag{6.11}$$

where

$$Q_j[u](\mathbf{x}) = \sum_{\boldsymbol{\alpha}<\mathbf{m}_j} \sum_{\boldsymbol{\beta}<\widehat{\mathbf{m}}_j} p_{\lambda\,\boldsymbol{\alpha}+\boldsymbol{\beta}}(\mathbf{x}) u^{(\boldsymbol{\alpha}+\boldsymbol{\beta})}$$

$$+\gamma_j \sum_{\boldsymbol{\alpha}<\widehat{\mathbf{m}}_j} p_{\lambda\,\boldsymbol{m}_j+\boldsymbol{\alpha}} u^{(\boldsymbol{\alpha})} + q^{(\mathbf{m})}(\mathbf{x}) \quad (j=1,\dots,n). \tag{6.12}$$

We prove the theorem by induction. The validity of Theorem 4.4 for $n = 2$ was already proved (see Theorem 4.2). Let $n \geq 3$, and let us assume that the theorem is true for $n-1$ dimensional problem. Then each problem (6.10)(6.11) is α–well–posed. Consequently, u admits the following representations:

$$u^{(\mathbf{m_j})}(\mathbf{x}) = \gamma_j\, u(\mathbf{x}) + \mathcal{G}_j(Q_j[u])(\mathbf{x}) \qquad (j=1,\dots,n), \tag{6.13}$$

where $\mathcal{G}_j$ is the Green's operator of the associated problem

$$v^{(\widehat{\mathbf{m}}_j)} = \sum_{\boldsymbol{\alpha}<\widehat{\mathbf{m}}_j} p_{\lambda\,\mathbf{m}_j+\boldsymbol{\alpha}}(\mathbf{x}) v^{(\boldsymbol{\alpha})}, \tag{6.14}$$

$$v(\mathbf{x}+\boldsymbol{\omega}_i) = v(\mathbf{x}) \quad (i \neq j). \tag{6.15}$$

In view of differentiability of coefficients p_α and Lemmas 5.4 and 5.5, from (6.12) and (6.13) we get

$$u^{(\mathbf{m_j})}(\mathbf{x}) = \gamma_j\, u(\mathbf{x}) + q^{(\mathbf{m_j})}(\mathbf{x}) + \mathcal{A}_j^\circ(u)(\mathbf{x}) + \mathcal{B}_j^\circ(q)(\mathbf{x}) \quad (j=1,\dots,n), \tag{6.16}$$

where $\mathcal{A}_j^\circ$ and $\mathcal{B}_j^\circ : C_{\boldsymbol{\omega}}(\mathbb{R}^n) \to C_{\boldsymbol{\omega}}(\mathbb{R}^n)$ are bounded linear operators. Hence

$$u(\mathbf{x}) = \mathcal{A}_j(u)(\mathbf{x}) + \mathcal{B}_j(q)(\mathbf{x}) \quad (j=1,\dots,n), \tag{6.17}$$

where

$$\mathcal{A}_j(u)(\mathbf{x}) = \int_0^{\omega_j} g_j(x_j,s_j)\mathcal{A}_j^\circ(u)(s_j,\hat{x}_j)ds_j \quad (j=1,\dots,n), \tag{6.18}$$

$$\mathcal{B}_j(q)(\mathbf{x}) = \int_0^{\omega_j} g_j(x_j,s_j)\big(q^{(\mathbf{m}_j)}(s_j,\widehat{\mathbf{x}}_j) + \mathcal{B}_j^\circ(q)(s_j,\widehat{\mathbf{x}}_j)\big)\, ds_j \quad (j=1,\dots,n), \tag{6.19}$$

and g_j is the Green's function of the problem

$$z^{(m_j)} = \gamma_j z, \quad z(x_j+\omega_j) = z(x_j). \tag{6.20}$$

Notice, that by Lemma 5.5, $\mathcal{B}_j$ is a bounded linear operator from $C_{\boldsymbol{\omega}}(\mathbb{R}^n)$ to $C_{\boldsymbol{\omega}}(\mathbb{R}^n)$.

(6.17) implies the representation

$$u(\mathbf{x}) = \mathcal{A}(u)(\mathbf{x}) + \mathcal{B}(q)(\mathbf{x}), \tag{6.21}$$

where

$$\mathcal{A} = \mathcal{A}_1 \circ \mathcal{A}_2 \circ \ldots \circ \mathcal{A}_n, \tag{6.22}$$

$$\mathcal{B} = \mathcal{B}_1 + \mathcal{A}_1 \circ \mathcal{B}_2 + \mathcal{A}_1 \circ \mathcal{A}_2 \circ \mathcal{B}_3 + \ldots + \mathcal{A}_1 \circ \mathcal{A}_2 \ldots \circ \mathcal{A}_{n-1} \circ \mathcal{B}_n. \tag{6.23}$$

From the construction of the operator $\mathcal{A}$ it is clear that $\mathcal{A} : C_{\boldsymbol{\omega}}(\mathbb{R}^n) \to C^{\mathbf{m}}_{\boldsymbol{\omega}}(\mathbb{R}^n)$ is a bounded linear operator. Consequently, $\mathcal{A} : C_{\boldsymbol{\omega}}(\mathbb{R}^n) \to C_{\boldsymbol{\omega}}(\mathbb{R}^n)$ is a compact operator.

Furthermore, (6.16) yields the estimates

$$\|u^{(\mathbf{m}_j+\boldsymbol{\beta})}\|_{C_{\boldsymbol{\omega}}} \le M\Big(\|q^{(\mathbf{m}_j+\boldsymbol{\beta})}\|_{C_{\boldsymbol{\omega}}} + \|q\|_{C_{\boldsymbol{\omega}}} + \|u\|_{C_{\boldsymbol{\omega}}}\Big) \qquad \boldsymbol{\beta} \le \widehat{\mathbf{m}}_{\boldsymbol{j}} - \widehat{\mathbf{1}}_j \quad (j = 1, \ldots, n), \tag{6.24}$$

where M is a positive number independent of u and q.

Now set

$$w_{ij}(\mathbf{x}) = u^{(\mathbf{m_{ij}})}(\mathbf{x}) \quad (\mathbf{m_{ij}} = \mathbf{m_i} + \mathbf{m_j}). \tag{6.25}$$

Then w_{ij} is a solution of the $n-2$ dimensional problem

$$w^{(\widehat{\mathbf{m}}_{ij})} = \sum_{\boldsymbol{\alpha} < \widehat{\mathbf{m}}_{ij}} p_{\lambda \mathbf{m}_{ij} + \boldsymbol{\alpha}}(\mathbf{x}) w^{(\boldsymbol{\alpha})} + Q_{ij}[u](\mathbf{x}), \tag{6.26}$$

$$w(\mathbf{x} + \boldsymbol{\omega}_l) = w(\mathbf{x}) \quad (l \ne i,\, j). \tag{6.27}$$

where

$$Q_{ij}[u](\mathbf{x}) = \sum_{\alpha < \mathbf{m}_{ij}} \sum_{\beta \le \widehat{\mathbf{m}}_{ij}} p_{\lambda\, \boldsymbol{\alpha} + \boldsymbol{\beta}}(\mathbf{x}) u^{(\alpha+\beta)} + q^{(\mathbf{m})}(\mathbf{x}).$$

Problem (6.26),(6.27) satisfies all of the conditions of Theorem 1.6. Since (6.26),(6.27) is an $n-2$ dimensional problem, by our assumption, it is α–well–posed. Therefore we have

$$u^{(\mathbf{m_{ij}})} = \mathcal{G}_{ij}\big(Q_{ij}[u]\big)(\mathbf{x}), \tag{6.28}$$

where $\mathcal{G}_{ij}$ is the Green's operator of the associated problem

$$w^{(\widehat{\mathbf{m}}_{ij})} = \sum_{\alpha<\widehat{\mathbf{m}}_{ij}} p_{\lambda\mathbf{m}_{ij}+\alpha}(\mathbf{x}) w^{(\alpha)}, \tag{6.29}$$

$$w(\mathbf{x}+\boldsymbol{\omega}_l) = w(\mathbf{x}) \quad (l \neq i,\, j). \tag{6.30}$$

By Lemma 5.5, we get:

$$u^{(\mathbf{m_{ij}})}(\mathbf{x}) \;=\; q^{(\mathbf{m_{ij}})}(\mathbf{x}) \;+\; \sum_{\boldsymbol{\alpha}<\mathbf{m}_{ij}} p_{\lambda\widehat{\mathbf{m}}_{ij}+\boldsymbol{\alpha}}(\mathbf{x}) u^{(\boldsymbol{\alpha})} \;+\; \mathcal{A}^{\circ}_{ij}(u)(\mathbf{x}) \;+\; \mathcal{B}^{\circ}_{ij}(q)(\mathbf{x}), \tag{6.31}$$

where $\mathcal{A}^{\circ}_{ij}$ and $\mathcal{B}^{\circ}_{ij} : C_{\boldsymbol{\omega}}(\mathbb{R}^n) \to C_{\boldsymbol{\omega}}(\mathbb{R}^n)$ are bounded linear operators.

Continuing this process step-by-step, one can obtain similar representation

$$u^{(\mathbf{m}_{\boldsymbol{\sigma}})}(\mathbf{x}) = q^{(\mathbf{m}_{\boldsymbol{\sigma}})}(\mathbf{x}) + \sum_{\boldsymbol{\alpha}<\mathbf{m}_{\boldsymbol{\sigma}}} p_{\lambda\mathbf{m}_{\widehat{\boldsymbol{\sigma}}}+\boldsymbol{\alpha}}(\mathbf{x}) u^{(\boldsymbol{\alpha})} + \mathcal{A}^{\circ}_{\boldsymbol{\sigma}}(u)(\mathbf{x}) + \mathcal{B}^{\circ}_{\boldsymbol{\sigma}}(q)(\mathbf{x}) \quad (\boldsymbol{\sigma} \in \boldsymbol{\Xi}), \tag{6.32}$$

where $\mathcal{A}^{\circ}_{\boldsymbol{\sigma}}$ and $\mathcal{B}^{\circ}_{\boldsymbol{\sigma}} : C_{\boldsymbol{\omega}}(\mathbb{R}^n) \to C_{\boldsymbol{\omega}}(\mathbb{R}^n)$ $(\boldsymbol{\sigma} \in \boldsymbol{\Xi})$ are bounded linear operators.

(6.24), (6.31) and (6.32) yield the estimates

$$\|u^{(\mathbf{m}_{\boldsymbol{\sigma}}+\boldsymbol{\beta})}\|_{C_{\boldsymbol{\omega}}} \leq M\Big(\|q^{(\mathbf{m}_{\boldsymbol{\sigma}}+\boldsymbol{\beta})}\|_{C_{\boldsymbol{\omega}}} + \|q\|_{C_{\boldsymbol{\omega}}} + \|u\|_{C_{\boldsymbol{\omega}}}\Big) \qquad \boldsymbol{\beta} \leq \mathbf{m}_{\widehat{\boldsymbol{\sigma}}} - \widehat{\boldsymbol{\sigma}} \quad (\boldsymbol{\sigma} \in \boldsymbol{\Xi}), \tag{6.33}$$

and, hence,

$$\|u\|_{C^{\mathbf{m}}_{\boldsymbol{\omega}}} \leq M\Big(\|q\|_{C^{\mathbf{m}}_{\boldsymbol{\omega}}} + \|u\|_{C_{\boldsymbol{\omega}}}\Big), \tag{6.34}$$

where M is a positive number independent of u and q.

Now we are ready to prove solvability of problem (4.1), (4.2).

Problem (4.8), (4.2) is well–posed for $\lambda = 0$. Therefore it is well–posed for $\lambda \in [0, \delta]$ for some $\delta \leq 1$. Our goal is to prove that $\delta = 1$. Assume the contrary. Then there exists $\lambda_0 \leq 1$ and a sequence $\lambda_k \nearrow \lambda_0$ such that

$$\lim_{k\to+\infty} \|u_k\|_{C^{\mathbf{m}}_{\boldsymbol{\omega}}} = +\infty, \tag{6.35}$$

where u_k is a solution of problem (4.8), (4.2) for $\lambda = \lambda_k$. Set:

$$\eta_k = \|u_k\|_{C^{\mathbf{m}}_{\boldsymbol{\omega}}}, \qquad \widetilde{u}_k(\mathbf{x}) = \frac{u_k(\mathbf{x})}{\eta_k}. \tag{6.36}$$

Then

$$\|\widetilde{u}_k\|_{C^{\mathbf{m}}_{\boldsymbol{\omega}}} = 1 \quad (k = 1, 2, \dots) \tag{6.37}$$

and $\widetilde{u_k}$ is a solution of the problem

$$u^{(\mathbf{m})} = \sum_{\boldsymbol{\alpha}<\mathbf{m}} p_{\lambda\,\boldsymbol{\alpha}}(\mathbf{x})u^{(\boldsymbol{\alpha})} + \frac{q(\mathbf{x})}{\eta_k}, \tag{6.38}$$

$$u(\mathbf{x} + \boldsymbol{\omega}_i) = u(\mathbf{x}) \quad (i = 1, \dots, n). \tag{6.39}$$

By virtue of (6.34),(6.37) and Arzela–Ascoli lemma, without loss of generality, one may assume that there exists $\widetilde{u} \in C^{\mathbf{m}}(\omega)$ such that

$$\lim_{k\to+\infty} \|\widetilde{u} - \widetilde{u}_k\|_{C^{\mathbf{m}}_{\boldsymbol{\omega}}} = 0 \tag{6.40}$$

and

$$\|\widetilde{u}\|_{C^{\mathbf{m}}_{\boldsymbol{\omega}}} = 1. \tag{6.41}$$

On the other hand, it is clear that $\widetilde{u}$ is a nonzero solution of problem $(4.8_0), (4.2)$ for $\lambda = \lambda_0$, which contradicts to the assumption that the homogenous problem $(4.8_0), (4.2)$ has only the trivial solution for every $\lambda \in [0, 1]$. The obtained contradiction proves the solvability of problem (4.1),(1.2) under the assumption that the homogeneous problem $(4.1_0), (4.2)$ has only the trivial solution.

At the same time,along with solvability of problem $(4.1), (4.2)$ we have also proved its α–well–posedness, by establishing the estimates (6.33) and (6.34).

In order to complete the proof, it remains to show that problem $(4.1_0), (4.2)$ has a finite dimensional space of solution. This follows from the fact, that every solution of problem $(4.1_0), (4.2)$ is also a solution of the equation

$$u(\mathbf{x}) = \mathcal{A}(u)(\mathbf{x}), \tag{6.42}$$

where $\mathcal{A} : C_{\boldsymbol{\omega}}(\mathbb{R}^n) \to C_{\boldsymbol{\omega}}(\mathbb{R}^n)$ is the compact operator defined by (6.22). $\square$

Proof of Theorem 4.5. Set:

$$\mathcal{L}_i = u^{(2m_i)} + (-1)^{m_i}u, \tag{6.43}$$

$$\mathcal{L} = \mathcal{L}_1 \circ \cdots \circ \mathcal{L}_n. \tag{6.44}$$

Let constants $p_{0\,\boldsymbol{\alpha}}$ are such that

$$\mathcal{L}u = u^{(2\mathbf{m})} - \sum_{\boldsymbol{\alpha}<\mathbf{m}} p_{0\,\boldsymbol{\alpha}} u^{(\mathbf{2}\boldsymbol{\alpha})}. \tag{6.45}$$

It is easy to verify that

$$(-1)^{\|\mathbf{m}\|+\|\boldsymbol{\alpha}\|-1} p_{0\,\boldsymbol{\alpha}} > 0. \tag{6.46}$$

Furthermore, since the problem

$$z^{(2m_i)} = (-1)^{m_i-1} z, \qquad z(x_i+\omega_i) = z(x_i) \tag{6.47}$$

has only the trivial solution for each $i \in \{1,\dots,n\}$, then, in view of (6.44), (6.45), by Theorems 4.3 and Lemma 5.4, the problem

$$u^{(2\mathbf{m})} = \sum_{\boldsymbol{\alpha}<\mathbf{m}} p_{0\,\boldsymbol{\alpha}} u^{(\mathbf{2}\boldsymbol{\alpha})} + q(\mathbf{x}), \tag{6.48}$$

$$u(\mathbf{x}+\boldsymbol{\omega}_i) = u(\mathbf{x}) \quad (i=1,\dots,n) \tag{6.49}$$

is α–well–posed.

Now consider the equations

$$u^{(2\mathbf{m})} = (1-\lambda) \sum_{\boldsymbol{\alpha}<\mathbf{m}} p_{0\,\boldsymbol{\alpha}} u^{(\mathbf{2}\boldsymbol{\alpha})} + \lambda \sum_{\boldsymbol{\alpha}+\boldsymbol{\beta}<2\mathbf{m}} \big(p_{\boldsymbol{\alpha}+\boldsymbol{\beta}}(\mathbf{x}) u^{(\boldsymbol{\alpha})}\big)^{(\boldsymbol{\beta})} + q(\mathbf{x}) \tag{6.50}$$

and

$$u^{(2\mathbf{m})} = (1-\lambda) \sum_{\boldsymbol{\alpha}<\mathbf{m}} p_{0\,\boldsymbol{\alpha}} u^{(\mathbf{2}\boldsymbol{\alpha})} + \lambda \sum_{\boldsymbol{\alpha}+\boldsymbol{\beta}<2\mathbf{m}} \big(p_{\boldsymbol{\alpha}+\boldsymbol{\beta}}(\mathbf{x}) u^{(\boldsymbol{\alpha})}\big)^{(\boldsymbol{\beta})}. \tag{6.50$_0$}$$

Let us show that the homogeneous problem $(6.50_0), (6.49)$ has only the trivial solution for each $\lambda \in [0,1]$. Indeed, let u be an arbitrary solution of problem $(6.50_0), (6.49)$. Multiply equation (6.50_0) by u and integrate over Ω. After integrating by parts multiple times and taking into account conditions (6.49), we get:

$$(-1)^{\|\mathbf{m}\|} \iint\limits_{\Omega} \big|u^{(\mathbf{m})}(\mathbf{x})\big|^2 \, d\mathbf{x} = (1-\lambda) \iint\limits_{\Omega} \sum_{\boldsymbol{\alpha}<\mathbf{m}} (-1)^{\|\boldsymbol{\alpha}\|} p_{0\,\boldsymbol{\alpha}} \big|u^{(\boldsymbol{\alpha})}(\mathbf{x})\big|^2 \, d\mathbf{x}$$
$$+ \lambda \iint\limits_{\Omega} \sum_{\boldsymbol{\alpha}+\boldsymbol{\beta}<2\mathbf{m}} (-1)^{\|\boldsymbol{\beta}\|} p_{\boldsymbol{\alpha}+\boldsymbol{\beta}}(\mathbf{x}) u^{(\boldsymbol{\alpha})}(\mathbf{x}) u^{(\boldsymbol{\alpha})}(\mathbf{x}) \, d\mathbf{x}. \tag{6.51}$$

(4.12), (6.46) and (6.51) immediately imply

$$u(\mathbf{x}) \equiv 0. \tag{6.52}$$

By Theorem 4.4, problem (4.9),(4.2) is well–posed. Moreover, if $p_{\boldsymbol{\alpha}+\boldsymbol{\beta}} \in C_{\boldsymbol{\omega}}^{\boldsymbol{\alpha}+\boldsymbol{\beta}}(\mathbb{R}^n)$ $(\boldsymbol{\alpha}+\boldsymbol{\beta} < 2\boldsymbol{m})$, then problem (4.9), (4.2) is α–well–posed. $\square$

Proof of Theorem 4.6. The proof is almost identical to the proof of Theorem 4.5.

Proof of Theorem 4.7. The proof is almost identical to the proof of Theorem 4.5.

Proof of Theorem 4.8. Let us prove first the existence of a solution to problem (4.17),(4.2). For the sake of technical simplicity, consider the particular case $m_1 = m_2 = 1$. The proof of the general case is similar.

Assume that conditions (5.50) hold. By Theorem 4.2 and Lemma 5.11, for every $\varepsilon > 0$ and $\lambda \in [0,1]$ the equation

$$u^{(2,2)} = \big(-\delta + \lambda(p_0(x_1,x_2)+\delta)\big)u + p_1(x_2)u^{(2,0)} + \varepsilon^2 u^{(0,2)} + q(x_1,x_2) \tag{6.53}$$

has a unique solution $u_{\varepsilon\lambda} \in C^{2,4}_{\omega_1\omega_2}(\mathbb{R}^2)$. By $\mathcal{A}_{\varepsilon\lambda}$ denote the operator assigning to every $q \in C^{2,2}_{\omega_1\omega_2}(\mathbb{R}^2)$ a solution $u_{\varepsilon\lambda}$ of problem (6.53), (4.2), i.e.,

$$u_{\varepsilon\lambda}(x_1,x_2) = \mathcal{A}_{\varepsilon\lambda}(q)(x_1,x_2). \tag{6.54}$$

According to Lemma 5.11, $\mathcal{A}_{\varepsilon\lambda}$ is a compact operator from $C^{2,2}_{\omega_1\omega_2}(\mathbb{R}^2)$ into itself. Moreover,

$$\mathcal{A}_{\varepsilon\lambda}(q)(x_1,x_2) = \mathcal{A}_{\varepsilon}\big(q + \lambda(p_0+\delta)\mathcal{A}_{\varepsilon\lambda}(q)\big)(x_1,x_2),$$

where $\mathcal{A}_\varepsilon = \mathcal{A}_{\varepsilon 0}$. Therefore

$$\mathcal{A}_{\varepsilon\lambda} = \mathcal{A}_\varepsilon + \lambda\mathcal{A}_\varepsilon\mathcal{P}\mathcal{A}_{\varepsilon\lambda}, \tag{6.55}$$

where $\mathcal{P}$ is the operator of multiplication by the function $p_0 + \delta$.

Note that by Theorem 4.2, for every $\lambda \in [0,1]$ the equation

$$u = \lambda\mathcal{A}_\varepsilon\mathcal{P}(u)$$

has only a trivial solution. Therefore, due to compactness of $\mathcal{A}_\varepsilon$, $\mathcal{I}-\lambda\mathcal{A}_\varepsilon\mathcal{P}$ has a bounded inverse. By this fact from, (6.55) we get

$$\mathcal{A}_{\varepsilon\lambda}=(\mathcal{I}-\lambda\mathcal{A}_\varepsilon\mathcal{P})^{-1}\mathcal{A}_\varepsilon.$$

By Lemma 5.11, condition (5.71) holds, where r_0 and ε_0 are positive numbers independent of λ and ε. On the other hand by Lemma 5.15 , condition (5.70) holds in the space $C^{2,2}_{\omega_1\omega_2}(\mathbb{R}^2)$. Therefore by Lemma 5.17, there exists a bounded linear operator $\mathcal{A}: C^{2,2}_{\omega_1\omega_2}(\mathbb{R}^2)\to C^{2,2}_{\omega_1\omega_2}(\mathbb{R}^2))$ satisfying (5.72), i.e., there exists $u\in C^{2,2}_{\omega_1\omega_2}(\mathbb{R}^2)$ such that

$$\lim_{\varepsilon\to 0}\|u_\varepsilon-u\|_{C^{2,2}_{\omega_1\omega_2}}=0, \tag{6.56}$$

where u_ε is a solution of problem (4.17_ε), (4.2). It is clear that u is a solution of problem (4.17),(4.2).

Finally let us show that problem (4.17),(4.2) has at most one solution, i.e., the equation

$$u^{(2,2)}=p_0(x_1,x_2)u+p_1(x_2)u^{(2,0)} \tag{6.57}$$

has only the trivial solution satisfying conditions (4.2). Indeed, let u be a solution of problem (6.57),(4.2). Multiplying (6.57) by $u(x,y)$ and integrating over the rectangle $[0,\omega_1]\times[0,\omega_2]$, we get

$$\int_0^{\omega_1}\int_0^{\omega_2}\left|u^{(1,1)}(x_1,x_2)\right|^2dx_2\,dx_1=\int_0^{\omega_1}\int_0^{\omega_2}p_0(x_1,x_2)u^2(x_1,x_2)\,dx_2\,dx_1$$
$$-\int_0^{\omega_1}\int_0^{\omega_2}p_1(x_2)\left|u^{(1,0)}(x_1,x_2)\right|^2dx_2\,dx_1.$$

From the latter equality, in view of (4.20) (4.21), we get $u(x,y)\equiv 0$. $\square$

The proof of Theorem 4.8 is similar to the proof of Theorem 4.9. The only difference is that instead of equation (3.8) the equation

$$u^{\mathbf{2m}}=(-\delta+\lambda(p_0(\mathbf{x})+\delta))u+\varepsilon^2u^{(2,0)}+\varepsilon^2u^{(0,2)}+q(\mathbf{x}),$$

should be considered, and instead of Lemmas 5.11 and 5.15, Lemmas 5.12 and 5.16 should be used.

CHAPTER III
Dirichlet type problem in a smooth convex domain

7. Formulation of the Main Results

Let D be a convex smooth domain ***inscribed*** in the rectangle $\Omega = [0,\omega_1]\times[0,\omega_2]$ such that

$$D = \{(x_1,x_2)\in\Omega : x_1\in[0,\omega_1],\ \ x_2\in[\gamma_1(x_1),\gamma_2(x_1)]\}$$
$$= \{(x_1,x_2)\in\Omega : x_2\in[0,\omega_2],\ \ x_1\in[\eta_1(x_2),\eta_2(x_2)]\},$$

where $\gamma_i\in C^2([0,\omega_1])$, $\eta_i\in C^2([0,\omega_2])$ $(i=1,2)$, and

$$\gamma_1(x_1^*)=0,\quad \gamma_2(x_2^*)=\omega_2,\quad \eta_1(y_1^*)=0,\quad \eta_2(y_2^*)=\omega_1$$

for some x_1^*, $x_2^*\in[0,\omega_1]$ and y_1^*, $y_2^*\in[0,\omega_2]$.

Consider the problem

$$u^{(2,2)} = \big(p_1(x_1,x_2)u^{(1,0)}\big)^{(1,0)} + \big(p_2(x_1,x_2)u^{(0,1)}\big)^{(0,1)} + p_0(x_1,x_2)u + q(x_1,x_2), \tag{7.1}$$

$$u(\eta_k(x_2),x_2)=\varphi_{1k}(x_2)\quad (k=1,2);\quad u^{(2,0)}(x_1,\gamma_k(x_1))=\varphi''_{2k}(x_1)\quad (k=1,2), \tag{7.2}$$

and its corresponding homogeneous problem

$$u^{(2,2)} = \big(p_1(x_1,x_2)u^{(1,0)}\big)^{(1,0)} + \big(p_2(x_1,x_2)u^{(0,1)}\big)^{(0,1)} + p_0(x_1,x_2)u, \tag{7.1$_0$}$$

$$u(\eta_k(x_2),x_2)=0\quad (k=1,2);\quad u^{(2,0)}(x_1,\gamma_k(x_1))=0\quad (k=1,2). \tag{7.2$_0$}$$

Theorem 7.1. *Let $p_1\in C^{2,0}(\Omega)$, $p_2\in C^{0,2}(\Omega)$, p_0, $q\in C(\Omega)$, $\varphi_{1k}\in C^2([0,\omega_2])$, $\varphi_{2k}\in C^2([0,\omega_1])$ $(k=1,2)$, and*

$$p_1(x_1,x_2)\ge 0,\qquad p_2(x_1,x_2)\ge 0\quad \text{for}\quad (x_1,x_2)\in D. \tag{7.3}$$

then problem (7.1), (7.2) *has the Fredholm property. Moreover, if in addition*

$$p_0(x_1,x_2)\le 0\quad \text{for}\quad (x_1,x_2)\in D, \tag{7.4}$$

then problem (7.1), (7.2) *is uniquely solvable.*

Remark 7.1. Notice that the solution of problem (7.1),(7.2) is defined not only in the domain D, but everywhere in Ω.

Remark 7.2. Smoothness of the boundary of the domain D is very important and cannot be relaxed. Indeed, let $D = [0,2] \times [0,1] \cup [0,1] \times [1,2]$. Consider the equation

$$u^{(2,2)} = 1. \tag{7.5}$$

Problems $(7.5), (7.2_0)$ only possible solution is

$$u(x_1, x_2) = \begin{cases} \int_0^2 g_2(x_1, s) u^{(2,0)}(s, x_2)\, ds & \text{for} \quad x_2 \in [0,1) \\ \int_0^1 g_1(x_1, s) u^{(2,0)}(s, x_2)\, ds & \text{for} \quad x_2 \in (1,2] \end{cases},$$

where

$$g_k(t, \tau) = \begin{cases} \frac{\tau(t-k)}{k} & \text{for} \quad 0 \le \tau \le t \\ \frac{t(\tau-k)}{k} & \text{for} \quad t \le \tau \le k \end{cases} \qquad (k = 1, 2).$$

One can see, that $u(x_1, x_2)$ is not a classical solution, since it is discontinuous along the line segment $0 \le x_1 \le 1,\ x_2 = 1$, and

$$u^{(2,0)}(x_1, x_2) = \begin{cases} \int_0^2 g_2(x_2, t)\, dt = \frac{x_2(x_2-2)}{2} & \text{for} \quad x_2 \in [0,1) \\ \int_0^1 g_1(x_2, t)\, dt = \frac{x_2(x_2-1)}{2} & \text{for} \quad x_2 \in (1,2] \end{cases},$$

is discontinuous along the line segment $x_1 = 1,\ 0 \le x_2 \le 1$.

Now let $n = 3$, and $\Pi = D \times [0, \omega_3]$, where D is the same as in Theorem 7.1, be the cylindrical domain inscribed in $\Omega = [0,\omega_1] \times [0,\omega_2] \times [0,\omega_3]$. Consider the initial–boundary value problem

$$\begin{aligned} u^{(2,2,m)} = \big(p_1(\mathbf{x}) u^{(1,0,m)}\big)^{(1,0,0)} + \big(p_2(\mathbf{x}) u^{(0,1,m)}\big)^{(0,1,0)} + p_0(\mathbf{x}) u^{(0,0,m)} \\ + \sum_{i=0}^{1} \sum_{j=0}^{1} \sum_{k=0}^{m-1} p_{ijk}(\mathbf{x}) u^{(i,j,k)} + q(\mathbf{x}), \end{aligned} \tag{7.6}$$

$$\begin{aligned} u(\eta_k(x_2), x_2, x_3) = \varphi_{1k}(x_2, x_3); \ \ u^{(2,0)}(x_1, \gamma_k(x_1), x_3) = \varphi_{2k}^{(2,0)}(x_1, x_3)\ (k = 1,2), \\ u^{(2,2,k-1)}(x_1, x_2, 0) = \varphi_{3k}^{(2,2)}(x_1, x_2) \quad (k = 1, \dots, m). \end{aligned} \tag{7.7}$$

Theorem 7.2. *Let* $p_1 \in C^{2,0,0}(\Omega)$, $p_2 \in C^{0,2,0}(\Omega)$, $p_0, p_{ijk},\ q \in C(\Omega)$, $\varphi_{ik} \in C^{2,m}(\widehat{\Omega}_i)$ $(k = 1,2;\ i = 1,2)$, $\varphi_{3k} \in C^{2,2}(\widehat{\Omega}_3)$ $(k = 1, \dots, m)$, *and*

$$p_1(\mathbf{x}) \ge 0, \ \ p_2(\mathbf{x}) \ge 0, \quad p_0(\mathbf{x}) \le 0 \quad \text{for} \quad \mathbf{x} \in \Pi. \tag{7.8}$$

Then problem $(7.6), (7.7)$ *is uniquely solvable.*

8. Auxiliary Statements

Let D be a convex smooth domain inscribed in the rectangle $\Omega = [0,\omega_1]\times[0,\omega_2]$ such that

$$D = \{(x_1,x_2)\in\Omega : x_1\in[0,\omega_1],\ \ x_2\in[\gamma_1(x_1),\gamma_2(x_1)]\}$$
$$= \{(x_1,x_2)\in\Omega : x_2\in[0,\omega_2],\ \ x_1\in[\eta_1(x_2),\eta_2(x_2)]\},$$

where $\gamma_i\in C^2([0,\omega_1])$, $\eta_i\in C^2([0,\omega_2])$ $(i=1,2)$, and

$$\gamma_1(x_1^*) = 0,\quad \gamma_2(x_2^*) = \omega_2,\quad \eta_1(y_1^*) = 0,\quad \eta_2(y_2^*) = \omega_1$$

for some x_1^*, $x_2^*\in[0,\omega_1]$ and y_1^*, $y_2^*\in[0,\omega_2]$.

Consider the boundary value problem

$$u^{(2,2)} = p_1(\mathbf{x})u^{(2,0)} + p_2(\mathbf{x})u^{(0,2)} + \sum_{j,k=0}^{1} p_{jk}(\mathbf{x})u^{(j,k)} + q(\mathbf{x}), \tag{8.1}$$

$$u(\eta_k(x_2),x_2) = 0\quad (k=1,2);\quad u^{(2,0)}(x_1,\gamma_k(x_1)) = 0\quad (k=1,2), \tag{8.2}$$

and set:

$$\rho_{11}(\mathbf{x}) = p_{11}(\mathbf{x}),\qquad\qquad \rho_{10}(\mathbf{x}) = p_{10}(\mathbf{x}),$$
$$\rho_{01}(\mathbf{x}) = p_{01}(\mathbf{x}) - 2p_2^{(0,1)}(\mathbf{x}),\qquad \rho_{00}(\mathbf{x}) = p_{00}(\mathbf{x}) + p_1(\mathbf{x})p_2(\mathbf{x}) - p_2^{(0,2)}(\mathbf{x}).$$

Along with (8.1),(8.2) consider the problems

$$v^{(2,0)} = p_2(\mathbf{x})v,\qquad v(\eta_k(x_2),x_2) = 0\quad (k=1,2), \tag{8.3}$$

and

$$w^{(0,2)} = p_1(\mathbf{x})w,\qquad w(x_1,\gamma_k(x_1)) = 0\quad (k=1,2). \tag{8.4}$$

By $g_1(x_1,s_1;x_2)$ and $g_2(x_2,s_2;x_1)$ denote the Green's functions of problems (8.3) and (8.4), respectively.

Lemma 8.1. *Let* $p_2\in C^{0,2}(\Omega)$,

$$p_1(x_1,x_2)\ge 0,\qquad p_2(x_1,x_2)\ge 0\quad \text{for}\quad (x_1,x_2)\in D. \tag{8.5}$$

Then problem (8.1), (8.2) *is equivalent to the integral equation*

$$u(x_1,x_2) = \int\limits_{\eta_1(x_2)}^{\eta_2(x_2)} \int\limits_{\gamma_1(x_1)}^{\gamma_2(x_1)} g_1(x_1,s_1;x_2) g_2(x_2,s_2;s_1) \Big(\sum_{j,k=0}^{1} \rho_{jk}(s_1,s_2) u^{(j,k)}(s_1,s_2) + q(s_1,s_2) \Big)\, ds_2\, ds_1, \quad (8.6)$$

in $C(D)$*, i.e., every solution* $u \in C^{2,2}(D)$ *of problem* (8.1), (8.2) *is a solution of equation* (8.6)*, and vice versa, every solution* $u \in C(D)$ *of equation* (8.6) *is a solution of problem* (8.1), (8.2)*.*

Proof. Let $u \in C^{2,2}(D)$ be a solution of problem (8.1),(8.2). Set

$$v(\mathbf{x}) = u^{(2,0)}(\mathbf{x}) - p_2(\mathbf{x}) u(\mathbf{x}).$$

Then v is a solution of the problem

$$v^{(0,2)} = p_1(\mathbf{x}) v + \sum_{j,k=0}^{1} \rho_{jk}(\mathbf{x}) u^{(j,k)}(\mathbf{x}) + q(\mathbf{x}), \quad (8.7)$$

$$v(x_1, \gamma_i(x_1)) = 0 \quad (i = 1,2), \quad x_1 \in [0, \omega_1]. \quad (8.8)$$

Hence we have

$$u^{(2,0)}(\mathbf{x}) - p_2(\mathbf{x}) u(\mathbf{x})) = \int\limits_{\gamma_1(x_1)}^{\gamma_2(x_1)} g_2(x_2,s_2;x_1) \Big(\sum_{j,k=0}^{1} \rho_{jk}(x_1,s_2) u^{(j,k)}(x_1,s_2) + q(x_1,s_2) \Big)\, ds_2. \quad (8.9)$$

(8.9), along with the condition

$$u(\eta_k(x_2), x_2) = 0 \quad (k = 1,2), \quad (8.10)$$

implies representation (8.6).

It can be easily verified that arbitrary solution $u \in C(D)$ of equation (8.6) indeed belongs to $C^{2,2}(D)$, and is a solution of problem (8.1), (8.2). □

Lemma 8.2. *Let $u \in C^{2,2}(D)$ be an arbitrary function satisfying conditions* (8.2). *Then*

$$\iint\limits_D u^{(2,2)}(\mathbf{x})u(\mathbf{x})\,d\mathbf{x} \geq \iint\limits_D \left(u^{(1,1)}(\mathbf{x})\right)^2 d\mathbf{x}. \tag{8.11}$$

Proof. Let $u \in C^{2,2}(D)$ be an arbitrary function satisfying conditions (8.2). Then

$$\frac{d}{dx_1}\big[u(x_1,\gamma_i(x_1))\big] = u^{(1,0)}(x_1,\gamma_i(x_1)) \\ + \gamma_i'(x_1)u^{(0,1)}(x_1,\gamma_i(x_1)) = 0 \quad \text{for} \quad x_1 \in [0,\omega_1], \quad (i=1,2), \tag{8.12}$$

and

$$\frac{d}{dx_1}\big[u^{(1,0)}(x_1,\gamma_i(x_1)\big] = u^{(2,0)}(x_1,\gamma_i(x_1)) + \gamma_i'(x_1)u^{(1,1)}(x_1,\gamma_i(x_1)) \\ = \gamma_i'(x_1)u^{(1,1)}(x_1,\gamma_i(x_1) \quad \text{for} \quad x_1 \in [0,\omega_1], \quad (i=1,2). \tag{8.13}$$

From (8.12) we get

$$u^{(1,0)}(x_1,\gamma_i(x_1)) = -\gamma_i'(x_1)u^{(0,1)}(x_1,\gamma_i(x_1)) \quad \text{for} \quad x_1 \in [0,\omega_1], \quad (i=1,2). \tag{8.14}$$

$$\iint\limits_D u^{(2,2)}(\mathbf{x})u(\mathbf{x})\,d\mathbf{x} = \int\limits_0^{\omega_2}\int\limits_{\eta_1(x_2)}^{\eta_2(x_2)} u^{(2,2)}(x_1,x_2)u(x_1,x_2)\,dx_1\,dx_2$$

$$= -\int\limits_0^{\omega_2}\int\limits_{\eta_1(x_2)}^{\eta_2(x_2)} u^{(1,2)}(x_1,x_2)u^{(1,0)}(x_1,x_2)\,dx_1\,dx_2 = -\iint\limits_D u^{(1,2)}(\mathbf{x})u^{(1,0)}(\mathbf{x})\,d\mathbf{x}$$

$$= -\int\limits_0^{\omega_1}\int\limits_{\gamma_1(x_1)}^{\gamma_2(x_1)} u^{(1,2)}(x_1,x_2)u^{(1,0)}(x_1,x_2)\,dx_1\,dx_2$$

$$= \int\limits_0^{\omega_1} u^{(1,1)}(x_1,\gamma_1(x_1))u^{(1,0)}(x_1,\gamma_1(x_1))\,dx_1 - \int\limits_0^{\omega_1} u^{(1,1)}(x_1,\gamma_2(x_1))u^{(1,0)}(x_1,\gamma_2(x_1))\,dx_1 \\ + \iint\limits_D \left(u^{(1,1)}(\mathbf{x})\right)^2 d\mathbf{x}. \tag{8.15}$$

Notice that convexity of the domain D implies the following inequalities

$$\gamma_1'(0) \leq 0, \quad \gamma_1'(\omega_1) \geq 0, \quad \gamma_1''(x_1) \geq 0 \quad \text{for} \quad x_1 \in [0,\omega_1], \tag{8.16}$$

and

$$\gamma_2'(0) \geq 0, \quad \gamma_2'(\omega_1) \leq 0, \quad \gamma_2''(x_1) \leq 0 \quad \text{for} \quad x_1 \in [0, \omega_1]. \tag{8.17}$$

In view of (8.13), (8.14) and (8.16), we have

$$\begin{aligned}
&\int_0^{\omega_1} u^{(1,1)}(x_1, \gamma_1(x_1)) u^{(1,0)}(x_1, \gamma_1(x_1))\, dx_1 \\
&= \int_0^{\omega_1} \gamma_1'(x_1) u^{(1,1)}(x_1, \gamma_1(x_1)) u^{(1,0)}(x_1, \gamma_1(x_1)) \frac{1}{\gamma_1'(x_1)}\, dx_1 \\
&= \frac{1}{2}\big(u^{(1,0)}(x_1, \gamma_1(x_1))\big)^2 \frac{1}{\gamma_1'(x_1)}\bigg|_0^{\omega_1} + \frac{1}{2}\int_0^{\omega_1} \big(u^{(1,0)}(x_1, \gamma_1(x_1))\big)^2 \frac{\gamma_1''(x_1)}{\gamma_1'^2(x_1)}\, dx_1 \\
&= \frac{1}{2}\big(\gamma_1'(x_1) u^{(0,1)}(x_1, \gamma_1(x_1))\big)^2 \frac{1}{\gamma_1'(x_1)}\bigg|_0^{\omega_1} + \frac{1}{2}\int_0^{\omega_1} \big(\gamma_1'(x_1) u^{(0,1)}(x_1, \gamma_1(x_1))\big)^2 \frac{\gamma_1''(x_1)}{\gamma_1'^2(x_1)}\, dx_1 \\
&= \frac{1}{2}\gamma_1'(\omega_1)\big(u^{(0,1)}(\omega_1, \gamma_1(\omega_1))\big)^2 - \frac{1}{2}\gamma_1'(0)\big(u^{(0,1)}(0, \gamma_1(0))\big)^2 \\
&\qquad + \frac{1}{2}\int_0^{\omega_1} \big(u^{(0,1)}(x_1, \gamma_1(x_1))\big)^2 \gamma_1''(x_1)\, dx_1 \geq 0.
\end{aligned} \tag{8.18}$$

Similarly, from (8.13), (8.14) and (8.17), we ge

$$\begin{aligned}
&\int_0^{\omega_2} u^{(1,1)}(x_1, \gamma_2(x_1)) u^{(1,0)}(x_1, \gamma_1 x_1)\, dx_1 \\
&= \frac{1}{2}\gamma_2'(\omega_1)\big(u^{(0,1)}(\omega_1, \gamma_2(\omega_1))\big)^2 - \frac{1}{2}\gamma_2'(0)\big(u^{(0,1)}(0, \gamma_2(0))\big)^2 \\
&\qquad + \frac{1}{2}\int_0^{\omega_1} \big(u^{(0,1)}(x_1, \gamma_2(x_1))\big)^2 \gamma_2''(x_1)\, dx_1 \leq 0.
\end{aligned} \tag{8.19}$$

(8.11) immediately follows from (8.15), (8.18) and (8.19). □

9. Proof of the Main Results

Proof of Theorem 7.1. First notice that it is sufficient to prove the theorem for problem (7.1), (7.2_0) only. Indeed, consider the problemss

$$v^{(2,0)} = 0, \qquad v(\eta_k(x_2), x_2) = 0 \quad (k = 1, 2), \tag{9.1}$$

and

$$w^{(0,2)} = 0, \qquad w(x_1, \gamma_k(x_1)) = 0 \quad (k = 1, 2). \tag{9.2}$$

By $\mathcal{G}_1$ (Γ_1) and $\mathcal{G}_2$ (Γ_2) denote the Green's operator (Green's boundary operator) of problems (9.1) and (9.2), respectively. Then it is clear that the function

$$u_0(\mathbf{x}) = \Gamma_1\big(\varphi_{1\,1}(x_2), \varphi_{1\,2}(x_2)\big)(x_1; x_2) + \mathcal{G}_1\Big(\Gamma_2\big(\varphi''_{2\,1}, \varphi''_{2\,2}\big)\Big)(\mathbf{x}) \tag{9.3}$$

is the unique solution of problem

$$u^{(2,2)} = 0, \tag{9.4}$$

$$u(\eta_k(x_2), x_2) = \varphi_{1k}(x_2) \quad (k = 1, 2); \quad u^{(2,0)}(x_1, \gamma_k(x_1)) = \varphi''_{2k}(x_1) \quad (k = 1, 2). \tag{9.5}$$

It is rather obvious that by means of function (9.3) one can always reduce problem (7.1), (7.2) to the problem (7.1).(7.2_0).

Equation (7.1) is a particular case of equation (8.1). Therefore, by Lemma 8.1, problem (7.1).(7.2_0) is equivalent to the Fredholm integral equation (8.6). Hence, for the unique solvability of problem (7.1), (7.2) it is sufficient to show that the homogeneous problem (7.1_0), (7.2_0) has only the trivial solution. Indeed, let u be an arbitrary solution of problem (7.1_0), (7.2_0). After multiplying equation (7.1_0) by u and integrating over D, we get

$$\iint\limits_D \Big(u^{(2,2)}(\mathbf{x})u(\mathbf{x}) + p_1(\mathbf{x})\big(u^{(1,0)}(\mathbf{x})\big)^2 + p_2(\mathbf{x})\big(u^{(0,1)}(\mathbf{x})\big)^2 - p_0(\mathbf{x})u^2(\mathbf{x})\Big)\, d\mathbf{x} = 0. \tag{9.6}$$

By Lemma 8.2, from (9.6) we get

$$\iint\limits_D \Big(\big(u^{(1,1)}(\mathbf{x})\big)^2 + p_1(\mathbf{x})\big(u^{(1,0)}(\mathbf{x})\big)^2 + p_2(\mathbf{x})\big(u^{(0,1)}(\mathbf{x})\big)^2 - p_0(\mathbf{x})u^2(\mathbf{x})\Big)\, d\mathbf{x} \le 0. \tag{9.7}$$

(9.7), (7.3) and (7.4) yield

$$u^{(1,1)}(\mathbf{x}) = 0 \quad \text{for} \quad \mathbf{x} \in D. \tag{9.8}$$

(9.8) and (7.2_0) imply

$$u(\mathbf{x}) \equiv 0. \quad \Box$$

The proof of Theorem 7.2 is similar to the proof of the Theorem 1.16.

CHAPTER IV
Quasi–Linear Problems

10. Formulation of the Main Results

10.1. **General boundary conditions.** In the domain Ω for the quasi–linear equation

$$u^{(\mathbf{m})} = \sum_{\boldsymbol{\alpha}<\mathbf{m}} p_{\boldsymbol{\alpha}}(\mathbf{x})u^{(\boldsymbol{\alpha})} + q(\mathbf{x}, \mathcal{D}^{\mathbf{m}\text{ - }\mathbf{1}}[u]) \tag{10.1}$$

consider the boundary conditions

$$h_{ik}\big(u^{(\mathbf{m}^{i-1})}(\bullet, \widehat{\mathbf{x}}_i)\big) = \varphi_{ik}^{(\mathbf{m}^{i-1})}(\widehat{\mathbf{x}}_i) \text{ for } \widehat{\mathbf{x}}_i \in \widehat{\Omega}_i \ \ (k = 1, \dots, m_i; \ \ i = 1, \dots, n). \tag{10.2}$$

Here $\mathcal{D}^{\mathbf{r}}[u] = \Big(u^{(\boldsymbol{\alpha})}\Big)_{\boldsymbol{\alpha}\le\mathbf{r}}$, and $q \in C(\Omega \times \mathbb{R}^{m_1-1\times\cdots\times m_n-1})$.

Set: $\mathbf{Z} = \big(z_{\boldsymbol{\alpha}}\big)_{\boldsymbol{\alpha}\le\mathbf{m}-\mathbf{1}}$; $q_{\boldsymbol{\alpha}}(\mathbf{x}, \mathbf{Z}) = \frac{\partial q(\mathbf{x},\mathbf{Z})}{\partial z_{\boldsymbol{\alpha}}}$.

Theorem 10.1. *Let the homogeneous problem* $(1.1_0), (1.2_0)$*, along with all associated problems* $(1.1_{\boldsymbol{\sigma}}), (1.2_{\boldsymbol{\sigma}})$ $(\boldsymbol{\sigma} \in \boldsymbol{\Xi})$*, be α–well–posed, let q be a continuously differentiable function with respect to the phase variables, and let there exist functions* $P_{i\boldsymbol{\alpha}} \in C(\Omega)$ $(\boldsymbol{\alpha} \le \mathbf{m} - \mathbf{1};\ i = 1, 2)$ *such that:*

(A_1) *the following inequalities hold*

$$P_{1\boldsymbol{\alpha}}(\mathbf{x}) \le q_{\boldsymbol{\alpha}}(\mathbf{x}, \mathbf{Z}) \le P_{2\boldsymbol{\alpha}}(\mathbf{x}) \quad \textit{for} \quad (\mathbf{x}, \mathbf{Z}) \in \Omega \times \mathbb{R}^{m_1-1\times\cdots\times m_n-1} \ \ (\boldsymbol{\alpha} \le \mathbf{m} - \mathbf{1};\ i = 1, 2) \tag{10.3}$$

(A_2) *for arbitrary measurable functions* $\widetilde{p}_{\boldsymbol{\alpha}} : \Omega \to \mathbb{R}$ *satisfying the inequalities*

$$P_{1\boldsymbol{\alpha}}(\mathbf{x}) \le \widetilde{p}_{\boldsymbol{\alpha}}(\mathbf{x}) \le P_{2\boldsymbol{\alpha}}(\mathbf{x}) \quad \textit{for} \quad \mathbf{x} \in \Omega \quad (\boldsymbol{\alpha} \le \mathbf{m} - \mathbf{1};\ i = 1, 2), \tag{10.4}$$

the problem

$$u^{(\boldsymbol{m})} = \sum_{\boldsymbol{\alpha}<\boldsymbol{m}} p_{\boldsymbol{\alpha}}(\mathbf{x})u^{(\boldsymbol{\alpha})} + \sum_{\boldsymbol{\alpha}\le\mathbf{m}-\mathbf{1}} \widetilde{p}_{\boldsymbol{\alpha}}(\mathbf{x})u^{(\boldsymbol{\alpha})}, \tag{10.5}$$

$$h_{ik}\big(u^{(\mathbf{m}^{i-1})}(\bullet, \widehat{\mathbf{x}}_i)\big) = 0 \ \textit{for} \ \widehat{\mathbf{x}}_i \in \widehat{\Omega}_i \ \ (k = 1, \dots, m_i;\ \ i = 1, \dots, n). \tag{10.2$_0$}$$

has only the trivial solution. Then problem $(10.1), (10.2)$ *is uniquely solvable.*

Theorem 10.2. *Let the homogeneous problem* $(1.1_0), (1.2_0)$*, be well–posed, and let*

$$\lim_{\|\mathbf{z}\|\to+\infty} \frac{|q(x,y,\mathbf{Z})|}{\|\mathbf{Z}\|} = 0 \tag{10.6}$$

uniformly on Ω*. Then problem* $(10.1), (10.2)$ *has at least one solution.*

For the equation

$$u^{(2\mathbf{m})} = \sum_{\boldsymbol{\alpha}+\boldsymbol{\beta}<2\mathbf{m}} \big(p_{\boldsymbol{\alpha}+\boldsymbol{\beta}}(\mathbf{x})u^{(\boldsymbol{\alpha})}\big)^{(\boldsymbol{\beta})} + q(\mathbf{x}, \mathcal{D}^{\mathbf{m}\text{ - }\mathbf{1}}[u]), \tag{10.7}$$

consider the Dirichlet boundary conditions

$$u^{((k-1)\mathbf{1}_i)}(0, \widehat{\mathbf{x}}_\mathbf{i}) = 0, \quad u^{((k-1)\mathbf{1}_i)}(\omega_i, \widehat{\mathbf{x}}_\mathbf{i}) = 0 \quad (k = 1, \dots, m_i;\ i = 1, \dots, n), \tag{10.8}$$

Corollary 10.1. *Let* $p_{\boldsymbol{\alpha}+\boldsymbol{\beta}} \in C^{\boldsymbol{m}}(\Omega)$ $(\boldsymbol{\alpha}+\boldsymbol{\beta} \in \boldsymbol{\Upsilon}_{2\boldsymbol{m}})$*,* $p_{\boldsymbol{\alpha}+\boldsymbol{\beta}} \in C^{\boldsymbol{\beta}}(\Omega)$ $(\boldsymbol{\alpha}+\boldsymbol{\beta} \notin \boldsymbol{\Upsilon}_{2\boldsymbol{m}})$*, and let the quadratic form nonnegative defined:*

$$\sum_{\boldsymbol{\alpha}+\boldsymbol{\beta}<2\mathbf{m}} (-1)^{\|\boldsymbol{m}\|+\|\boldsymbol{\beta}\|-1} p_{\boldsymbol{\alpha}+\boldsymbol{\beta}}(\mathbf{x}) z_{\boldsymbol{\alpha}} z_{\boldsymbol{\beta}} \geq 0 \quad \textit{for} \quad \mathbf{x} \in \Omega, \quad z_{\boldsymbol{\alpha}} \in \mathbb{R} \ \ (\boldsymbol{\alpha} < 2\mathbf{m}). \tag{10.9}$$

Moreover, let q *satisfy equality* (10.6) *uniformly on* Ω*. Then problem* $(10.7), (10.8)$ *has at least one solution.*

10.2. **Problem on Periodic Solutions.** For the equations (10.1) and (10.8) consider the problem on $\boldsymbol{\omega}$–periodic solutions

$$u(\mathbf{x} + \boldsymbol{\omega}_i) = u(\mathbf{x}) \quad (i = 1, \dots, n). \tag{10.10}$$

Theorem 10.3. *Let* $p_{\boldsymbol{\alpha}} \in C_{\boldsymbol{\omega}}(\mathbb{R}^n)$ $(\boldsymbol{\alpha} < \boldsymbol{m})$*,* q *be* $\boldsymbol{\omega}$*–periodic with respect to* $\boldsymbol{x}$*, and let the homogeneous problem* $(4.1_0), (4.2)$ *be well–posed. Moreover, let* q *satisfy equality* (10.6) *uniformly on* Ω*. Then problem* $(10.1), (10.10)$ *has at least one solution.*

Corollary 10.2. *Let* $p_{\boldsymbol{\alpha}+\boldsymbol{\beta}} \in C^{\boldsymbol{\beta}}_{\boldsymbol{\omega}}(\mathbb{R}^n)$ $(\boldsymbol{\alpha}+\boldsymbol{\beta} < 2\boldsymbol{m})$*,* q *be* $\boldsymbol{\omega}$*–periodic with respect to* $\boldsymbol{x}$*, and let there exist* $\delta > 0$ *such that*

$$\sum_{\boldsymbol{\alpha}+\boldsymbol{\beta}<2\mathbf{m}} (-1)^{\|\boldsymbol{m}\|+\|\boldsymbol{\beta}\|-1} p_{\boldsymbol{\alpha}+\boldsymbol{\beta}}(\mathbf{x}) z_{\boldsymbol{\alpha}} z_{\boldsymbol{\beta}} \geq \delta \sum_{\boldsymbol{\alpha}<2\mathbf{m}} z_{\boldsymbol{\alpha}}^2 > 0 \quad \textit{for} \quad \mathbf{x} \in \Omega. \tag{10.11}$$

Moreover, let q *satisfy equality* (10.6) *uniformly on* Ω*. Then problem* $(10.7), (10.10)$ *has at least one solution.*

10.3. **Dirichlet type problem in a smooth convex domain.** Let D be a convex smooth domain ***inscribed*** in the rectangle $\Omega = [0,\omega_1] \times [0,\omega_2]$ such that

$$D = \{(x_1,x_2) \in \Omega : x_1 \in [0,\omega_1],\ \ x_2 \in [\gamma_1(x_1),\gamma_2(x_1)]\}$$
$$= \{(x_1,x_2) \in \Omega : x_2 \in [0,\omega_2],\ \ x_1 \in [\eta_1(x_2),\eta_2(x_2)]\},$$

where $\gamma_i \in C^2([0,\omega_1])$, $\eta_i \in C^2([0,\omega_2])$ $(i = 1,2)$, and

$$\gamma_1(x_1^*) = 0, \quad \gamma_2(x_2^*) = \omega_2, \quad \eta_1(y_1^*) = 0, \quad \eta_2(y_2^*) = \omega_1$$

for some x_1^*, $x_2^* \in [0,\omega_1]$ and y_1^*, $y_2^* \in [0,\omega_2]$.

Consider the problem

$$u^{(2,2)} = \big(p_1(x_1,x_2)u^{(1,0)}\big)^{(1,0)} + \big(p_2(x_1,x_2)u^{(0,1)}\big)^{(0,1)} + p_0(x_1,x_2)u$$
$$+q(x_1,x_2,u,u^{(1,0)},u^{(0,1)},u^{(1,1)}), \tag{10.12}$$

$$u(\eta_k(x_2),x_2) = \varphi_{1k}(x_2) \quad (k = 1,2); \quad u^{(2,0)}(x_1,\gamma_k(x_1)) = \varphi_{2k}''(x_1) \quad (k = 1,2). \tag{10.13}$$

Theorem 10.4. *Let $p_1 \in C^{2,0}(\Omega)$, $p_2 \in C^{0,2}(\Omega)$, p_0, $q \in C(\Omega)$, $\varphi_{1k} \in C^2([0,\omega_2])$, $\varphi_{2k} \in C^2([0,\omega_1])$ $(k = 1,2)$, and let the inequalities hold*

$$p_1(x_1,x_2) \geq 0, \qquad p_2(x_1,x_2) \geq 0, \qquad p_0(x_1,x_2) \leq 0 \quad \text{for} \quad (x_1,x_2) \in D. \tag{10.14}$$

Moreover, let q satisfy equality (10.6) *uniformly in Ω. Then problem* (10.12), (10.13) *has at least one solution.*

11. Auxiliary Statements

Lemma 11.1. *Let $p_l \in C(\Omega)$ $(l = 1, 2, \dots)$, and let there exist $p \in L^\infty(\Omega)$ and $M > 0$ such that*

$$\|p_l\|_{C(\Omega)} \le M \qquad (l = 1, 2, \dots), \tag{11.1}$$

$$\lim_{l\to\infty} \int\limits_0^{x_1} \dots \int\limits_0^{x_n} \big(p_l(s_1, \dots, s_n) - p(s_1, \dots, s_n)\big)\, ds_n \dots ds_1 = 0 \quad \text{uniformly on} \quad \Omega. \tag{11.2}$$

Then for arbitrary $z \in C^{\mathbf{1}}(\Omega)$ we have

$$\lim_{l\to\infty} \int\limits_0^{x_1} \dots \int\limits_0^{x_n} \big(p_l(s_1, \dots, s_n) - p(s_1, \dots, s_n)\big) z(s_1, \dots, s_n)\, ds_n \dots ds_1 = 0 \quad \text{uniformly on} \quad \Omega. \tag{11.3}$$

Lemma 11.1 can be easily proved by integration by parts multiple times.

Consider the sequence of the following equations

$$u^{(\mathbf{m})} = \sum_{\boldsymbol{\alpha} < \mathbf{m}} p_{\boldsymbol{\alpha}}(\mathbf{x}) u^{(\boldsymbol{\alpha})} + q_l(\mathbf{x}), \tag{11.4}$$

where $q_l \in C(\Omega)$ $(l = 1, 2, \dots)$.

Lemma 11.2. *Let problem $(1.1_0), (1.2_0)$ be α–well–posed, $p_l \in C(\Omega)$ $(l = 1, 2, \dots)$, let there exist $q \in L^\infty(\Omega)$ and $M > 0$ such that*

$$\|q_l\|_{C(\Omega)} \le M \qquad (l = 1, 2, \dots), \tag{11.5}$$

$$\lim_{l\to\infty} \int\limits_0^{x_1} \dots \int\limits_0^{x_n} \big(q_l(s_1, \dots, s_n) - q(s_1, \dots, s_n)\big)\, ds_n \dots ds_1 = 0 \quad \text{uniformly on} \quad \Omega. \tag{11.6}$$

Then

$$\lim_{l\to\infty} \|u_l - u\|_{C^{\mathbf{m}-\mathbf{1}}(\Omega)} = 0, \tag{11.7}$$

where $u_l \in C^{\mathbf{m}}(\Omega)$ is a solution of problem $(11.4), (1.2_0)$, and $u \in AC^{\mathbf{m}-\mathbf{1}}(\Omega)$ is a solution of problem $(1.1), (1.2_0)$.

Proof. Set:

$$Q_l(\mathbf{x}) = \int\limits_0^{x_1} \dots \int\limits_0^{x_n} \frac{(x_1 - s_1)^{m_1-1}}{(m_1-1)!} \cdots \frac{(x_n - s_n)^{m_n-1}}{(m_n-1)!} q_l(s_1, \dots, s_n)\, ds_n \dots ds_1, \tag{11.8}$$

$$Q(\mathbf{x}) = \int\limits_0^{x_1} \dots \int\limits_0^{x_n} \frac{(x_1 - s_1)^{m_1-1}}{(m_1-1)!} \cdots \frac{(x_n - s_n)^{m_n-1}}{(m_n-1)!} q(s_1, \dots, s_n)\, ds_n \dots ds_1. \tag{11.9}$$

Then

$$q_l(\mathbf{x}) = Q_l^{(\mathbf{m})}(\mathbf{x}), \qquad q(\mathbf{x}) = Q^{(\mathbf{m})}(\mathbf{x}). \tag{11.10}$$

Let u_l be a solution of problem $(11.4), (1.2_0)$. Due to (11.5) we have

$$\|u_l\|_{C^{\mathbf{m}}(\Omega)} \le M_1 \qquad (l = 1, 2, \dots), \tag{11.11}$$

where M_1 is a positive number independent of l. In view of (11.11) and Arzela–Ascoli lemma, without loss of generality, one may assume that there exists $u \in AC^{\mathbf{m-1}}(\Omega)$ such that

$$\|u^{(\boldsymbol{\alpha})}\|_{L^\infty(\Omega)} \le M_1 \qquad (\boldsymbol{\alpha} \in \boldsymbol{\Upsilon}_{\mathbf{m}}), \tag{11.12}$$

and (11.7) holds.

Representations (3.39), (3.54) and (3.55) immediately imply that u is a solution of problem $(1.1), (1.2_0)$. $\square$

Lemma 11.3. *Let problem* $(1.1_0), (1.2_0)$ *be* α*–well–posed, and let there exist functions* $P_{i\boldsymbol{\alpha}} \in C(\Omega)$ $(\boldsymbol{\alpha} \le \mathbf{m} - \mathbf{1};\ i = 1, 2)$ *such that for arbitrary measurable functions* $\widetilde{p}_{\boldsymbol{\alpha}} : \Omega \to \mathbb{R}$ *satisfying the inequalities*

$$P_{1\boldsymbol{\alpha}}(\mathbf{x}) \le \widetilde{p}_{\boldsymbol{\alpha}}(\mathbf{x}) \le P_{2\boldsymbol{\alpha}}(\mathbf{x}) \quad \textit{for} \quad \mathbf{x} \in \Omega \quad (\boldsymbol{\alpha} \le \mathbf{m} - \mathbf{1};\ i = 1, 2), \tag{11.13}$$

the problem

$$u^{(\boldsymbol{m})} = \sum_{\boldsymbol{\alpha} < \boldsymbol{m}} p_{\boldsymbol{\alpha}}(\mathbf{x}) u^{(\boldsymbol{\alpha})} + \sum_{\boldsymbol{\alpha} \le \mathbf{m-1}} \widetilde{p}_{\boldsymbol{\alpha}}(\mathbf{x}) u^{(\boldsymbol{\alpha})}, \tag{11.14}$$

$$h_{ik}\big(u^{(\mathbf{m}^{i-1})}(\bullet, \widehat{\mathbf{x}}_i)\big) = 0 \ \textit{ for } \widehat{\mathbf{x}}_i \in \widehat{\Omega}_i \ \ (k = 1, \dots, m_i;\ \ i = 1, \dots, n) \tag{11.15}$$

has only the trivial solution. Then a solution of the problem

$$u^{(\boldsymbol{m})} = \sum_{\boldsymbol{\alpha}<\boldsymbol{m}} p_{\boldsymbol{\alpha}}(\mathbf{x})u^{(\boldsymbol{\alpha})} + \sum_{\boldsymbol{\alpha}\leq\mathbf{m-1}} \widetilde{p}_{\boldsymbol{\alpha}}(\mathbf{x})u^{(\boldsymbol{\alpha})} + q(\mathbf{x}), \tag{11.16}$$

$$h_{ik}\big(u^{(\mathbf{m}^{i-1})}(\bullet, \widehat{\mathbf{x}}_i)\big) = 0 \ \ for \ \ \widehat{\mathbf{x}}_i \in \widehat{\Omega}_i \ \ (k = 1, \dots, m_i; \ \ i = 1, \dots, n) \tag{11.17}$$

admits the estimate

$$\|u\|_{C^{\mathbf{m-1}}(\Omega)} \leq M\|q\|_{C(\Omega)}, \tag{11.18}$$

where M is a positive number independent of functions $\widetilde{p}_{\boldsymbol{\alpha}}$ $(\boldsymbol{\alpha} \leq \mathbf{m-1})$.

Proof. Assume the contrary: let there exist $q \in C(\Omega)$, and $\widetilde{p}_{l\,\boldsymbol{\alpha}} \in C(\Omega)$ $(\boldsymbol{\alpha} \leq \mathbf{m-1})$ $(l = 1, 2, \dots)$ satisfying the inequalities

$$P_{1\boldsymbol{\alpha}}(\mathbf{x}) \leq \widetilde{p}_{l\,\boldsymbol{\alpha}}(\mathbf{x}) \leq P_{2\boldsymbol{\alpha}}(\mathbf{x}) \quad \text{for} \quad \mathbf{x} \in \Omega \ \ (\boldsymbol{\alpha} \leq \mathbf{m-1}; \ i = 1, 2; \ \ l = 1, 2, \dots), \tag{11.19}$$

such that

$$\|u_l\|_{C^{\mathbf{m-1}(\Omega)}} = \eta_l, \qquad \lim_{l\to\infty} \eta_l = +\infty, \tag{11.20}$$

where u_l is a solution of the equation

$$u^{(\mathbf{m})} = \sum_{\boldsymbol{\alpha}<\mathbf{m}} p_{\boldsymbol{\alpha}}(\mathbf{x})u^{(\boldsymbol{\alpha})} + \sum_{\boldsymbol{\alpha}\leq\mathbf{m-1}} \widetilde{p}_{l\,\boldsymbol{\alpha}}(\mathbf{x})u^{(\boldsymbol{\alpha})} + q(\mathbf{x}) \tag{11.16$_l$}$$

satisfying conditions (11.17).

Due to inequalities (11.19), without loss of generality, one may assume that there exist measurable functions $\widetilde{p}_{\boldsymbol{\alpha}}$ $(\boldsymbol{\alpha} \leq \mathbf{m-1})$ satisfying inequalities (11.13) such that

$$\lim_{l\to\infty} \int_0^{x_1} \dots \int_0^{x_n} \big(p_{l\boldsymbol{\alpha}}(s_1, \dots, s_n) - p_{\boldsymbol{\alpha}}(s_1, \dots, s_n)\big)\, ds_n \dots ds_1$$
$$= 0 \quad \text{uniformly on} \quad \Omega. \tag{11.21}$$

Set

$$\widetilde{u}_l(\mathbf{x}) = \frac{u_l(\mathbf{x})}{\eta_l}. \tag{11.22}$$

Then, by the estimate (3.57), we have

$$\|\widetilde{u}_l\|_{C^{\mathbf{m-1}(\Omega)}} = 1, \tag{11.23}$$

$$\|\widetilde{u}_l\|_{C^{\mathbf{m}(\Omega)}} \leq M\Big(1 + \frac{1}{\eta_l}\|q\|_{C(\Omega)}\Big), \tag{11.24}$$

where M is a positive constant independent of q and l.

In view of (11.23), without loss of generality, one may assume that there exists $u \in AC^{\mathbf{m-1}}(\Omega)$ such that

$$\|u^{(\boldsymbol{\alpha})}\|_{L^\infty(\Omega)} \leq M \qquad (\boldsymbol{\alpha} \in \boldsymbol{\Upsilon}_{\mathbf{m}}), \tag{11.25}$$

and (11.7) holds.

By virtue of representations (3.39), (3.54) and (3.55), equality (11.23) and Lemma 11.1, we get that u is a *nontrivial* solution of problem $(1.1), (1.2_0)$. The obtained contradiction completes the proof of the lemma. □

12. Proofs of the Main Results

Proof of Theorem 10.1. First notice that it is sufficient to prove the theorem for problem $(10.1), (10.2_0)$ only.

Let $z \in C^{\mathbf{m-1}}(\Omega)$ be an arbitrary function, and set:

$$\widetilde{p}_\alpha[z](\mathbf{x}) = \int_0^1 q_{\boldsymbol{\alpha}}(\mathbf{x},\, t\,\mathcal{D}^{\mathbf{m\,-\,1}}[z](\mathbf{x}))\,dt \quad (\boldsymbol{\alpha} \leq \mathbf{m-1}), \tag{12.1}$$

$$q_0(\mathbf{x}) = q(\mathbf{x}, \mathbf{0}). \tag{12.2}$$

Consider the problem

$$u^{(\mathbf{m})} = \sum_{\boldsymbol{\alpha}<\mathbf{m}} p_{\boldsymbol{\alpha}}(\mathbf{x})u^{(\boldsymbol{\alpha})} + \sum_{\boldsymbol{\alpha}\leq\mathbf{m-1}} \widetilde{p}_{\boldsymbol{\alpha}}[z](\mathbf{x})u^{(\boldsymbol{\alpha})} + q_0(\mathbf{x}), \tag{12.3}$$

$$h_{ik}\big(u^{(\mathbf{m}^{i-1})}(\bullet, \widehat{\mathbf{x}}_i)\big) = 0 \ \text{ for } \ \widehat{\mathbf{x}}_i \in \widehat{\Omega}_i \ \ (k = 1, \ldots, m_i; \ \ i = 1, \ldots, n). \tag{12.4}$$

By the assumption of the Theorem 10.1, problem (12.3),(12.4) is well–posed, whereas, by Lemma 11.3, a solution u of problem (12.3),(12.4) admits the estimate

$$\|u\|_{C^{\mathbf{m-1}}(\Omega)} \leq M\|q_0\|_{C(\Omega)}, \tag{12.5}$$

where M is a positive constant independent of $z \in C^{\mathbf{m-1}}(\Omega)$. It is rather obvious that the operator $\mathcal{A} : z \to u$ is a continuous operator from $C^{\mathbf{m-1}}(\Omega)$ into $C^{\mathbf{m}}(\Omega)$. Thus $\mathcal{A}$ is a compact operator from $C^{\mathbf{m-1}}(\Omega)$ to $C^{\mathbf{m-1}}(\Omega)$. Furthermore, $\mathcal{A}$ maps $C^{\mathbf{m-1}}(\Omega)$ into a ball of radius M. By Schauder's fixed point theorem, $\mathcal{A}$ has at least one fixed point u. Therefore, u is a solution of the equation

$$u^{(\mathbf{m})} = \sum_{\boldsymbol{\alpha}<\mathbf{m}} p_{\boldsymbol{\alpha}}(\mathbf{x})u^{(\boldsymbol{\alpha})} + \sum_{\boldsymbol{\alpha}\leq\mathbf{m-1}} \widetilde{p}_{\boldsymbol{\alpha}}[u](\mathbf{x})u^{(\boldsymbol{\alpha})} + q_0(\mathbf{x}), \tag{12.6}$$

and, consequently, of equation (10.1), since

$$q(\mathbf{x}, \mathbf{Z}) = \sum_{\boldsymbol{\alpha}\leq\mathbf{m-1}} \int_0^1 q_{\boldsymbol{\alpha}}(\mathbf{x},\, t\,\mathbf{Z})\,dt\ z_{\boldsymbol{\alpha}} \,+\, q_0(\mathbf{x}). \tag{12.7}$$

Thus, solvability of problem (10.1),(10.2) is proved. Let us prove uniqueness of a solution. Let u_1 and u_2 be arbitrary solutions of problem (10.1),(10.2), and let

$u = u_2 - u_1$. Then u is a solution of problem $(10.5), (10.2_0)$, where

$$\widetilde{p}_\alpha(\mathbf{x}) = \int_0^1 q_{\boldsymbol{\alpha}}\Big(\mathbf{x},\, (1-t)\,\mathcal{D}^{\mathbf{m}\,\text{-}\,\mathbf{1}}[u_1](\mathbf{x}) + t\,\mathcal{D}^{\mathbf{m}\,\text{-}\,\mathbf{1}}[u_2](\mathbf{x})\Big)\,dt \quad (\boldsymbol{\alpha} \le \mathbf{m} - \mathbf{1}). \qquad (12.8)$$

By condition (A_2) of Theorem 10.1, $u(\mathbf{x}) \equiv 0$. $\square$

Proof of Theorem 10.1. As in case of Theorem 10.1, it is sufficient to prove the theorem for problem $(10.1), (10.2_0)$ only.

Let u be a solution of problem $(10.1), (10.2_0)$. Then u admits the representation

$$u(x) = \mathcal{A}(u), \qquad (12.9)$$

where

$$\mathcal{A}(u) = \mathcal{G}\big(q(\mathcal{D}^{\mathbf{m}\,\text{-}\,\mathbf{1}}[u])\big), \qquad (12.10)$$

and $\mathcal{G}$ is the Green's operator of problem $(1.1_0), (1.2_0)$.

Since $\mathcal{G} : C(\Omega) \to C^{\mathbf{m}}(\Omega)$ is a bounded linear operator, then $\mathcal{A}$ is a continuous operator from $C^{\mathbf{m}-\mathbf{1}}(\Omega)$ to $C^{\mathbf{m}}(\Omega)$. Consequently, $\mathcal{A}$ is a compact operator from $C^{\mathbf{m}-\mathbf{1}}(\Omega)$ into $C^{\mathbf{m}-\mathbf{1}}(\Omega)$.

Let us show that $\mathcal{A}$ maps the ball $B_r^{\mathbf{m}-\mathbf{1}}$ of sufficiently large radius r into itself. Indeed, due to (10.6), for arbitrary $\varepsilon > 0$ the function q admits the estimate

$$|q(\mathbf{x}, \mathbf{Z})| \le M_\varepsilon + \varepsilon \|\mathbf{Z}\|, \qquad (12.11)$$

where M_ε is a positive number independent of $\mathbf{Z}$. Let

$$M_0 = \|\mathcal{G}\|, \qquad \varepsilon = \frac{1}{2M_0}, \qquad r = 2M_0 M_\varepsilon. \qquad (12.12)$$

Then

$$\|\mathcal{A}(z)\|_{C^{\mathbf{m}-\mathbf{1}}(\Omega)} \le M_0\big(M_\varepsilon + \varepsilon \|z\|_{C^{\mathbf{m}-\mathbf{1}}(\Omega)}\big) = M_0 M_\varepsilon + \frac{1}{2}\|z\|_{C^{\mathbf{m}-\mathbf{1}}(\Omega)}. \qquad (12.13)$$

From the latter inequality it follows that

$$\|\mathcal{A}(z)\|_{C^{\mathbf{m}-\mathbf{1}}(\Omega)} \le 2M_0 M_\varepsilon, \qquad (12.14)$$

if

$$\|z\|_{C^{\mathbf{m}-\mathbf{1}}(\Omega)} \le 2M_0 M_\varepsilon. \qquad (12.15)$$

Consequently, $\mathcal{A}$ maps the closed ball $B_r^{\mathbf{m-1}}$ of the space $C^{\mathbf{m-1}}(\Omega)$ into itself for $r = 2M_0M_\varepsilon$. By Schauder's fixed point theorem, $\mathcal{A}$ has at least one fixed point $u \in B_r^{\mathbf{m-1}}$. It is clear that u is a solution of problem $(10.1), (10.2_0)$. $\square$

Corollary 10.1 follows from Theorems 10.2 and 1.7.

The proofs of Theorems 10.3 and 10.4 are similar to the proof of Theorem 10.2.

Corollary 10.2 follows from Theorems 10.3 and 4.5

References

[1] A. T. Asanova and D. S. Dzhumabaev, Correct solvability of nonlocal boundary value problems for systems of hyperbolic equations. (Russian) ; translated from *Differ. Uravn.* **41** (2005), no. 3, 337–346, *Differ. Equ.* **41** (2005), no. 3, 352–363.

[2] A. T. Asanova, On a nonlocal boundary value problem for systems of quasilinear hyperbolic equations. (Russian) *Dokl. Akad. Nauk* **411** (2006), no. 1, 7–11.

[3] A. T. Asanova and D. S. Dzhumabaev, Well-posedness of nonlocal boundary value problems with integral condition for the system of hyperbolic equations. *J. Math. Anal. Appl.* **402** (2013), No. 1, 167–178.

[4] A.K.Aziz,Periodic solutions of hyperbolic partial differential equations. *Proc. Amer. Math.Soc.* **17**(1966), No 3, 557–566.

[5] A.K.Aziz and S.L.Brodsky, Periodic solutions of a class of weakly nonlinear hyperbolic partial differential equations. *SIAM J. Math. Anal.* **3**(1972), No 2, 300–313.

[6] A.K.Aziz and M.G.Horak, Periodic solutions of hyperbolic partial differential equations in the large. *SIAM J. Math. Anal.* **3**(1972), No 1, 176–182.

[7] A.K.Aziz and A.M.Meyers, Periodic solutions of hyperbolic partial differential equations in a strip. *Trans. Amer. Math. Soc.* **146**(1969), 167–178.

[8] L.Cesari, Periodic solutions of hyperbolic partial differential equations. *Proc. Internat. Sympos. Non-linear Vibrations* (*Kiev* 1961), vol. 2, *Izd. Akad. Nauk Ukrain. SSR, Kiev,* 1963, 440–457.

[9] L.Cesari, A criterion for the existence in a strip of periodic solutions of hyperbolic partial differential equations. *Rend. Circ. Mat. Palermo* (2) **14**(1965), 95–118.

[10] L.Cesari, Existence in the large of periodic solutions of hyperbolic partial differential equations. *Arch. Rational Mech. Anal.* **20**(1965), No 2, 170–190.

[11] L.Cesari, A boundary value problem for quasilinear hyperbolic systems in Schauder's canonic form. *Ann. Scuola norm. super. Pisa* **1**(1974), No 3-4, 311–358.

[12] L.Cesari, Un problema ai limiti per sistemi di equazioni iperboliche quasi lineari nella forma canonica di Shauder. *Rend. Accad. Naz. Lincei. Cl. Sc. fis. mat. e natur.* **57**(1974), No 5, 303–307.

[13] D. Colton, Pseudoparabolic equations in one space variable. *J. Differential Equations.* **12** (1972), No. 3, 559–565.

[14] K. Deimling, A Carathéodory theory for systems of integral equations. *Ann. mat. pura ed appl.* **86**(1970), 217–260.

[15] 13. K.Deimling, Absolutely continuous solutions of Cauchy problem for $u_{xy} = f(x, y, u, u_x, u_y)$. *Ann. mat. pura ed appl.* **89**(1971), 381–391.

[16] M.P. Grigolia, On a generalized characteristic boundary value problem for hyperbolic systems. (Russian) *Differentsial'nye Uravneniya* **21**(1985), No 4, 678–686.

[17] J.K.Hale, Periodic solutions of a class of hyperbolic equations containing a small parameter. *Arch. Rat. Mech. Anal.* **23**(1967), No 5, 380–398.

[18] L.V.Kantorovich and G.P.Akilov, Functional analysis. (Russian) Nauka, Moscow, 1977.

[19] I. T. Kiguradze, Boundary value problems for systems of ordinary differential equations. (Russian) *Itogi Nauki Tekh., Ser. Sovrem. Probl. Mat. Noveishie Dostizh.* **30** (1987), 3–103; English transl.: *J. Sov.Math.* **43** (1988), No 2, 2259–2339.

[20] I. T. Kiguradze and T. Kusano, On periodic solutions of higher-order nonautonomous ordinary differential equations. (Russian) Differ. Uravn. 35 (1999), no. 1, 72–78, 142; translation in Differential Equations 35 (1999), no. 1, 71-77

[21] I. Kiguradze, On periodic solutions of nth order ordinary differential equations. *Nonlinear Anal.* **40** (2000), No. 1-8, Ser. A: Theory Methods, 309-321.

[22] I. Kiguradze and T. Kiguradze, On solvability of boundary value problems for higher order nonlinear hyperbolic equations. *Nonlinear Analysis* **69**(2008), 1914-1933.

[23] I. Kiguradze and A. Lomtatidze, Periodic solutions of nonautonomous ordinary differential equations. *Monatsh. Math.* **159** (2010), No. 3, 235–252.

[24] T. Kiguradze, Some boundary value problems for systems of linear partial differential equations of hyperbolic type. *Mem. Differential Equations Math. Phys.* **1** (1994), 1–144.

[25] T. Kiguradze, On solutions that are bounded and periodic in a strip of quasilinear hyperbolic systems. (English. Russian original) *Differential Equations* **30** (1994), No 10, 1628–1639; translation from *Differ. Uravn.* **30** (1994), No 10, 1760–1773.

[26] T. Kiguradze, On bounded in a strip solutions of quasilinear partial differential equations of hyperbolic type. *Appl. Anal.* **58** (1995), No 3–4, 199-214.

[27] T. Kiguradze, On periodic in the plane solutions of nonlinear hyperbolic equations. *Nonlinear Analysis* **39** (2000), 173–185.

[28] T. Kiguradze and V. Lakshmikantham, On doubly periodic solutions of fourth-order linear hyperbolic equations. *Nonlinear Analysis* **49** (2002), 87–112.

[29] T. Kiguradze and V. Lakshmikantham, On the Dirichlet problem for fourth-order linear hyperbolic equations. *Nonlinear Analysis* **49** (2002), 197 219.

[30] T. Kiguradze and V. Lakshmikantham, On the Dirichlet problem in a characteristic rectangle for higher order linear hyperbolic equations. *Nonlinear Analysis* **50** (2002) 1153–1178.

[31] T. Kiguradze and T. Kusano, On the well-posedness of initial-boundary value problems for higher-order linear hyperbolic equations with two independent variables. (Russian) *Differ. Uravn.* **39** (2003), No. 4, 516–526; translation in *Differ. Equ.* **39** (2003), No. 4, 553–563.

[32] T. Kiguradze and T. Kusano, On ill-posed initial-boundary value problems for higher-order linear hyperbolic equations with two independent variables. (Russian) *Differ. Uravn.* **39** (2003), No. 10, 1379–1394; translation in *Differ. Equ.* **39** (2003), no. 10, 1454–1470.

[33] T. Kiguradze and T. Kusano, On bounded and periodic in a strip solutions of nonlinear hyperbolic systems with two independent variables. *Comput. and Math.* **49** (2005), 335–364.

[34] T. Kiguradze and T. Smith, On doubly periodic solutions of quasilinear hyperbolic equations of the fourth order. *Proc. Confer. Differential* & *Difference Equations Appl.* (Melbourne, Florida), 2006, 541–553.

[35] T. Kiguradze and V. Lakshmikantham, On initial–boundary value problems in bounded and unbounded domains for a class of nonlinear hyperbolic equations of the third order. *J Math. Anal. Appl.* **324** (2006), 1242–1261.

[36] T. Kiguradze and R. Ben-Rabha, On strong well–posedness of initial–boundary value problems for higher order nonlinear hyperbolic equations with two independent variables. *Georgian Math. J.* 24 (2017), No. 3, 409–428. ?????????????

[37] T. Kiguradze and N. Al-Jaber, Multi dimensional boundary value problems for linear hyperbolic equations of higher order. *International Workshop QUALITDE* 2016, Tbilisi, Georgia, 127–131. (http://rmi.tsu.ge/eng/QUALITDE-2016/workshop_2016.htm 127–131).

[38] T. Kiguradze and N. Al-Jaber, Ill–Posed Multi Dimensional Boundary Value Problems for Linear Hyperbolic Equations of Higher Order. *International Workshop QUALITDE* 2017, Tbilisi, Georgia, 92–96. (http://rmi.tsu.ge/eng/QUALITDE-2017/Abstracts_workshop_2017.pdf).

[39] T. Kiguradze and N. Al-Jaber, Multidimensional periodic problems for higher order linear hyperbolic equations. *Georgian Math. J.* (to appear).

[40] T. Kiguradze and N. Al-Jaber, Well–posed boundary value problems in a multidimensional box for higher order linear hyperbolic equations (in preparation).

[41] V.Lakshmikantham and S.G.Pandit, Periodic solutions of hyperbolic partial differential equations. *Comput. and Math.* **11** (1985), No 1-3, 249–259.

[42] Liu Baoping, The integral operator method for finding almost-periodic solutions of nonlinear wave equations. *Nonlinear Anal.* **11** (1987), No 5, 553–564.

[43] Yu.A.Mitropolsky and G.P.Khoma, On periodic solutions of the second order wave equations. I. (Russian) *Ukrain. Mat. Z.* **38**(1986), No 5. 593–600.

[44] Yu.A.Mitropolsky and G.P.Khoma, On periodic solutions of the second order wave equations. II. (Russian) *Ukrain. Mat. Z.* **38**(1986), No 6, 733–739.

[45] Yu.A.Mitropolsky and G.P.Khoma, On periodic solutions of the second order wave equations. III. (Russian) *Ukrain. Mat. Z.* **39**(1987), No 3, 347–353.

[46] Yu.A.Mitropolsky and B.P.Tkach, Periodic solutions of nonlinear systems of partial differential equations of neutral type. (Russian) *Ukrain. Mat. Z.* **21**(1969), No 4, 475–486.

[47] Yu.A.Mitropolsky and L.B.Urmancheva, On two-point boundary value problem for systems of hyperbolic equations. (Russian) *Ukrain Mat. Z.* **42**(1990), No 12, 1657–1663.

[48] B.I.Ptashnyck, Ill–posed boundary value problems for partial differential equations. (Russian) *Naukova Dumka, Kiev,* 1984.

[49] P.H.Rabinowitz, Periodic solutions of nonlinear hyperbolic partial differential equations. *Comm. Pure Appl. Math.* **20**(1967), 145–205.

[50] P.H.Rabinowitz, Periodic solutions of nonlinear hyperbolic partial differential equations. II. *Comm. Pure Appl. Math.* **22**(1969), 15–39.

[51] P.H.Rabinowitz, Time periodic solutions of nonlinear wave equations. *Manuscripta Math.* **5**(1971), 165–194.

[52] B.P.Tkach, Periodic solutions of systems of partial differential equations of neutral type. (Russian) *Differentsial'nye Uravneniya* **5**(1969), No 4, 735–748.

[53] B.P.Tkach, Periodic and quasiperiodic solutions of systems with distributed parameters. (Russian) *Doctoral thesis, Institute of Mathematics of the Ukrainean Academy of Sciences, Kiev,* 1991.

[54] S.Walczak, Absolutely continuous functions of several variables and their application to differential equations. *Bull. Pol. Acad. Sci. Math.* **35**(1987), No 11-12, 733–744.

[55] S.V.Žestkov, On periodic solutions of nonlinear hyperbolic systems in a strip. (Russian) *Differentsial'nye Uravneniya* **25**(1989), No 10, 1806–1807.