Optimizing Dynamic Range for
Micro-Electro-Mechanical Resonators

by

Douglas John Schoeller

Bachelor of Science
Department of Mechanical and Aerospace Engineering
Florida Institute of Technology
2017

A thesis
submitted to Florida Institute of Technology
in partial fulfillment of the requirements
for the degree of

Master of Science
in
Mechanical and Aerospace Engineering

Melbourne, Florida
July, 2018
We the undersigned committee hereby approve the attached thesis.

Optimizing Dynamic Range for Micro-Electro-Mechanical Resonators by Douglas John Schoeller

Dr. Steven Shaw
Professor
Mechanical and Aerospace Engineering
Committee Chair

Dr. Ersoy Subasi
Assistant Professor
Engineering Systems
Outside Committee Member

Dr. Hector Gutierrez
Professor
Mechanical and Aerospace Engineering
Committee Member

Dr. Hamid Hefazi
Professor and Department Head
Mechanical and Aerospace Engineering
ABSTRACT

Title:
Optimizing Dynamic Range for
Micro-Electro-Mechanical Resonators

Author:
Douglas John Schoeller

Major Advisor:
Dr. Steven Shaw

The performance of microelectromechanical systems (MEMS) is a topic that attracts attention from the scientific community due to their diverse applications. The use of MEMS has been applied to areas of medicine, timing, sensing, and microfluids to name a few. This work seeks to enhance the performance of a MEM resonator by optimizing its linear dynamic range (LDR) through the idea of combining the mixed nonlinear effects that stem from mechanical and electrostatic forces. The modeling procedure in this work incorporates the third-order Duffing model and includes an electrostatic effect that is expanded to the fifth order nonlinearity. When involving the nonlinear effects simultaneously, this no longer allows for closed-form solutions and the approach to finding an optimal performance state resorts to numerical methods. It has been customary to apply a DC bias on the resonator that is directed to cancel out the cubic nonlinearity composed of both mechanical and electrostatic terms, but this study shows that it is possible to increase the LDR still more.
# Table of Contents

Abstract iii

List of Figures x

List of Tables x

Acknowledgments xi

1 Introduction of the Field 1
   1.1 Historical Background and Development . . . . . . . . . . . . . . . 1
   1.2 The MEM Resonator . . . . . . . . . . . . . . . . . . . . . . . . . 4

2 Modeling 7
   2.1 Equation of Motion Setup . . . . . . . . . . . . . . . . . . . . . . . 7
   2.2 Forces from Mechanics . . . . . . . . . . . . . . . . . . . . . . . . 9
   2.3 Forces from Electrostatics . . . . . . . . . . . . . . . . . . . . . . 14
   2.4 Electrostatic Driving Force . . . . . . . . . . . . . . . . . . . . . . 18
   2.5 Damping . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . 19
   2.6 Equation of Motion . . . . . . . . . . . . . . . . . . . . . . . . . . 22
   2.7 Varying the Nonlinear Coefficients . . . . . . . . . . . . . . . . . . 23

3 Analysis of the Equation of Motion 28
3.1 The Method of Averaging ........................................ 29
3.2 Definitions of Desired Quantities ............................... 34

4 Optimization Procedure ........................................... 40
  4.1 Determining the Critical States ................................. 42
  4.2 Finding and Verifying the Optimal Operating Point .......... 48

5 Main Results .......................................................... 55

6 Discussion and Future Work ....................................... 60

References .............................................................. 63
List of Figures

1.1 Frequency response diagrams for three different systems, each at three different drive levels. The solid lines in each figure show the system at the maximum LDR for systems that show (a) a pure hardening response, (b) a pure softening response, and (c) a combination of both hardening and softening effects. Figure from Fisher [7] . . . . 6

2.1 (a) A schematic of a single-anchored double-ended tuning fork with labels on the location of the biased anchor, drive electrodes, and the sense electrode. (b) The DETF with a color gradient showing the relative displacement of the resonance mode shape, with red being the most extreme displacement; here the electrodes have been removed for clarity. The features are similar to that of other MEMS devices, with beam thickness $h = 6\mu m$, width $b = 20\mu m$, and length $\ell = 200\mu m$. Figure from Ghaffari et al. 2013 [8]. . . . . . . . . . . . 8

2.2 Example of a clamped-clamped 200 kHz MEMS resonator, with electrode gap size $d = 2\mu m$, beam thickness $h = 5\mu m$, width $b = 25\mu m$, and length $\ell = 446\mu m$. Figure from Shao et al. 2008 [20]. . . . . 9

2.3 A deformed differential element in a beam under a tension $\tau$ . . . . . . 11
2.4 A schematic showing the orientation of the electrodes at the clamped-clamped beam, each of which is carrying a voltage to actuate and detect the beam motion. Electrode gap $d$ is greatly exaggerated for clarity; typically it is less than $h$. Figure from Shaw et al. [21].

2.5 A typical linear harmonic response curve of the dynamic modification factor (DMF) versus frequency ratio. Note the quality factor $1/2\zeta$ is shown as the height of the response. Figure from Tedesco et al. [24].

2.6 The line in the $(\delta, \gamma)$ plane consisting of the values of $\delta$ and $\gamma$ for $v$ increasing from zero to unity.

2.7 The variation of slope of the parametric line as a function of $r$ has a unique maximum at $r = \sqrt{35}/6$.

3.1 Illustration of the critical state of a typical frequency response curve operated at the critical driving force $f_{cr}$. The point $(\sigma_{cr}, \Upsilon_{cr})$ is located at the onset of bistability.

3.2 A typical steady state curve $H(\Upsilon)$ as a function of $\Upsilon$ for a system tuned to a $\sigma$ value that yields two SN points, i.e., bistability. When the system driving force level, corresponding to $F$, is varied, intersection points $H(\Upsilon) = F$ correspond to steady-state response conditions. Here as $F$ is increased from zero, two SN bifurcations are encountered, at the levels indicated, leading to bistability (three intersection points) over a range of drive levels between the SNs.
3.3 \( H(\Upsilon) \) versus \( \Upsilon \) for a critical value of the drive frequency, corresponding to \( \sigma_{cr} \). The location of the critical cusp point is indicated with amplitude \( \Upsilon_{cr} \) and drive level \( F_{cr} \). This satisfies both the first and second derivative conditions for a cusp. ................. 38

3.4 \( H(\Upsilon) \) versus \( \Upsilon \) for a critical value of the drive frequency, corresponding to \( \sigma_{cr} \). The location of the critical cusp point is indicated with amplitude \( \Upsilon_{cr} \) and drive level \( F_{cr} \). This satisfies both the first and second derivative conditions for a cusp. ................. 38

4.1 Surfaces of \( \Upsilon_{cusp} \) calculated numerically for a range of \( \delta \) and \( \gamma \). The red and green curves are the closed-form curves produced by Fisher [7] for the pure quintic and pure cubic cases, respectively, which the surface converges to in the appropriate limits. ................. 44

4.2 A typical cross section of the \( \Upsilon_{cusp} \) surfaces that corresponds to a parameter line for a given \( r \). Points 1-6 are points consisting of components \( (\delta, \gamma, \Upsilon_{cusp}) \) that are in the neighborhood of interest and express different behaviors in the state of the system. Point 4 is before the onset of bistability; points 2 and 5 are at the critical point, and points 1, 3, and 6 are beyond the onset of bistability. . . . 47

4.3 Parameter lines in the \( (\delta, \gamma) \) plane with \( r = 0.9, \sqrt{35}/6, \) and 1.1, with \( v \) going from zero to unity. Note that with \( r_{opt} = \sqrt{35}/6 \) the slope of the parameter line is the largest, which coincides with the largest \( \Upsilon_{cr} \) that can be achieved. ......................... 49

viii
4.4 Corresponding cross sections of the $\Upsilon_{cusp}$ surfaces for $r = 0.9, \sqrt{35}/6$, and 1.1, showing the corresponding locations of $(v_{cr}, \Upsilon_{dc}, \Upsilon_{cr})$ as the normalized voltage $v$ is varied, where $\Upsilon_{dc}$ is at the degenerate cusp, and $\Upsilon_{cr}$ is at the corresponding upper cusp point. The point on the upper blue curve, for $r = r_{opt}$, is the $\Upsilon_{cr}$ that corresponds to the desired solution to the min-max problem. Here $\Gamma = 0.01$.

4.5 The cross section of the $\Upsilon_{cusp}$ surfaces and the optimal parametric line at $r = \sqrt{35}/6$ labeled in the same manner as Figure 4.2, where the point located at the point 5 location is the optimal operating point. Here $\Gamma = 0.01$.

4.6 $H(\Upsilon)$ versus $\Upsilon$ at the critical conditions. The cusp conditions indicate $\Upsilon_{cr}$ and $F_{cr} = H(\Upsilon_{cr})$, where $F_{cr} = f_{cr}^2$. This plot confirms that the point found numerically satisfies the derivative constraints imposed by the optimization problem. Here $\Gamma = 0.01$.

4.7 The frequency response at the critical conditions, where $\Upsilon$ is plotted versus the frequency detuning $\sigma$. The critical values $\sigma_{cr}, \Upsilon_{cr}$ and the peak conditions $\sigma_{max}, \Upsilon_{max}$ are indicated. Taken at $\Gamma = 0.01$.

5.1 Curve-fitting second, third, and fourth order polynomials in the logarithmic domain to voltage level versus damping data. Note as $\Gamma$ goes to zero, the functions converge to their leading terms, which vary around $v = 1$. Also note the fourth order curve is dashed to illustrate it varies little from the third order fit.
5.2 Curve-fitting second, third, and fourth order polynomials in the logarithmic domain to amplitude ratio \( R_{\text{max}}/R_{q} = \sqrt{\Upsilon_{\text{max}}/\Upsilon_{q}} \) versus damping data. In a similar way to Figure 5.1, the fourth order fit does not vary much from the third order fit.

List of Tables

3.1 Closed-form results of pure quintic and pure cubic cases. Note that the nonlinear parameters \( \gamma \) and \( \delta \) must be taken to be at their respective intercept points. Note also that the natural frequency has been normalized to unity in the present work.

5.1 Electrostatic tuning and critical quantities for optimal performance for different values of \( \Gamma \). Note that the values of \( \Gamma \), \( \delta \), and \( \gamma \) have been normalized using the \( v \)-dependent natural frequency \( \omega_{n} \), as defined in Chapter 3.

5.2 Electrostatic tuning and the desired maximum quantities at different values of \( \Gamma \).
Acknowledgements

I’d like to thank Dr. Shaw for his guidance and patience (definitely patience) in taking me up as his graduate assistant. I’ll be forever grateful for his help in getting me through graduate school.
Chapter 1

Introduction of the Field

1.1 Historical Background and Development

From the beginning of Richard Feynman’s look into the miniaturization of systems in the late 1950’s [5] to the rapid advancement of technology in the following decades, the extent of using small electrical devices, most notably printed integrated circuits (ICs), has sent a wave of intrigue to the engineering community.

The benefit of using very small electrical components has manifested itself in the modern computer era of the twenty-first century, with an ability to store and process information that vastly outperforms the computing capacity of three decades ago. Small components in ICs such as transistors do not require as much energy and respond quicker than their bulky discrete predecessors [17]. Photolithography improvements in the 20th century went from less than 10 transistors to an IC in the 1960s to as many as 10 million by the 1990s [18]. As of June 2017, IBM has managed to put 30 billion transistors onto a 5 nm chip [1], in an effort to keep up with the pace set by Moore’s Law, which states that the number of transistors in
an IC will double every two years. As time moves on, the trend set by Moore’s Law is getting more difficult to match. Research has resorted to different approaches to modeling these circuits, involving things like quantum computing and carbon nanotubes for transistors. Shalf and Leland [19] outline a three-dimensional approach of alternative methods to continue advancing in microelectronic technology, which includes building new devices, building new architectures with or without new devices, and developing new types of computational models. All this effort in perseverance to match the increasing demand for computational efficiency from a constantly advancing society.

The ability to scale down electrical devices to the extent that technology can provide has become the fuel for today’s technological revolution, with a looming increase on the dependency on the properties of electrical components including energy consumption, precision, and manufacturability. It is imperative that future devices carry properties that not only provide reliable information sustainably, but are easy to incorporate into the market. Therefore it was only natural to also apply the idea of miniaturization to mechanical systems, to machinery, which Feynman shared his fascination for in his notable lecture *There’s Plenty of Room at the Bottom* [5] and continued to do so in his sequel lecture *Infinitesimal Machinery* [6]. With these talks, Feynman incited conversation in the scientific community about the possible uses for these “tiny machines” that are actuated by electrostatics. Presently, the field of microelectromechanical systems (MEMS), the topic of this thesis, has developed and has now found itself as a particular area of interest in the science and engineering communities [17] [18]. Petersen [16] shows how since the 1980s, the manufacturing of these physical mechanisms could be done inexpensively by exploiting silicon for its mechanical properties, that on a small
scale with no stress concentrations, silicon maintains itself as a suitable, durable building material.

Since *Infinitesimal Machinery* in the 1980s, there has obviously been only a climb in the production of electronics, and along with that, a developed curiosity and eventual practicality for MEMS. For instance, MEMS are being applied in accelerometers in smartphones and several other electronic accessories along with applications to inkjet printers for precise ink-dispensing applications [7]. Grayson et al. [9] discussed that due to their small size, compatibility with electronics, and ease of operation on small time scales, MEMS are useful in the ability to create retinal implants, neural implants to work with the central nervous system, and small needles to help with vaccinations. They also have the potential to assist in drug delivery to the nervous and endocrine system and serve as long-term biosensors for monitoring. MEMS have been used in as assortment of applications like serving as micropumps [15], laser scanners [10], and of course as resonators for frequency reference and time-keeping [3] [7].

MEMS are now being used as a replacement for quartz resonators in oscillators to provide a steady source of time for high-speed computer operations [3]. This is important because MEMS devices can be directly integrated onto circuits, while quartz resonators must be custom fitted for certain applications, remarkably similar to ICs replacing discrete components in computers that began to happen in the mid-twentieth century [3]. MEMS in the same way as integrated transistors use less material and do not require as much power [7].

From the beginning, Feynman inquired about the physical properties of these tiny machines, in particular how their behavior would differ from the familiar and more understood macroscale devices. This initial inquiry ultimately brought
about modern experimentation and simulation to determine MEMS properties, in an effort to eventually optimize their performance [7] [17] [20] [21].

1.2 The MEM Resonator

The focus of this work expands upon previous studies on modeling MEM resonators, focusing on limitations imposed by nonlinear behavior, particularly for the clamped-clamped beam form of a MEM resonator. Like any measurement device it is important that the signal generated from the resonator deviates clearly from the surrounding noise, which is to have a large signal to noise ratio (SNR). This would mean to either limit noise, or to enlarge the amplitude of the device, the latter becoming the more reasonable course to take, since noise can only be minimized to a certain extent. Past work has shown the limitations of resonator amplitude due to nonlinear effects brought about by mechanical and electrostatic factors [7] [17] [21]. Nonlinearity in the stiffness of the system causes bistability, which is to have more than one vibratory state at a given drive frequency. Bistability is not desired from a resonator, since the objective is to maintain a steady-state vibration at an amplitude that is consistently above the noise level. If bistability exists, noise or other disturbances may cause the response to move to another operating point, a completely unacceptable situation. Research has worked toward achieving the maximum linear dynamic range (LDR) of the device, which is the largest amplitude possible before the system reaches a bistable state [20] [14].

In a recent study, Fisher [7] examined the extent of cubic and fifth order (quintic) nonlinear effects separately that are incurred on the system up to the maximum possible LDR, and was able to obtain closed-form expressions for the maximum
amplitude, the associated magnitude of the driving force, and the frequency at which the state of the system is obtained for each effect. Mechanical factors on the device produce a “hardening” response in the system while electrostatic factors produce a “softening” response [7] [14] [17]. The visualization of bistability due to each effect is shown in the frequency-response diagrams in Figure 1.1. A similar approach is taken in this work to model the system, however the cubic and quintic nonlinear effects are incorporated simultaneously, no longer allowing for closed-form expressions of the desired dynamic quantities. The motivation for this move is to confirm the idea that the mechanical and electrostatic effects can be combined in such a way to not only raise the amplitude of the device but to also keep the response of the system single-valued, to resemble the solid line in Figure 1.1(c) [20].
Figure 1.1: Frequency response diagrams for three different systems, each at three different drive levels. The solid lines in each figure show the system at the maximum LDR for systems that show (a) a pure hardening response, (b) a pure softening response, and (c) a combination of both hardening and softening effects. Figure from Fisher [7].
Chapter 2

Modeling

This chapter is dedicated to the derivation of the equation of motion of a prototypical MEM resonator element, a clamped-clamped beam acted on by symmetrically placed electrodes, and examining the effects of various parameters. The details of this specific device are not essential, however, since any electrostatically actuated mechanical resonator will have the same qualitative features, and the results presented here can be generalized to a wide variety of devices. One of the types of resonators relevant to the present work is the single-anchored double-ended tuning fork resonator (DETF) shown in Figure 2.1. The mode of interest here involves two beams coupled at both ends through a common base, with electrodes on both the inner and outer sides of both beams [8].

2.1 Equation of Motion Setup

A prototypcial example of the type of MEM oscillator considered in this work is shown in Figure 2.2. It is oriented symmetrically about the center, with a pair
Figure 2.1: (a) A schematic of a single-anchored double-ended tuning fork with labels on the location of the biased anchor, drive electrodes, and the sense electrode. (b) The DETF with a color gradient showing the relative displacement of the resonance mode shape, with red being the most extreme displacement; here the electrodes have been removed for clarity. The features are similar to that of other MEMS devices, with beam thickness $h = 6 \mu m$, width $b = 20 \mu m$, and length $\ell = 200 \mu m$. Figure from Ghaffari et al. 2013 [8].

of electrodes used for DC tuning and for AC drive. The collection of forces in an equation of motion follows Newton’s second law, accounting for the external forces acting on the beam.

The external forces that are present in this system are: (1) a restoring force due to mechanical stiffness stemming from the elasticity of the material, (2) electrostatic forces imposed by a DC bias voltage on the electrodes, which result in a “negative” restoring force, (3) an electrostatic driving force from an AC voltage on the electrodes that maintains oscillation, and (4) a force that resists motion and results in dissipation of energy. Each of these forces are incorporated into the following equation of motion, Equation 2.1.

$$ F_{\text{inertia}} = F_{\text{mech}} + F_{\text{elecDC}} + F_{\text{elecAC}} + F_{\text{damping}} \quad (2.1) $$
The remainder of this chapter addresses each component of force to ultimately produce an equation of motion of the system that contains measured and derived variables.

2.2 Forces from Mechanics

The small amplitude transverse vibrational response of a clamped-clamped beam can be modeled by the following partial differential equation (PDE) and boundary conditions, from Meirovitch [13],

\[
\rho \frac{\partial^2 u}{\partial T^2} - \tau \frac{\partial^2 u}{\partial x^2} + EI \frac{\partial^4 u}{\partial x^4} = 0
\]  

(2.2)
\begin{align}
  u(0, T) &= u(\ell, T) = 0 \\
  \frac{\partial u}{\partial x}(0, T) &= \frac{\partial u}{\partial x}(\ell, T) = 0
\end{align}

where \( u \) is the beam displacement, \( T \) is time, \( x \) is the spatial dimension along the beam, \( \rho \) is the mass per unit length, \( \ell \) is the length of the beam, \( \tau \) is the tension in the beam, \( E \) is the modulus of elasticity, and \( I \) is the second moment of the cross-sectional area. The cross section of the MEM resonator in consideration is a constant with rectangular cross-sectional area \( A = bh \) and second moment of area \( I \) equal to

\[ I = \frac{bh^3}{12}, \]

noting that \( h \) is the dimension in the direction of motion.

The approach taken to obtain an ordinary differential equation (ODE) for the mode of interest from the beam equation follows the assumed modes method, for example, as described in many texts and, for a MEM resonator, by Shaw [21]. In the assumed modes method, an analytical function that satisfies the imposed boundary conditions serves as the spatial shape of the vibrational mode. The beam equation is then projected onto that mode shape. This is under the assumption that the system is not going to significantly interact with any other modes of vibration. To this end, let the solution to the beam equation take the form

\[ u(x, T) = \phi(x)q(T)h \]

where \( q(T) \) is a non-dimensional modal displacement, \( \phi(x) \) is a mode shape, and the beam thickness \( h \) serves to scale the solution down to the
microscale. The form of the assumed mode used in this study is given by:

\[ \phi(x) = \sin^2 \left( \frac{\pi x}{\ell} \right), \quad x = [0, \ell] \] \hspace{1cm} (2.6)

where \( \ell \) is the length of the beam.

With this information, provided the values of the beam properties \( \rho, E, \) and \( I, \) the only remaining value to be found in order to proceed with the beam equation is the tension \( \tau. \) The tension is generally composed of two terms, a constant term \( \tau_0, \) imposed, for example, by thermal effects, which is taken to be zero in the analysis in subsequent chapters, since it simply shifts the mechanical part of the natural frequency, and a displacement-dependent term that stems from mid-line stretching of the beam when it undergoes transverse displacement. This latter term is derived with the use of an infinitesimal element of the beam, shown in Figure 2.3. This

![Figure 2.3: A deformed differential element in a beam under a tension \( \tau. \)](image)

tension is a force along the axis of the beam defined using the relationship between

11
stress and area

\[ \tau = \psi A \]  \hspace{1cm} (2.7)

where \( \psi \) is the stress on the cross section of the beam. Since the beam is made of an amorphous, homogeneous material, the beam has linear elastic properties [5] [18] and can be modeled with Hooke’s Law,

\[ \psi = E \varepsilon \]  \hspace{1cm} (2.8)

where the strain \( \varepsilon \) is in turn a function of the deformation \( \lambda \) that occurs in the differential element, which is given by

\[ \varepsilon = \frac{\lambda}{\ell} \]  \hspace{1cm} (2.9)

The deformation found from Figure 2.3 is

\[ \lambda = \sqrt{dx^2 + du^2} - dx. \]  \hspace{1cm} (2.10)

Combining Equations 2.7-2.10 produces the local tension for a single differential element induced by stretching, given by

\[ \Delta \tau = \frac{EA}{\ell} \left( \sqrt{dx^2 + du^2} - dx \right). \]  \hspace{1cm} (2.11)

In order to determine the total tension, assuming this tension is immediately present along the entire length of the beam due to deformation (that is, wave speeds along the beam are taken to be large compared to the time scale of transverse vibration), the differential tension is integrated over the length of the beam, which
is not trivial in the form of Equation 2.11. The deformation can be approximated with the use of a binomial expansion

$$\sqrt{dx^2 + du^2} - dx = \left(\sqrt{1 + \left(\frac{du}{dx}\right)^2} - 1\right) dx = \frac{1}{2} \left(\frac{du}{dx}\right)^2 dx + ... \quad (2.12)$$

The leading term is kept and after integration over the length of the beam, the resulting total tension is formed as

$$\tau = \tau_0 + \frac{EA}{2\ell} \int_0^\ell \left(\frac{du}{dx}\right)^2 dx. \quad (2.13)$$

Now that an equation for tension has been formed, the mode shape $\phi(x)$ and the rest of the parameters are substituted into the beam equation. The resulting equation is projected onto the mode shape by multiplying the PDE by the mode shape $\phi(x)$ and integrating over the length of the beam $\ell$ and dividing by $\ell$. This is essentially a dot product taken over all the differential elements of the beam. The result of the projection takes the form of a generic cubic nonlinear, second order ODE. Dividing the equation by the second order coefficient, which is the effective mass of the mode, $m_{eff} = 3h\rho/8$, yields

$$q'' + \omega_m^2 q + \gamma_m q^3 = 0. \quad (2.14)$$

where $q'$ denotes $dq/dT$ and $\omega_m^2$ and $\gamma_m$ are given by

$$\omega_m^2 = \frac{4\nu^2 h^2 \pi^4}{9\ell^4} \quad (2.15)$$
\[ \gamma_m = \frac{\nu^2 h^2 \pi^4}{3 \ell^4}. \]  

(2.16)

and \( \nu = \sqrt{E/\rho} \) is the speed of sound of the beam material and subscript \( m \) denotes a property that stems from mechanics.

Equation 2.14 serves as the free vibration equation of motion involving mechanical nonlinearity, with the third order nonlinear term, commonly referred to as the Duffing nonlinearity. This equation, unlike approaches in classical linear beam theory, accounts for midline stretching. As the deflection in the beam increases, so does the tension, and therefore increases the beam’s stiffness, resulting in the “hardening” effect illustrated in Chapter 1 (since \( \gamma_m > 0 \)). The physical effect of this is that the frequency of free vibration increases as a function of amplitude.

### 2.3 Forces from Electrostatics

One of the main concepts of MEM devices discussed by Feynman is that they can be actuated with the use of electrostatics. In order to obtain an accurate model for a capacitive clamped-clamped beam at such a small scale, it becomes necessary to include their nonlinear effects in the equation of motion.

This development follows the methodology taken by Fisher [7] and Shaw et al. [21], which includes a fifth order (quintic) nonlinearity due to electrostatic effects, as is observed in devices such at the DETF [17]. Stemming from fundamental principles of physics, the electrostatic model begins with Coulomb’s Law, which shows the relationship between two neighboring electrical charges in Equation 2.17.

\[ F_e = k_e \frac{q_1 q_2}{r^2} \]  

(2.17)
In the context of the present system, the charges involved come from the two electrodes that are placed symmetrically at the center of the beam shown in Figure 2.2. A schematic of the system is shown in Figure 2.4, where the electrode gaps are greatly exaggerated for clarity in the diagram [21]. The charges that are found in Coulomb's Law can be exchanged for potential differences that exist in each electrode. The resultant force of the electrostatics is defined as the sum of the forces that come from each electrode. As it vibrates, the distance of beam elements from the electrodes deviates from the gap $d$ at equilibrium. When the beam displaces closer to one electrode at a value specified by the mode shape $\phi(x)$, the same value is the same amount farther from the other electrode. The electrodes attract in opposite directions relative to each other and together attract the beam away from its free vibrating equilibrium position; these forces combat the mechanical stiffness forces that are found within the beam. Putting all these pieces together provides an equation for the electrostatic force present on a single element at a
position $x$ along the beam,

$$F_{elec} = -\frac{\epsilon_0 b}{2} \left[ \frac{V_{b1}^2}{(d - u)^2} - \frac{V_{b2}^2}{(d + u)^2} \right]$$  \hspace{1cm} (2.18)$$

where $V_{bi}$ are the bias voltages applied to the electrodes. Since this work focuses on the symmetrical clamped-clamped beam, the present voltages are taken to be equal, $V_{b1} = V_{b2} = V_b$, so that Equation 2.18 simplifies to:

$$F_{elec} = -2\epsilon_0 bV_b^2 ud \frac{1}{(d^2 - u^2)^2}.$$  \hspace{1cm} (2.19)$$

and with the substitution of the assumed form of $u$ produces

$$F_{elec} = -2\epsilon_0 bV_b^2 dhq \sin^2 \left(\frac{\pi x}{\ell}\right) \frac{1}{(d^2 - h^2q^2 \sin^4 \left(\frac{\pi x}{\ell}\right))^2}.$$  \hspace{1cm} (2.20)$$

Equation 2.20 is for an element and must be projected onto the mode of interest in order to incorporate it into the differential equation produced in Section 2.2. In fact, it would be difficult to determine analytically the extent the electrostatic force would have on the beam when vibrating in a given mode, since the electrostatics actually influence the mode shape. In order to provide a more direct comparison to its mechanical nonlinear counterpart, an approximation is employed. Equation 2.20 is expanded as a Taylor series to the fifth order in terms of $q$, which is given by

$$F_{elec} = \frac{-2b\epsilon_0 hV_b^2 q}{d^3} \left[ \sin\left(\frac{\pi x}{\ell}\right)^2 + \frac{2h^2q^2 \sin\left(\frac{\pi x}{\ell}\right)^6}{d^2} + \frac{3h^4q^4 \sin\left(\frac{\pi x}{\ell}\right)^{10}}{d^4} + \cdots \right].$$  \hspace{1cm} (2.21)$$

This is an electrostatic force that takes place on a beam element at $x$ along the
beam at displacement $u = \phi(x)qh$ from the equilibrium position, when the beam is deformed in the mode shape $\phi(x)$. In a process similar to that used for the tension in Section 2.2, which results in the effective force along the entire beam, the element force is projected onto the mode shape by multiplying by $\phi(x)$ and integrated over the length of the beam and divided by $\ell$, which results in

$$F_{DC} = \frac{-bh\varepsilon_0 V_b^2 q}{4d^3} \left[ 3 + \frac{35h^2q^2}{8d^2} + \frac{693h^4q^4}{128d^4} + \cdots \right]$$ (2.22) 

Before this force can be added into the equation of motion derived in Section 2.2, it must be normalized by the modal mass, $m_{eff}$. In addition, for convenience in the subsequent analysis, the nondimensional gap parameter $r = d/h$ is introduced as the ratio between the electrode gap size and the thickness of the beam. These steps modify Equation 2.22 into the effective electrostatic force that can be added into the equation of motion, given by

$$\frac{F_{DC}}{m_{eff}} = -\frac{b\varepsilon_0 V_b^2 q}{h^3\ell pr^3} \left[ 2 + \frac{35q^2}{12r^2} + \frac{231q^4}{64r^4} + \cdots \right].$$ (2.23) 

We will ignore the higher order terms in the rest of the analysis.

At this point we can define the coefficients of the powers of $q$ of Equation 2.23, as a means to compare them to the similar terms in Equation 2.14:

$$\omega_e^2 = -\frac{2b\varepsilon_0 V_b^2}{h^3\ell pr^3}$$ (2.24) 

$$\gamma_e = -\frac{35b\varepsilon_0 V_b^2}{12h^3\ell pr^5}$$ (2.25)
\[ \delta_e = -\frac{231b_0V_b^2}{64h^3lpr^2} \]  

(2.26)

noting that there is now a fifth order term in the equation of motion given by \( \delta_e q^5 \), and that subscript \( e \) denotes electrostatic terms.

It is now very easy to collect the like terms from each contribution of force in the full equation of motion. Equations 2.27 and 2.28 illustrate the combining of mechanical and electrostatic effects into coefficients to be used in Chapter 3:

\[ \omega_n^2 = \omega_m^2 + \omega_e^2 \]  

(2.27)

\[ \dot{\gamma} = \gamma_m + \gamma_e \]  

(2.28)

where \( \delta_e \) is the only quintic term involved in this model and it is zero or negative. Note that these coefficients all depend on the bias voltage \( V_b \). Also, note that \( \omega_n^2 \) must be positive for vibrations to occur, and if the DC bias \( V_b \) is sufficiently large to cause \( \omega_n^2 \) to be zero, electrostatic buckling will occur, which is not of interest here. However, bias levels that cause \( \dot{\gamma} \) to be zero are of interest and will be discussed below.

### 2.4 Electrostatic Driving Force

An alternating external energy source is necessary if the system experiences energy loss and a steady-state vibration is desired. The MEM devices considered in this work are driven electrostatically by an alternating current applied to one or both electrodes. This can be modeled with a frequency \( \omega \) and an effective force
amplitude \( f \), also obtained by projection on the mode shape and divided by the modal mass, to make the following expression for the effective drive force:

\[
F_{elec,AC} = -\hat{f} \cos (\Omega T). \tag{2.29}
\]

Note that \( \hat{f} \) has units of frequency squared, since it has been divided by the modal mass and a length scale. This model, likewise to the formation of the electrostatic force model with the use of a Taylor series, is in a form that is convenient to proceed with the solving process covered in Chapter 3. Its form is not crucial, since one can note that there is some monotonic relationship between the AC current and \( \hat{f} \). This work concerns when the driving frequency is close to that of the resonant frequency of the system.

## 2.5 Damping

All other quantities up to this point have been able to be derived from first principles, but damping is more challenging to model. Since damping can come from a number of sources such as fluid friction, material hysteresis, and anchor losses, the approach to modeling damping is heuristic and based on matching experimental tests to a common model.

The model used for resonator damping in this work is a linear dissipation model in the form of an effective force on the mode, given by

\[
F_{damping} = 2\zeta \omega_n q'. \tag{2.30}
\]

where \( \omega_n \) is the natural frequency and \( \zeta \) is the damping ratio. This results in an
exponential decay of the vibration amplitude. The damping ratio can be found experimentally by applying the standard logarithmic decrement method to the amplitude decay of the system. This method of modeling damping is common in the fields of structural dynamics and control systems.

In the MEMS community, as in the EE community, it is common to use the quality factor $Q$ as the parameter describing damping. It is defined as the ratio of energy stored to the energy loss per cycle due to dissipative forces and takes on the following form:

$$Q = \frac{1}{2\zeta}. \quad (2.31)$$

Since $Q$ is inversely proportional to $\zeta$, therefore a higher quality factor means the system experiences less loss per cycle. For MEMS $Q$ is usually in the range $10^2 < Q < 10^6$.

Another form for damping is to use the decay time $\hat{\Gamma}^{-1}$ as the variable representing damping in the equation of motion. The decay rate (the inverse of the decay time) is given by

$$\hat{\Gamma} = \frac{\omega_n}{2Q}. \quad (2.32)$$

Substituting Equation 2.30 for Equations 2.31 and 2.32 yields a convenient expression for the damping force:

$$F_{damping} = 2\hat{\Gamma}q'. \quad (2.33)$$

It is important to understand the impact of damping on the linear frequency response of a system. Figure 2.5 shows the the bandwidth, bounded by the half-power points, defined as the points such that the output response is cut in half. The frequencies that are associated with the half power points can be put in terms of the damping ratio, and for $\zeta \ll 1$, the bandwidth is the difference in these frequencies.
can be estimated as $2\zeta = Q^{-1}$. Therefore a larger damping ratio has the ability to expand the bandwidth and make the system less prone to the nonlinear property of bistability, however at the cost of lessening the response amplitude of the device. The low damping ratios of typical MEM resonators requires much more careful maintenance of stability.
2.6 Equation of Motion

At this point all of the quantities from Equation 2.1 have been addressed. In this section all the variables are brought together to complete the equation of motion which will be used for the analysis, which is given by,

\[ q'' + 2\hat{\Gamma}q' + \omega_n^2 q + \hat{\gamma}q^3 + \delta_e q^5 = \hat{f} \cos(\Omega T) \]  

(2.34)

Again, the equation of motion is presented per unit mass so that \( \omega_n^2 \), \( \hat{\gamma} \), and \( \delta_e \) are the manifestations of the combinations of linear, third order, and fifth order terms, respectively, that were present in the mechanical and electrostatic stiffnesses. It is important to note that \( \delta_e \) is negative, since the only fifth order term is due to electrostatics.

This equation of motion can be made simpler and more convenient to use by rescaling time with the following definition of the time scale \( t \):

\[ t = \omega_n T. \]  

(2.35)

When \( q \) is established with this new time scale, its time derivatives result as

\[ \frac{dq}{dT} = \omega_n \frac{dq}{dt} \]  

(2.36)

\[ \frac{d^2q}{dT^2} = \omega_n^2 \frac{d^2q}{dt^2} \]  

(2.37)
This modifies the equation of motion to become

\[ \omega_n^2 \ddot{q} + 2\hat{\Gamma} \omega_n \dot{q} + \omega_n^2 q + \hat{\gamma} q^3 + \delta e q^5 = \hat{f} \cos \left( \frac{\Omega}{\omega_n} t \right) \]  

(2.38)

where \( \dot{q} \) denotes \( dq/dt \). Dividing through by \( \omega_n^2 \) yields the completely non-dimensional equation of motion

\[ \ddot{q} + 2\Gamma \dot{q} + q + \gamma q^3 + \delta q^5 = f \cos (\omega t) \]  

(2.39)

where \( \Gamma = \frac{\hat{\Gamma}}{\omega_n}, \gamma = \frac{\hat{\gamma}}{\omega_n^2}, \delta = \frac{\delta e}{\omega_n^2}, \) and \( f = \frac{\hat{f}}{\omega_n^2} \). Note that this allows the coefficient to the linear stiffness term to be unity, which will be useful moving forward. Also, note that the damping coefficient in this form is \( 2\Gamma = Q^{-1} \). The remainder of the work is with respect to this time scaling.

### 2.7 Varying the Nonlinear Coefficients

In practice, once a device is fabricated the only things that can be changed are the DC voltage \( V_b \), the AC voltage amplitude, expressed here as \( f \), and the AC frequency, normalized to \( \omega \). So, even after the device is built, one can tune the natural frequency and nonlinear parameters using the DC bias, and then vary the drive amplitude and frequency to meet the desired operating conditions. Therefore, it is important to understand the nature of how the linear and nonlinear device parameters vary as a function of the DC bias. For \( V_b = 0 \) the system has only a hardening cubic term, that is \( \gamma = \gamma_m > 0 \), and the quintic term is zero, \( \delta = 0 \), which is the standard Duffing model with a purely mechanical natural frequency, \( \omega_m \). As the DC voltage is increased, all stiffness terms are decreased, including
the linear frequency and both nonlinear terms, since the DC bias effect is purely softening. Here the focus is centered on the nonlinear terms, which have been normalized by the bias-dependent natural frequency $\omega_n$. For convenience, these are provided explicitly here:

$$\gamma = \frac{12h^5\nu^2\pi^4r^5\rho - 105b\varepsilon_0V_b^2}{16h^5\nu^2\pi^4r^5\rho - 72b\varepsilon_0r^2V_b^2}$$  (2.40)

$$\delta = -\frac{2079b\varepsilon_0V_b^2}{256h^5\nu^2\pi^4r^7\rho - 1152b\varepsilon_0r^4V_b^2}$$

As the DC voltage is increased, $\delta$ and $\gamma$ vary along a line in the $(\delta, \gamma)$ plane that starts at $(0, \frac{3}{4})$, from which both parameters decrease. There is an important value of the DC bias at which the cubic term is zero, determined by solving Equation 2.40 for $V_b$ when $\gamma = 0$. This value is given by

$$V_{b0} = 2\pi^2\nu \sqrt{\frac{\rho h^5r^5}{35b\varepsilon_0\ell^2}}.$$  (2.41)

It is convenient to introduce the normalized DC bias, $v = V_b/V_{b0}$, which reduces $\delta$ and $\gamma$ to the following forms

$$\gamma = \frac{105(v^2 - 1)}{4(18r^2v^2 - 35)}$$  (2.42)

$$\delta = \frac{2079v^2}{64r^2(18r^2v^2 - 35)}$$  (2.43)

which conveniently depend on only $r$ and $v$. It was already established that when $v = 0$ ($V_b = 0$), $\delta = 0$ is zero and the point $(0, \frac{3}{4})$ is one of the intercepts in the $(\delta, \gamma)$ plane. The point $v = 1$ ($V_b = V_{b0}$) corresponds to the point when the cubic
term no longer exists and there is purely a nonlinear softening effect due to the quintic term. This point, the other intercept on the line in the \((\delta, \gamma)\) plane, is \((\delta_0, 0)\) where \(\delta_0\) is seen from Equation 2.43 to be

\[
\delta_0 = \frac{2079}{64r^2(18r^2 - 35)}.
\]  

(2.44)

The condition at which the cubic term is zero, \(v = 1\), is the tuning point suggested by most previous studies [20]. However, as it will be shown, this is not necessarily optimal since competition between hardening and softening can be used to find a better operating point. Our main goal is to seek conditions that can improve upon those previous results by judicious selection of the DC bias and device parameters.

A useful view of the nonlinear parameters as \(V_b\) is varied is summarized by Figure 2.6, which shows the line traversed in the \((\delta, \gamma)\) plane as \(v = V_b/V_{b0}\) is varied from zero to unity. The slope of this line given by \(\delta\) is

\[
\frac{\gamma - \frac{3}{4}}{\delta} = \frac{80r^2}{99} - \frac{32r^4}{77}
\]  

(2.45)

which, surprisingly, and very conveniently, is a function of only the normalized gap parameter \(r\). The slope as a function of \(r\) is shown in Figure 2.7, where it is noted that it has a unique maximum at \(r_{max} = \sqrt{35}/6\), which is when the electrode gap is 98.6% of the beam thickness in the direction of its vibration. As will be seen in Chapter 4, this observation also greatly facilitates the optimization process.

These facts suggest that the parameters \(r\) and \(v\) are natural choices for the optimization process, and it will turn out that \(r_{max}\) provides one optimality condition, so that one need only find the value of \(v\) for optimality. Once these are determined, the other physical parameters can be selected to meet other design criteria. For
Figure 2.6: The line in the \((\delta, \gamma)\) plane consisting of the values of \(\delta\) and \(\gamma\) for \(v\) increasing from zero to unity.

In example, one can vary the other beam dimensions, length and width, in order to achieve a desired resonance frequency \(\omega_n\) at the optimal operating point.

It should also be noted that this entire process must be carried out for a given value of \(\Gamma\), or equivalently, \(Q\), since in practice \(\Gamma\) is always minimized (\(Q\) is maximized) in order to increase the sharpness and amplitude at resonance. Typical MEM resonators have \(Qs\) in the range \(10^2 - 10^6\).

We now turn to finding approximate solutions for the equation of motion, in a form amenable for the optimization process.
Fig 2.7: The variation of slope of the parametric line as a function of $r$ has a unique maximum at $r = \frac{\sqrt{35}}{6}$
Chapter 3

Analysis of the Equation of Motion

Now that a nondimensional equation of motion has been derived to model the system dynamics, we can proceed with the goal of optimizing the linear dynamic range (LDR). This chapter walks through the process of the method of averaging, which is a powerful approximation technique that has the ability to transform the current nonlinear, time-periodic equation of motion into averaged equations that no longer depend on time. These averaged equations govern the slowly varying amplitude and phase, or, equivalently, the quadratures (that is, the sine and cosine components), of the response near resonance under certain conditions. They can be used to derive an equation that directly relates the steady-state response amplitude and phase of the response to the magnitude $f$ and frequency $\omega$ of the driving force and device parameters.
3.1 The Method of Averaging

The method of averaging is appropriate to be employed on the MEM oscillator model because it exhibits two different time scales, one of which the system dynamics are rapidly changing in time (called *fast* time), and the other which is on a more prolonged, long term scale (*slow* time). Overall, while the oscillator is rapidly vibrating back and forth, its amplitude and phase change very little from one fast oscillation to the next, resulting in slow changes over this relatively gradual time scale. This is a common trait shared by oscillators with small damping, small nonlinearity, and driven near resonance [23]. In this method, combined with the method of variation of parameters in differential equations, these time scales are exploited and treated as if they were independent variables to ultimately produce autonomous averaged equations that are very convenient for analyzing the steady state system response. The methodology and notation used in this chapter is similar to previous works [7] [21].

The method begins with the equation of motion derived in Chapter 2, where we keep a frequency parameter $\omega_0$ in the equation in order to see the role of the natural frequency, which is set to unity after its role is clear, since this coefficient has been normalized out. Thus, the equation is

$$\ddot{q}(t) + 2\Gamma \dot{q}(t) + \omega_0^2 q(t) + \gamma \dot{q}^3(t) + \delta q^5(t) = f \cos(\omega t). \quad (3.1)$$

To initiate the method, $q$ will be subjected to a coordinate transformation into time dependent Cartesian components $s(t)$ and $v(t)$ as follows:

$$q = s(t) \cos(\omega t) - v(t) \sin(\omega t). \quad (3.2)$$
Applying a method akin to the variation of parameters, the following constraint is imposed to simplify the process:

\[ \dot{s}(t) \cos(\omega t) - \dot{v}(t) \sin(\omega t) = 0 \]  \hspace{1cm} (3.3)

Consequently this makes \( \dot{q} \) only composed of two terms, shown in Equation 3.4:

\[ \dot{q} = -s(t)\omega \sin(\omega t) - v(t)\omega \cos(\omega t) \]  \hspace{1cm} (3.4)

This in turn makes \( \ddot{q} \) of a form that involves only first derivatives, as follows:

\[ \ddot{q} = -s(t)\omega^2 \cos(\omega t) - \dot{s}(t)\omega \sin(\omega t) + v(t)\omega^2 \sin(\omega t) - \dot{v}(t)\omega \cos(\omega t). \]  \hspace{1cm} (3.5)

This is incredibly useful, because now Equations 3.2, 3.4, and 3.5 can be substituted into Equation 2.34 and it yields a linear equation in terms of \( \dot{s}(t) \) and \( \dot{v}(t) \). This equation, in addition to the constraint in Equation 3.3, also linear in \( \dot{s}(t) \) and \( \dot{v}(t) \), provides two equations with two unknowns. Solving yields expressions for \( \dot{s}(t) \) and \( \dot{v}(t) \) in terms of \( s(t) \) and \( v(t) \) and the system parameters. These expressions involve terms that are slow in time but oscillate with multiple harmonics of \( \omega \).

In the next step, the signature step of averaging, is applied on the equations \( \dot{s}(t) \) and \( \dot{v}(t) \), where they are integrated over a single period of oscillation \( 2\pi/\omega \), where \( s(t) \) and \( v(t) \) are taken to be constant over the interval. This reveals the autonomous averaged equations for \( \dot{s}(t) \) and \( \dot{v}(t) \) that describe the slow time dynamics:

\[ \dot{s}(t) = -\frac{5s^4v\delta + 2s^2(3v\gamma + 5v^3\delta) + 16s\Gamma\omega + v(6v^2\gamma + 5v^4\delta - 8\omega^2 + 8\omega_0^2)}{16\omega} \]  \hspace{1cm} (3.6)
\[
\dot{v}(t) = \frac{-8f + 5s^5\delta + 2s^3(3\gamma + 5v^2\delta) - 16v\Gamma\omega + s(6v^2\gamma + 5v^4\delta - 8\omega^2 + 8\omega_0^2)}{16\omega} \tag{3.7}
\]

At this point, knowing the context of a resonator, it becomes desirable to transform these quantities into terms of amplitude and phase. This can be done quite simply with the fundamental relationships between Cartesian and polar coordinates, in Equations 3.8 and 3.9:

\[
R^2 = s^2 + v^2 \tag{3.8}
\]

\[
\phi = \arctan\left(\frac{v}{s}\right). \tag{3.9}
\]

Taking time derivatives of these two functions and solving for \(\dot{R}\) and \(\dot{\phi}\) produces slow time dynamic equations now for amplitude and phase:

\[
\dot{R} = \frac{ss\dot{s} + v\dot{v}}{R} \tag{3.10}
\]

\[
\dot{\phi} = \frac{s\dot{v} - v\dot{s}}{R^2} \tag{3.11}
\]

with \(\dot{s}\) and \(\dot{v}\) as defined in Equations 3.8 and 3.9. Also, in order to fully convert Equations 3.10 and 3.11 into terms of amplitude and phase, it is useful to substitute \(s = R \cos(\omega t)\) and \(v = R \sin(\omega t)\). This now makes \(\dot{R}\) and \(\dot{\phi}\) both functions of \(R, \phi,\) and the design parameters, and with some simplifying, yields Equations 3.12 and 3.13:

\[
\dot{R} = -R\Gamma - \frac{f \sin(\omega t)}{2\omega} \tag{3.12}
\]
Equations 3.12 and 3.13 are fully developed averaged equations illustrating the time change in magnitude and phase. Earlier it was understood that these are slow-changing equations. Much like any dynamical system, if a time-dependent quantity, such as a change in a mass flow in fluids analysis, is slow-changing or has a very limited impact on the results of the system, it can be assumed to be zero to yield an equation of steady-state (mass flow in equals mass flow out). In a similar way, setting Equations 3.12 and 3.13 to zero yields the steady-state conditions of the MEM resonator model.

Conveniently, each equation contains a piece of a Pythagorean identity \( \sin^2(x) + \cos^2(x) = 1 \), which can be used to advance the analysis further. Now that the equations have been set to zero, solving for the sine term in Equation 3.12 and the cosine term in Equation 3.13 shows Equations 3.14 and 3.15:

\[
\sin(\omega t) = -\frac{2R\Gamma\omega}{f} \tag{3.14}
\]

\[
\cos(\omega t) = \frac{R(6R^3\gamma + 5R^4\delta + 8(\omega_0^2 - \omega^2))}{8f} \tag{3.15}
\]

Applying the Pythagorean identity reveals a sought-after, single steady-state equation that directly relates the response amplitude to magnitude of the driving force,
the driving frequency, and the device parameters:

\[
f^2 = \frac{1}{64} R^2 (60R^6\gamma\delta + 25R^8\delta^2 - 96R^2\gamma(\omega^2 - \omega_0^2) + 64(4\Gamma^2\omega^2 + (\omega^2 - \omega_0^2)^2) + 4R^4(9\gamma^2 + 20\delta(2\omega^2 - \omega_0^2)))
\] (3.16)

A couple of substitutions are used in Equation 3.16 to facilitate the solution process. The nondimensional frequency tuning variable \( \sigma \) is introduced, representing the relative value of the drive frequency to the linear resonance frequency, as follows:

\[
\sigma = \omega^2 - \omega_0^2 = \frac{\Omega^2 - \omega_n^2}{\omega_n^2}
\] (3.17)

recalling that \( \omega_0 = 1 \). Also since the equation is tenth order in \( R \) and each instance of \( R \) is at an even power, it is helpful to introduce \( \Upsilon \) as the square of the amplitude:

\[
\Upsilon = R^2.
\] (3.18)

In addition, at this order we can replace \( \omega \) by 1, since near resonance \( \sigma \ll \omega \approx \omega_0 = 1 \). Using this substitution, plus those in 3.17 and 3.18, into 3.16, produces the steady state condition that is central in the remainder of this work:

\[
f^2 = \frac{1}{64} \Upsilon (64\gamma^2 + 36\gamma^2\Upsilon^2 + 60\gamma\delta\Upsilon^3 + 25\delta^2\Upsilon^4 - 16\sigma\Upsilon (6\gamma + 5\delta\Upsilon) + 256\Gamma^2).
\] (3.19)

The steady state equation in Equation 3.19 is the essential equation from which branches the remaining analysis in this work. This analysis considers the nonlinear parameters \( \gamma \) and \( \delta \) simultaneously, requiring numerical methods to produce results for the optimal response, whereas Fisher [7] considered them independently.
3.2 Definitions of Desired Quantities

Fisher [7] solved for five quantities that were essential in summarizing the functionality of a MEM resonator in terms of its LDR. They include (1) the critical response amplitude, (2) the frequency tuning at which the critical amplitude occurs, (3) the critical driving force, (4) the maximum amplitude, and (5) the frequency tuning at which the maximum amplitude occurs. Each of these quantities are addressed and defined in this section, to provide context for the optimization problem described in Chapter 4.

The critical state of the MEM resonator is the point at which the device is at the onset of bistability. Bistability is not allowed since noise can push the response between two (or more) stable branches, if they exist. This occurs at such a combination of values of the device parameters and a driving force $f$ that brings the device to its maximum LDR, as shown from the solid curves in Figure 1.1. The five desired quantities are extracted at this state. The critical amplitude is the amplitude that has a necessary property of having a vertical tangent to the frequency response curve, where an example is illustrated in Figure 3.1. But a number of saddle node (SN) bifurcation points share this same characteristic on frequency response curves that can also exhibit bistability. That can be seen from Figure 1.1 on the larger dashed curves that have such points. In Figure 1.1(a), for a hardening system, there are two SN points that exist at two frequencies greater than $\omega_0$, and in Figure 1.1(b), for softening systems, the larger dashed curves have SN points at two frequencies that are less than $\omega_0$.

What is special about the critical amplitude at the critical state is that it is a singular SN point with a cubic tangency that occurs at the special condition of the merging of the two SN points with quadratic tangency. This is best illustrated in
Figure 3.1: Illustration of the critical state of a typical frequency response curve operated at the critical driving force $f_{cr}$. The point $(\sigma_{cr}, \Upsilon_{cr})$ is located at the onset of bistability.

Figures 3.2 and 3.3. The cusp point in Figure 3.3 corresponds to a drive level $f_{cr}$ that is the maximum amount of force that can be applied before causing bistability. This quantity will be maximized in the optimization problem, which will provide conditions for $\Upsilon_{max}$.

Fisher [7] was able to find a unique cusp condition that can be solved for in closed form in the pure quintic and pure cubic cases. But with both nonlinearities present, the situation resembles Figure 1.1(c), where more than one cusp condition can occur. The objective moving forward is to find the one that first causes the onset of bistability as the drive level is increased, and determine the drive level at that point.

An alternate view of the cusp condition will be useful for the optimization process presented in Chapter 4. Consider the steady-state condition given in Equation 3.19, which expresses the force as a function of the amplitude $\Upsilon$ and system and drive parameters. If one plots the right hand side of Equation 3.19, call this $H(\Upsilon)$,
Figure 3.2: A typical steady state curve $H(\Upsilon)$ as a function of $\Upsilon$ for a system tuned to a $\sigma$ value that yields two SN points, i.e., bistability. When the system driving force level, corresponding to $F$, is varied, intersection points $H(\Upsilon) = F$ correspond to steady-state response conditions. Here as $F$ is increased from zero, two SN bifurcations are encountered, at the levels indicated, leading to bistability (three intersection points) over a range of drive levels between the SNs.

versus $\Upsilon$, the steady-state amplitudes for a given force level are the places where $F = H(\Upsilon)$, where we have introduced $F$ for convenience. Note that $H(0) = 0$ so that the response starts with $\Upsilon = 0$ for $F = 0$. As $F$ is increased, this root moves out, and in some cases, such as the usual Duffing equation, two more roots appear, and then two disappear, in saddle-node (SN) bifurcations, as depicted in Figure 3.2. Note that at the SN conditions, if one perturbs the parameters the system will go from having zero solutions near the SN point to having two solutions, with one (local) solution occurring only at the SN condition. For a cusp condition to occur, the parameters must be such that $H(\Upsilon)$ has an inflection point, such as shown in Figure 3.3, so that if one perturbs away from this point the system goes
from having one root to three roots. This is precisely what occurs when the two
SNs coincide. The conditions for this cusp, are, therefore, \( F = H(\Upsilon), \ \frac{dF}{d\Upsilon} = 0, \)
and \( \frac{d^2 F}{d\Upsilon^2} = 0. \) These conditions will be used as constraints in the optimization
procedure.

The maximum amplitude and its associated frequency at \( f_{cr} \) are the quantities
that describe the best performance of the resonator. The point \((\sigma_{max}, \Upsilon_{max})\) is the
upper bound of the LDR, or the point that distances itself best from surrounding
noise, producing the largest SNR without bistability. Its graphical representation
is shown in Figure 3.4. The method to solve for these values begins with a partial
derivative in \( H \) in terms of \( \sigma \) and setting it to zero, which produces the following
expression for \( \sigma_{max} \):

\[
\sigma_{max} = \frac{1}{8} \Upsilon (6\gamma + 5\delta \Upsilon) \quad (3.20)
\]

This equation can be inserted back into \( H \), and knowing the appropriate values
for \( \delta, \gamma \) and \( f_{cr} \) (\( f_{cr} \) is already as large as it can be, so it is effectively \( f_{max} \)), a
polynomial can be solved for \( \Upsilon_{max} \). \( \Upsilon_{max} \) can then be put into Equation 3.20 to
produce \( \sigma_{max} \).

Fisher [7] solved for each of these critical and maximum quantities consider-
ing the purely cubic \((v = 0)\) and quintic \((v = 1)\) nonlinearities separately and
found closed-form expressions for each quantity in each case in terms of the design
parameters. The cubic case was known, but the result for the quintic case was
original. The results for the cubic and quintic critical cases are provided in Table
3.1, converted to the present notation for comparison.

Note that in Table 3.1 the expression for \( \Upsilon_{cr} \) for the pure quintic case is inversely
proportional to \( \sqrt{-\delta} \) where \( \delta \) is negative by definition, and in the pure cubic case
\( \Upsilon_{cr} \) is inversely proportional to \( \gamma \). Also note that for the quintic case, \( v = 1, \) so
Figure 3.3: $H(\Upsilon)$ versus $\Upsilon$ for a critical value of the drive frequency, corresponding to $\sigma_{cr}$. The location of the critical cusp point is indicated with amplitude $\Upsilon_{cr}$ and drive level $F_{cr}$. This satisfies both the first and second derivative conditions for a cusp.

Figure 3.4: $H(\Upsilon)$ versus $\Upsilon$ for a critical value of the drive frequency, corresponding to $\sigma_{cr}$. The location of the critical cusp point is indicated with amplitude $\Upsilon_{cr}$ and drive level $F_{cr}$. This satisfies both the first and second derivative conditions for a cusp.
Table 3.1: Closed-form results of pure quintic and pure cubic cases. Note that the nonlinear parameters $\gamma$ and $\delta$ must be taken to be at their respective intercept points. Note also that the natural frequency has been normalized to unity in the present work.

<table>
<thead>
<tr>
<th>Quantity</th>
<th>Critical</th>
<th>Maximum</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\Upsilon$</td>
<td>$\frac{2\sqrt{6}\Gamma}{5^{3/4}\sqrt{-\delta}}$</td>
<td>$\frac{12\sqrt{6}\Gamma}{5\times5^{3/4}\sqrt{-\delta}}$</td>
</tr>
<tr>
<td>$\sigma$</td>
<td>$-\sqrt{5}\Gamma$</td>
<td>$-\frac{108\sqrt{\Gamma}}{25\sqrt{5}}$</td>
</tr>
<tr>
<td>$f^2$</td>
<td>$\frac{48\sqrt{6}}{5\times5^{3/4}\sqrt{-\delta}}$</td>
<td>$\frac{48\sqrt{6}(\Gamma)^{5/2}}{5\times5^{3/4}\sqrt{-\delta}}$</td>
</tr>
</tbody>
</table>

that $\delta = \delta_0$, and for the cubic case, $v = 0$, so that $\gamma = \gamma_m$.

In Chapter 4 the quantities $\Upsilon_{cr}$, $\sigma_{cr}$, $f_{cr}$, $\Upsilon_{max}$, and $\sigma_{max}$ are considered for the general case with both cubic and quintic terms, $0 < v < 1$, which requires numerical calculation since no closed form results are possible. In this case, there is a competition between cubic (hardening) and quintic (softening) effects that results in maximum amplitudes that are larger than either of the limiting cases given in Table 3.1. These calculations for $\Upsilon_{cr}$ should match the results from the expressions in Table 3.1 in the $v = 0, 1$ limiting conditions in order to verify the validity of the numerical method.
Chapter 4

Optimization Procedure

In this chapter the accumulation of Chapters 2-3 are brought together to create an optimization problem that is at the heart of this work. Finding the maximum amplitude possible without bistability is precisely finding the maximum value of $\Upsilon$ that can exist without bistability. The onset of bistability is a cusp point, which frames the optimization problem of finding the $\Upsilon_{\text{max}}$ corresponding to finding a certain cusp condition. In the general case there will be more than one cusp condition possible, so we must find the first one that occurs as $f$ (or, equivalently, $F = f^2$) is increased from zero. Also, it is known that $\Upsilon_{\text{max}}$ is monotonic in $f$, so that finding the optimal operating point boils down to finding the desired $f_{cr}$ [7]. One condition that clearly must be satisfied is 3.19, which relates $f^2$, defined here as $F$, to $\Upsilon$, and other parameters through the function $H(\Upsilon)$, which is given by the right hand side of 3.19. The conditions that define the existence of cusp points, such as shown in Figures 3.1 and 3.3, can be expressed in terms of derivatives, as described in Chapter 3. However, there will be multiple solutions to the cusp conditions (up to three) and the correct one must be selected. A graphical
understanding of the conditions involved, and the preliminary observations made in Sections 2.7, will greatly assist this process.

The physics of the problem outlined in Chapter 2 suggests that one optimize the LDR by varying the normalized DC bias $v$, which can range from 0 to 1, and the normalized gap $r$, which can vary upward from some minimum value $r_m$, set by device fabrication tolerances. Given that, our goal is to maximize the minimum cusp condition on $\Upsilon$, which will provide an bound on the conditions for avoiding bistability. The results will also naturally provide the desired operating conditions.

This optimization problem can be stated, in terms of the function $H$ introduced in Chapter 3, as:

$$\max_{v \in [0,1]} \min_{r > r_m} \Upsilon$$

subject to,

$$\Upsilon > 0$$

$$H(\Upsilon) = F$$

$$\frac{\partial F}{\partial \Upsilon} = 0$$

$$\frac{\partial^2 F}{\partial \Upsilon^2} = 0$$

Solution of this optimization problem involves computing the desired cusp point with components $(\Upsilon_{cr}, \sigma_{cr}, f_{cr})$. For the general case with both $\gamma$ and $\delta$ nonzero, it is not possible to solve this completely in closed form, as it was for the purely cubic and quintic cases; see Table 3.1. What is similar is that once the correct cusp condition and the corresponding $r$ and $v$ values are determined, this consequently yields the desired components $(\Upsilon_{max}, \sigma_{max}, f_{cr})$, which provide the desired
maximum amplitude and the drive amplitude and frequency required to achieve it. So, it turns out that one only need find the optimal values for $r$ and $v$ and the remaining parameters are determined.

This process can be simplified, rather than done in a brute force manner, by doing some calculations based on insight from the model, as described next.

4.1 Determining the Critical States

Beginning the effort to find the critical components ($\Upsilon_{cr}, \sigma_{cr}, f_{cr}$), the constraint derivatives are taken:

$$\frac{\partial H}{\partial \Upsilon} = \sigma^2 + \frac{27}{16} \gamma^2 \Upsilon^2 + \frac{15}{4} \gamma \delta \Upsilon^3 + \frac{125}{64} \delta^2 \Upsilon^4 - \frac{3}{4} \sigma \Upsilon (4 \gamma + 5 \delta \Upsilon) + 4 \Gamma^2 \omega^2 = 0 \quad (4.6)$$

$$\frac{\partial^2 H}{\partial \Upsilon^2} = \frac{1}{16} (54 \gamma^2 \Upsilon + 5 \delta \Upsilon (-24 \sigma + 25 \delta \Upsilon^2) + \gamma (-48 \sigma + 180 \delta \Upsilon^2)) = 0. \quad (4.7)$$

Notice that the second derivative equation is linear in $\sigma$, which makes it accessible to solve for in terms of the other variables and parameters, with the result

$$\sigma = \Upsilon \left( \frac{54 \gamma^2 + 180 \gamma \delta \Upsilon + 125 \delta^2 \Upsilon^2}{24(2 \gamma + 5 \delta \Upsilon)} \right) \quad (4.8)$$

Substituting Equation 4.8 into 4.6 yields an equation for the cusp amplitudes $\Upsilon_{cusp}$ in terms of $\delta$ and $\gamma$, along with $\Gamma$. We shall refer to this equation, which is not given due to its length, as “the combined constraint.” This equation is sixth order in $\Upsilon$, with all powers present, so no closed form solution is directly possible.
The approach taken in this work is to numerically search through values of \( \gamma \) and \( \delta \) in the combined constraint and repeatedly solve for \( \Upsilon \), which are the \( \Upsilon_{cusp} \) values, since they are in the context of the cusp constraints. It is observed that this equation has up to three real, positive roots. This forms surfaces of the values of \( \Upsilon_{cusp} \) in an array of \( \gamma \) and \( \delta \) values, from which we can determine the optimal condition using the max-min condition. This way of mapping \( \Upsilon_{cusp} \) proves to be useful in conceptually understanding the behavior of the objective function and assists in visualizing where the optimization point will occur. A sample resulting set of \( \Upsilon_{cusp} \) values over an array of \( \delta \) and \( \gamma \) values is shown in Figure 4.1, for \( \Gamma = 0.01 \). The nature of the surfaces representing \( \Upsilon_{cusp} \) are central to the optimization process. With values of \( \Upsilon_{cusp} \) in hand, we can search for the max-min value, which will be associated with a critical point at which a double cusp root occurs, a point we refer to as a “degenerate cusp.” This point occurs at a voltage \( v_{cr} \) and an amplitude \( \Upsilon_{dc} \), is used to determine the corresponding value of \( v_{cr} \) and the desired cusp amplitude \( \Upsilon_{cr} \), which, as will be seen, differs from \( \Upsilon_{dc} \). The associated values for \( \sigma_{cr} \) can be found with the use of Equation 4.8, and the collection of quantities \( (\Upsilon_{cr}, \sigma_{cr}, \gamma, \delta, \Gamma) \) can then together be used to calculate the associated \( f_{cr} \) using Equation 3.19, recalling that \( F = f_{cr}^2 \). As shown below, these values are crucial for determining the max-min value at the desired point, \( \Upsilon_{max} \). A series of Figures that follow will clarify the approach.

The \( \Upsilon_{cusp} \) surfaces reveal the non-convexity of the problem, by the fact that it was produced by solving for zeros in the combined constraint equation for ranges of values of \( \delta \) and \( \gamma \), which results in the surfaces shown. The nature of these surfaces implies that the solution to the max-min problem will likely occur at a degenerate cusp point, which is not convenient for conventional gradient-based optimization.
Figure 4.1: Surfaces of $\Upsilon_{\text{cusp}}$ calculated numerically for a range of $\delta$ and $\gamma$. The red and green curves are the closed-form curves produced by Fisher [7] for the pure quintic and pure cubic cases, respectively, which the surface converges to in the appropriate limits.

It is possible to map the values of $\Upsilon_{\text{cusp}}$ in ranges of $\gamma$ and $\delta$ that are feasible by a MEM device. This was discussed in Chapter 2, where the parametric equations that express $\delta$ and $\gamma$ in terms of $r$ and $v$ produced a line that was defined by $v$ going from zero to unity. This line is a manifestation of all the points for $\delta$ and $\gamma$ that exist for a device as $v$ is varied. As noted in Chapter 2, the slope of this line depends solely on the normalized electrode gap parameter $r$.

Based on the surfaces shown in Figure 4.1, it is seen that the surfaces of $\Upsilon_{\text{cusp}}$ generally increase as one gets closer to the origin, which makes sense since that is pushing the system towards linearity. Thus, there is a motivation to maximize
the slope of the line, so that it can coincide with a set of $\Upsilon_{cusp}$ values that are larger in value. As shown in Chapter 2, not only was a general equation of the line produced, but the process of maximizing the slope was conducted to demonstrate that the steepest slope of the line occurs at a unique $r$ equal to $r_{opt} = \sqrt{35}/6$. Therefore the largest $\Upsilon_{cusp}$ exists along this optimal parametric line. Thus, the problem has been reduced to finding the value of $v$ that produces the max-min value of $\Upsilon_{cusp}$.

A typical set of $\Upsilon_{cusp}$ that is associated with the parametric line of $\delta$ and $\gamma$ at a value of $r$ is shown in Figure 4.2, which is a cross section of the surface pictured in Figure 4.1. Points 1-6 correspond to different cusp conditions the MEM resonator can exhibit as the parameter $v$ is varied. Note that the upper curve (or surface in Figure 4.1) is monotonic in $v$, and at the critical point of interest there appears under this surface a pair of coincident roots to the combined constraint equation, which occur at the degenerate cusp point, labeled as point 2 in the Figure. This puts point 5 in the spotlight as the optimal point to seek out, that is, it gives $\Upsilon_{cr}$, since it is the point at the brink of the introduction of multiple SN points. Specifically, it is the max-min point for $\Upsilon$ along the line set by the value of $r$. We denote the values at point 2 as $(v_{cr}, \Upsilon_{dc})$ as the degenerate cusp, and the value at point 5 is the desired max cusp $\Upsilon_{cr}$. The only free parameter in this process is the damping $\Gamma$, which will dictate the optimal solution. It is important to note that at a given value of $v$ the stiffness parameters $\gamma$ and $\delta$ are fixed, but if there exist multiple cusp points, each such point has associated with it a different drive level $f$ and a frequency detuning $\sigma$, set by $\gamma$, $\delta$, and the associated $\Upsilon_{cusp}$.

The point of vertical tangency, which is point 2 in Figure 4.2, is crucial in determining the optimal operating condition. This is a point where two cusps are
coincident, which is the degenerate cusp point $\Upsilon_{dc}$ referred to above. This point is determined by numerically solving the simultaneous equations for the cusp condition and an infinite slope condition. In both these equations the nonlinearity parameters $\gamma$ and $\delta$ are parameterized by $v$, the normalized voltage, and the solution provides the voltage and amplitude at this point, which we label as $(v_{cr}, \Upsilon_{dc})$. From this point the amplitude $\Upsilon_{cr}$ is computed from the upper branch of the $\Upsilon_{cusp}$ solutions at $v_{cr}$. The infinite slope condition is conveniently described by thinking of $v$ as a function of $\Upsilon$, in which case the slope condition becomes zero. We can express these two equations, with $v(\Upsilon)$, as $W = 0$ and $\frac{dW}{d\Upsilon} = 0$ and numerically solve them. This set of equations has many roots ($\approx 20$) but the relevant one is easily identified.

In order to verify this methodology, we first plot optimal parametric line at $r_{opt} = \sqrt{35}/6$ alongside other parametric lines with neighboring values of $r$. In addition, their projections onto the $\Upsilon_{cusp}$ surface were plotted to visualize the corresponding sets of $\Upsilon_{cusp}$ that are associated with the set of points along each line. This process is executed in the next section.
Figure 4.2: A typical cross section of the $\Upsilon_{cusp}$ surfaces that corresponds to a parameter line for a given $r$. Points 1-6 are points consisting of components $(\delta, \gamma, \Upsilon_{cusp})$ that are in the neighborhood of interest and express different behaviors in the state of the system. Point 4 is before the onset of bistability; points 2 and 5 are at the critical point, and points 1, 3, and 6 are beyond the onset of bistability.
4.2 Finding and Verifying the Optimal Operating Point

In Figure 4.3, the optimal parametric line at \( r = r_{opt} = \sqrt{35/6} \approx 0.986 \) is shown along with lines with values of \( r = 0.9 \) and 1.1. This helps better understand that the function of the slope in terms of \( r \) is not monotonic, since the lines involving \( r = 0.9 \) and 1.1 are both more shallow than the line for \( r_{opt} \).

The next step is to plot the corresponding cross sections of the \( \Upsilon_{cusp} \) surfaces along these parametric lines, to show that the parametric line with \( r_{opt} \) generates a corresponding values of \( \Upsilon_{cusp} \) that cannot be improved upon with any other nearby line. The plot in Figure 4.4 was produced to compare the cross sections of the surfaces for each of the three sample parametric lines. The points that are equivalent of point 2 in Figure 4.2 are marked with their respective values of \( v_{cr} \) and \( \Upsilon_{dc} \), with respect to their parametric line. Directly above these points are the points corresponding to point 5 in Figure 4.2, at the amplitude \( \Upsilon_{cr} \). It is shown to confirm that the optimal parametric line at \( r_{opt} \) has the largest corresponding \( \Upsilon_{cr} \), which is maximized in terms of \( r \). Figure 4.5 shows the \( \Upsilon_{cusp} \) cross section of
Figure 4.3: Parameter lines in the \((\delta, \gamma)\) plane with \(r = r_{\text{opt}} = 0.9, 1.1, \sqrt{35}/6\), with \(v\) going from zero to unity. Note that with \(r_{\text{opt}} = \sqrt{35}/6\) the slope of the parameter line is the largest, which coincides with the largest \(\Upsilon_{cr}\) that can be achieved.
Figure 4.4: Corresponding cross sections of the $\Upsilon_{cusp}$ surfaces for $r = 0.9, \sqrt{35}/6$, and 1.1, showing the corresponding locations of $(v_{cr}, \Upsilon_{dc}, \Upsilon_{cr})$ as the normalized voltage $v$ is varied, where $\Upsilon_{dc}$ is at the degenerate cusp, and $\Upsilon_{cr}$ is at the corresponding upper cusp point. The point on the upper blue curve, for $r = r_{opt}$, is the $\Upsilon_{cr}$ that corresponds to the desired solution to the min-max problem. Here $\Gamma = 0.01$. 
the optimal parametric curve in 3D space for $\Gamma = 0.01$, with labels of points that correspond to points 1-6 in Figure 4.2.

In order to thoroughly verify the nature of the results that have been described in this section, and to verify that the numerical computation is accurate, sample plots of $H(\Upsilon)$ versus $\Upsilon$ and frequency response diagrams of $\Upsilon$ versus $\sigma$ are produced to illustrate the understanding that the point located at the point 5 position is the max-min point for $\Upsilon$. In producing these plots, it is required that $\sigma_{cr}$ and $f_{cr}$ be calculated using Equations 4.8 and 3.19, respectively, for the desired operating condition, using the $\delta, \gamma, \Upsilon_{cr}$ values that are numerically determined from the critical point.

In producing an $H(\Upsilon) - \Upsilon$ plot, values for $(\delta, \gamma, \sigma_{cr})$ must be applied to $H(\Upsilon)$ and the plot is the result of varying $\Upsilon$. The results resemble Figure 3.3 and shows that for a $\delta$ and $\gamma$, at the proper frequency tuning $\sigma_{cr}$, an $\Upsilon_{cr}$ is achieved at the drive level $F_{cr}$, where $F_{cr} = f_{cr}^2$. This follows since the derivative constraints provided by the optimization problem are satisfied. The $H(\Upsilon) - \Upsilon$ plot for the optimal point on the optimal parameter line is shown in Figure 4.6, indicating the required inflection point.

To produce a frequency response diagram at the critical condition, $H(\Upsilon)$ is considered as a function of the frequency detuning $\sigma$. Since $H(\Upsilon)$ is quadratic in $\sigma$, $\sigma$ can easily be expressed in terms of the other variables. Consequently, the expressions obtained for the two branches of $\sigma$, which are in terms of $(\delta, \gamma, f, \Upsilon)$, can be combined to produce the frequency response diagram. The frequency response at the critical condition is obtained using parameter values at the critical condition. A sample of the frequency response diagram for the critical point is shown in Figure 4.7. Note that the upper cusp point is close to the peak amplitude $\Upsilon_{\max}$ and that
Figure 4.5: The cross section of the $\Upsilon_{cusp}$ surfaces and the optimal parametric line at $r = \sqrt{35}/6$ labeled in the same manner as Figure 4.2, where the point located at the point 5 location is the optimal operating point. Here $\Gamma = 0.01$. 
Figure 4.6: $H(\Upsilon)$ versus $\Upsilon$ at the critical conditions. The cusp conditions indicate $\Upsilon_{cr}$ and $F_{cr} = H(\Upsilon_{cr})$, where $F_{cr} = f_{cr}^2$. This plot confirms that the point found numerically satisfies the derivative constraints imposed by the optimization problem. Here $\Gamma = 0.01$.

the corresponding frequencies are also close to one another. Also note that there is a remnant of another cusp at a lower amplitude, which exists due to being close to other cusp conditions that occur at a lower drive level for a slightly smaller value of $\nu$; see points 1 and 3 Figure 4.2.

Based on the shapes of the $H(\Upsilon) - \Upsilon$ plot and the frequency response diagram for the optimal point, it is clear that the algorithm that generated the $\Upsilon_{cusp}$ surface is accurate and the approach for incorporating the optimal parameter line and the location of point 5 (the degenerate cusp point) as the solution to the min-max optimization problem are correct.

In Chapter 5 similar results are provided and discussed for a wide range of $\Gamma$ values relevant to MEM resonators.
Figure 4.7: The frequency response at the critical conditions, where $\Upsilon$ is plotted versus the frequency detuning $\sigma$. The critical values $\sigma_{cr}, \Upsilon_{cr}$ and the peak conditions $\sigma_{max}, \Upsilon_{max}$ are indicated. Taken at $\Gamma = 0.01$. 

0.3906 $\Upsilon_{max}$

0.3578 $\Upsilon_{cr}$
Chapter 5

Main Results

Table 5.1 provides the tuning parameters that optimize the MEM resonator’s performance at the critical operating point for different values of $\Gamma$, along with the values for quantities that result from the critical state. Note that the values of $\delta$ and $\gamma$ are normalized by the natural frequency squared $\omega_n^2$, and $v$ is defined as the normalized electrostatic voltage as defined in Chapter 2. Since $\delta$ and $\gamma$ are functions of only $v$ and $r$, once $r_{opt} = \sqrt{35}/6$ is adopted, $v$ single-handedly determines the desired DC bias required to achieve optimal performance. Table 5.2 provides the maximum values, $\Upsilon_{\text{max}}$ and $\sigma_{\text{max}}$, that correspond to the maximum amplitude at the critical condition. This is the maximum possible amplitude that can be achieved without bistability for the given level of damping $\Gamma$.

An important quantity is introduced in Table 5.2, namely the Gain Ratio, which is a measure of the improvement in the quantity $\Upsilon_{\text{max}}$ derived in this work as compared to the corresponding amplitude achieved in the pure quintic case, labeled $\Upsilon_q$, which is the common electrostatic tuning used to zero out the cubic term, that is, at the $v = 1$ operating point [11]. The Gain Ratio is a ratio of these
Table 5.1: Electrostatic tuning and critical quantities for optimal performance for different values of $\Gamma$. Note that the values of $\Gamma$, $\delta$, and $\gamma$ have been normalized using the $v$-dependent natural frequency $\omega_n$, as defined in Chapter 3.

<table>
<thead>
<tr>
<th>Damping</th>
<th>DC Bias</th>
<th>Nonlinearities</th>
<th>Cusp$_{deg}$</th>
<th>Critical Quantities</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\Gamma$</td>
<td>$v_{cr}$</td>
<td>$\gamma$</td>
<td>$\delta$</td>
<td>$\Upsilon_{dc}$</td>
</tr>
<tr>
<td>0.05</td>
<td>0.5868</td>
<td>0.5940</td>
<td>-0.3971</td>
<td>0.4489</td>
</tr>
<tr>
<td>0.01</td>
<td>0.7987</td>
<td>0.3987</td>
<td>-0.8942</td>
<td>0.1338</td>
</tr>
<tr>
<td>0.005</td>
<td>0.8574</td>
<td>0.3141</td>
<td>-1.1097</td>
<td>0.08491</td>
</tr>
<tr>
<td>0.001</td>
<td>0.937</td>
<td>0.1631</td>
<td>-1.4940</td>
<td>0.03272</td>
</tr>
<tr>
<td>0.0001</td>
<td>0.9804</td>
<td>0.0560</td>
<td>-1.7666</td>
<td>0.00952</td>
</tr>
<tr>
<td>0.00001</td>
<td>0.9938</td>
<td>0.0183</td>
<td>-1.8627</td>
<td>0.00293</td>
</tr>
</tbody>
</table>

$\Upsilon$ values, which is the amplitude squared, and therefore the raw amplitude gain from optimization improves upon the previous approach by a factor ranging from around 1.46 to 2.15, depending on the damping. This is a significant improvement, achieved by simply backing off the DC gain from the previous approaches.

Graphical representations of the data contained in Table 5.2 are shown in Figures 5.1 and 5.2, which show curve fits for how the critical voltage $v_{cr}$ and the amplitude gain $R_{max}/R_q = \sqrt{\Upsilon_{max}/\Upsilon_q}$ vary with damping $\Gamma$. 

56
Table 5.2: Electrostatic tuning and the desired maximum quantities at different values of $\Gamma$.

<table>
<thead>
<tr>
<th>Damping</th>
<th>DC Bias</th>
<th>Nonlinearities</th>
<th>Maximum Quantities</th>
<th>Gain Ratio</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\Gamma$</td>
<td>$v_{cr}$</td>
<td>$\gamma$</td>
<td>$\delta$</td>
<td>$\Upsilon_{max}$</td>
</tr>
<tr>
<td>0.05</td>
<td>0.5868</td>
<td>0.5940</td>
<td>-0.3971</td>
<td>1.3108</td>
</tr>
<tr>
<td>0.01</td>
<td>0.7987</td>
<td>0.3987</td>
<td>-0.8942</td>
<td>0.3906</td>
</tr>
<tr>
<td>0.005</td>
<td>0.8574</td>
<td>0.3141</td>
<td>-1.1097</td>
<td>0.24787</td>
</tr>
<tr>
<td>0.001</td>
<td>0.937</td>
<td>0.1631</td>
<td>-1.4940</td>
<td>0.09545</td>
</tr>
<tr>
<td>0.0001</td>
<td>0.9804</td>
<td>0.0560</td>
<td>-1.7666</td>
<td>0.02774</td>
</tr>
<tr>
<td>0.00001</td>
<td>0.9938</td>
<td>0.0183</td>
<td>-1.8627</td>
<td>0.00852</td>
</tr>
</tbody>
</table>
Figure 5.1: Curve-fitting second, third, and fourth order polynomials in the logarithmic domain to voltage level versus damping data. Note as $\Gamma$ goes to zero, the functions converge to their leading terms, which vary around $v = 1$. Also note the fourth order curve is dashed to illustrate it varies little from the third order fit.

\[ f_2(\Gamma) = 1.1042 + 1.0290 \ln(\Gamma) - 1.4928 \ln(\Gamma)^2 \]
\[ f_3(\Gamma) = 0.9011 - 2.7038 \ln(\Gamma) - 21.6872 \ln(\Gamma)^2 - 32.5214 \ln(\Gamma)^3 \]
\[ f_4(\Gamma) = 0.9054 - 2.5944 \ln(\Gamma) - 20.7034 \ln(\Gamma)^2 - 28.8708 \ln(\Gamma)^3 + 4.7084 \ln(\Gamma)^4 \]
Figure 5.2: Curve-fitting second, third, and fourth order polynomials in the logarithmic domain to amplitude ratio $R_{\text{max}}/R_q = \sqrt{\Upsilon_{\text{max}}/\Upsilon_q}$ versus damping data. In a similar way to Figure 5.1, the fourth order fit does not vary much from the third order fit.
Chapter 6

Discussion and Future Work

This work is notable as the first optimal solution for a system that involves both softening and hardening effects, which as previously mentioned improves the amplitude of the MEM resonator, and therefore improves the SNR. The optimization has a greater effect for systems with smaller quality factors (larger $\Gamma$). This makes sense, considering that a large quality factor $Q$ has the effect of increasing the sharpness of the resonance peak of the device, therefore making it more susceptible to nonlinear effects.

With the optimal conditions in hand, it is useful to summarize the design process for a resonator, which will necessarily be iterative. This is so, since varying the design will alter $Q$, which influences the optimal operating conditions. Also, there is a target natural frequency, $\omega_n$, which will depend on the bias voltage $V_b$, which in turn depends on the beam dimensions. This interdependence will make the process iterative, but with a clear target. One simplifying feature of the process is that once the beam parameters are set, the electrode gap is easily found to be 98.6% of the beam thickness $h$. This overall process is not worth pursuing in detail.
for the simple configuration considered, since in practice it will typically involve
detailed finite element models for the mechanics and electrostatics.

In this work, linear damping was considered, which was the only component in
the modeling procedure not strictly derived from fundamental principles. Future
work may look into incorporating nonlinear damping into the model. In addition,
it is known that systems with asymmetric nonlinear stiffness result in mixed hard-
ening and softening and may benefit from similar analyses. Also, solid modeling
of complex geometries and nonlinear computational methods could certainly ad-
vance upon the optimization process introduced here, and make it relevant to more
realistic geometries, such as the DETF shown in Chapter 1.

Also in this work, the electrostatic model was truncated at fifth order nonlin-
earity, and it is quite possible to obtain even higher order terms to better represent
the effects brought about by Coulomb’s Law, although this may significantly com-
plicate the optimization process.

Most importantly, the modeling and optimization technique provided in this
work is ripe for experimental verification using simple test systems based on the
clamped-clamped beam configuration considered. This is suggested to be the first
next step, in order to validate the basic approach. The first experiments suggested
could be simply to vary the DC bias in existing resonators in order to observe where
the maximum dynamic range is achieved, and compare it with the existing theory,
even if the parameters, such as the electrode gap, do not meet the optimality
conditions. Such experiments may motivate further work in the design of MEM
resonators for optimal LDR. This work also sets a foundation for approaching other
problems that aim at combating nonlinearities.

Since resonance is not desired in most aerospace, civil, and mechanical engi-
neering structures, perhaps another promising area in this regard is in the design of resonant circuits.
Bibliography


