On the Qualitative Theory of the Nonlinear Degenerate Second Order Parabolic Equations Modeling Reaction-Diffusion-Convection Processes

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ABSTRACT

On the Qualitative Theory of the Nonlinear Degenerate Second Order Parabolic Equations Modeling Reaction-Diffusion-Convection Processes

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We consider nonlinear second order degenerate or singular parabolic equation

$$u_t - a(u^m)_{xx} + bu^\beta + c(u^p)_x = 0, \ a, m, \beta, p > 0, b, c \in \mathbb{R}$$

describing reaction-diffusion-convection processes arising in many areas of science and engineering, such as filtration of oil or gas in porous media, transport of thermal energy in plasma physics, flow of chemically reacting fluid, evolution of populations in mathematical biology etc. We apply the methods developed in *U.G. Abdulla, Journal of Differential Equations, 164, 2(2000), 321-354* for the reaction-diffusion equation ($c = 0$) and prove the existence, uniqueness, boundary regularity and comparison theorems for the initial-boundary value problems in non-cylindrical domains with non-smooth boundary curves under the minimal restriction on the boundary. Constructed weak solutions are continuous up to the non-smooth boundary if at each interior point the left modulus of the lower (respectively upper) semicontinuity of the left (respectively right) boundary curve satisfies an upper (respectively lower) Hölder condition near zero with Hölder exponent $\nu > \frac{1}{2}$. The value $\frac{1}{2}$ is critical as in the classical theory of heat equation, and is independent of nonlinearity parameters m, β, p, and from the degeneration or singularity of the PDE. General theory is applied to the problem on the initial development and asymptotics of the interfaces and local solutions near the interfaces for the reaction-diffusion-convection equation with compactly supported initial function. Depending on the relative strength of three competing forces such as diffusion, convection, and reaction, the interface may expand, shrink or remain stationary. The methods used are rescaling and blow-up techniques for the identification of the asymptotics of the solution along the class of interface type curves, construction of the barriers and application of the comparison theorem in non-cylindrical domains with characteristic boundary curves, as they are developed in papers *U.G. Abdulla & J.King, SIAM J. Math. Anal., 32, 2(2000), 235-260; U.G. Abdulla, Nonlinear Analysis, 50, 4(2002), 541-560.*

Contents

List of Notations

- $\mathbb{R}^n$ is n-dimensional Euclidean space; $x = (x_1, x_2, ..., x_n)$ is arbitrary point.

- $\mathbb{R}^{n+1}$ is $(n+1)$-dimensional Euclidean space with point notation (x, t), where $x \in \mathbb{R}^n,\ 0 < t \leq T \leq +\infty$.

- Ω is bounded domain in $\mathbb{R}^n$.

- Q_T is a cylinder $\Omega \times (0, T]$ in $\mathbb{R}^n$, or collection of points $(x, t) \in \mathbb{R}^{n+1}$ with $x \in \mathbb{R}^n,\ 0 < t \leq T \leq +\infty$.

- S_T is a lateral boundary $\partial\Omega \times (0, T]$ of Q_T, or collection of points $(x, t) \in \mathbb{R}^{n+1}$ with $x \in \partial\Omega,\ t \in (0, T]$.

- $E = \Big\{(x, t) : \ \phi_1(t) < x < \phi_2(t), \quad 0 < t \leq T\Big\}$ is noncylindrical domain in $\mathbb{R}^2$, where $\phi_i(t), i = 1, 2$ are continuous functions on $[0, T]$.

- $D = \Big\{(x, t) : \ \phi(t) < x < +\infty, \quad 0 < t \leq T\Big\}$ is unbounded noncylindrical domain in $\mathbb{R}^2$, where $\phi(t)$ is continuous function on $[0, T]$.

- $(x)_+ = \max\{x, 0\}, \quad x \in \mathbb{R}^1$.

- $C[0,T]$ is Banach space of continous functions on $[0,T]$ with the norm

$$\|\phi\|_{C[0,1]} = \max_{0\le t\le T} |\phi(t)|.$$

- $\omega^-_{t_0}(\phi;\delta) = \max\Big(\phi(t_0) - \phi(t) : \ t_0 - \delta \le t \le t_0\Big)$ is the left modulus of lower semicontinuity of the function $\phi \in C[0,T]$ at the point $t_0 \in (0,T]$.

- $\omega^+_{t_0}(\phi;\delta) = \min\Big(\phi(t_0) - \phi(t) : \ t_0 - \delta \le t \le t_0\Big)$ is the left modulus of upper semicontinuity of the function $\phi \in C[0,T]$ at the point $t_0 \in (0,T]$.

- $C^k[0,T], \ k \in \mathbb{Z}^+$ is Banach space of k-times continuously differentiable functions on $[0,T]$ with the norm

$$\|\phi\|_{C^k[0,T]} = \sum_{j=0}^{k} \max_{0\le t\le T} \left|\phi^{(j)}(t)\right|.$$

- $C^\infty[0,T] = \bigcap_{k=0}^{\infty} C^k[0,T]$ is Banach space of infinitely differentiable functions on $[0,T]$.

- $C_0^\infty(\Omega)$ is Banach space of infinitely differentiable functions with compact support in $\Omega \subset \mathbb{R}^n$.

- $C(\bar{E})$ is Banach space of continuous functions on $\bar{E}$ with the norm

$$\|u\|_{C(\bar{E})} = \max_{(x,t)\in\bar{E}} |u(x,t)|.$$

- $C^{2,1}_{x,t}(\bar{E})$ is Banach space of continuous functions on $\bar{E}$ with continuous $x-$ derivatives up to the order 2, and continuous $t-$ derivative with the norm

$$\|u\|_{C^{2,1}_{x,t}(\bar{E})} = \sum_{j=0}^{2} \Big\|\frac{\partial^j u}{\partial x^j}\Big\|_{C(\bar{E})} + \Big\|\frac{\partial u}{\partial t}\Big\|_{C(\bar{E})}.$$

- $C_{x,t}^{2+\mu,1+\frac{\mu}{2}}(\bar{E})$ is Banach space of elements of $C_{x,t}^{2,1}(\bar{E})$ such that $\frac{\partial^2 u}{\partial x^2}$ and $\frac{\partial u}{\partial t}$ satisfy Hölder condition in $\bar{E}$ with exponent $\mu \in (0,1)$ and $\frac{\mu}{2}$ respectively.

- $L_p(\Omega)$ is Banach space of Lebesgue measurable functions with finite norm

$$\|u\|_{L_p(\Omega)} := \left(\int_\Omega |u|^p \; dx\right)^{\frac{1}{p}}, \quad 1 \le p < \infty$$

.

- $L_2(\Omega)$ is Hilbert space with inner product $(u,v) = \int_\Omega uvdx$.

- $L_\infty(\Omega)$ is Banach space of bounded Lebesgue measurable functions with the norm

$$\|u\|_{L_\infty(\Omega)} := ess \; sup_\Omega \; |u|.$$

- $H^1(\Omega)$ is Hilbert-Sobolev space of weakly differentiable functions with inner product

$$(u,v) = \int_\Omega u(x)v(x) + \nabla u \cdot \nabla v dx.$$

- $H_0^1(\Omega)$ is Hilbert space which is a closure of $C_0^\infty(\Omega)$ in $H^1(\Omega)$ norm.

- $\mathrm{L}_2\big(0,\mathrm{T};H_0^1(D)\big)$ is Banach space of all strongly measurable maps $u:[0,T] \to H_0^1(D)$ with the finite norm

$$\|u\|_{\mathrm{L}_2\left(0,\mathrm{T};H_0^1(D)\right)} = \left(\int_0^T \|u\|_{H_0^1(D)}^2 dt\right)^{1/2} = \left(\int_0^T \int_D (|u|^2 + |\nabla u|^2) dxdt\right)^{1/2}.$$

- $W_q^{2,1}(\Omega)$ is Sobolev space of elements of $L_q(\Omega)$, weakly differentiable up to order two with respect to x, and once with respect to t, with finite norm

$$\|u\|_{W_q^{2,1}(\Omega)} = \left(\int_\Omega |u|^q + \left|\frac{\partial u}{\partial x}\right|^q + \left|\frac{\partial^2 u}{\partial x^2}\right|^q + \left|\frac{\partial u}{\partial t}\right|^q \right)^{1/q}.$$

List of Figures

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Chapter 1

Introduction and Main Results

1.1 Nonlinear Diffusion Problem

1.1.1 Physical Motivation

Consider the flow of ideal gas in a homogeneous porous medium [26, 93, 110]. Let $\rho = \rho(x,t)$ be a density, $v = v(x,t)$ be a velocity, and $p = p(x,t)$ be a pressure of the gas at every point x and time t. This flow can be mathematically described by the following three equations:

(A conservation of mass equation) $\quad f\rho_t + div\,(\rho\vec{v}) = 0$

(The Darcy's law) $\quad \vec{v} = -\frac{\kappa}{\mu}\nabla p,$

(Equation of state) $\quad \rho = \rho_0 p^{\gamma},$

where $f \in (0,1]$ is a porosity of the media, $\kappa \in \mathbb{R}^+$ is the permeability and $\mu \in \mathbb{R}^+$

is the viscosity of the gas . By eliminating v and p from the system we derive the following PDE for ρ,

$$f\rho_t - \frac{k}{\mu}\frac{1}{\rho^{\frac{1}{\gamma}}}\frac{1}{\gamma}\frac{1}{1+\frac{1}{\gamma}}div\nabla\rho^{\frac{1}{\gamma}+1} = 0$$

and by scaling time variable we derive the following nonlinear diffusion equation

$$\rho_t = \Delta\rho^m,\; m > 1 \tag{1.1}$$

which is a second order nonlinear degenerate parabolic equation with implicit degeneracy. The equation (1.1) and it's generalization arise in many applications in mathematical physics, biology, chemistry and many other fields of science and engineering.

1.1.2 Instantaneous Point Source Solution

Consider the instantaneous point-source problem for the diffusion equation:

$$\begin{cases} u_t = div(u^\sigma \nabla u) & x \in \mathbb{R}^N,\; t > 0, \\ u(x,0) = \delta(x) & x \in \mathbb{R}^N \\ \int_{\mathbb{R}^N} u(x,t)\, dx = 1 & t \geq 0 \end{cases}$$

where $\delta(\cdot)$ be a Dirac measure with support at $\{0\}$. It has a solution

$$u_*(x,t) = t^{-\frac{N}{2+N\sigma}} \left[\frac{\sigma}{2(2+N\sigma)} \left(\eta_0^2 - |x|^2 t^{-\frac{2}{2+N\sigma}} \right)_+ \right]^{\frac{1}{\sigma}}. \tag{1.2}$$

where $(\chi)_+ = max(\chi;0)$. The solution (1.2) is called the instantaneous point-source solution of the nonlinear diffusion equation, or so called Barenblatt solution

[26, 111]. Figure 1.1 demonstrates the shape of Barenblatt's solution in various time instances. There are two significant features of the Barenblatt's solution which played a significant role in the development of the theory of nonlinear second order degenerate parabolic equations.

- Barenblatt solution has a finite speed of propagation, that is to say at every finite time instant, the support of the solution is compact:

  $$\mathrm{spt}u = \overline{\{(x,t) : u(x,t) > 0\}} = \{|x| \leq \zeta_0 t^{\frac{1}{2+N\sigma}}\}.$$

  Hence, the solution of the nonlinear diffusion equation present the hyperbolic behavior characteristic for the linear wave equation. This behavior is in stark contrast to the same problem for the linear heat equation: with any compactly supported initial data, the solution will become positive everywhere at any positive time, or there is an infinite speed of propagation. From the standpoint of applications it is natural to expect that diffusion propagates in space with finite speed. Hence, nonlinear diffusion equation presents the phenomena which is more accurate reflection of the diffusion process in nature. That was a reason why starting from 1960s mathematical theory of nonlinear diffusion type equations became one of the central problems in the theory of nonlinear PDEs.

- Despite being more relevant for the applications, Barenblatt solution is not a classical solution. The first order time and second order space derivatives

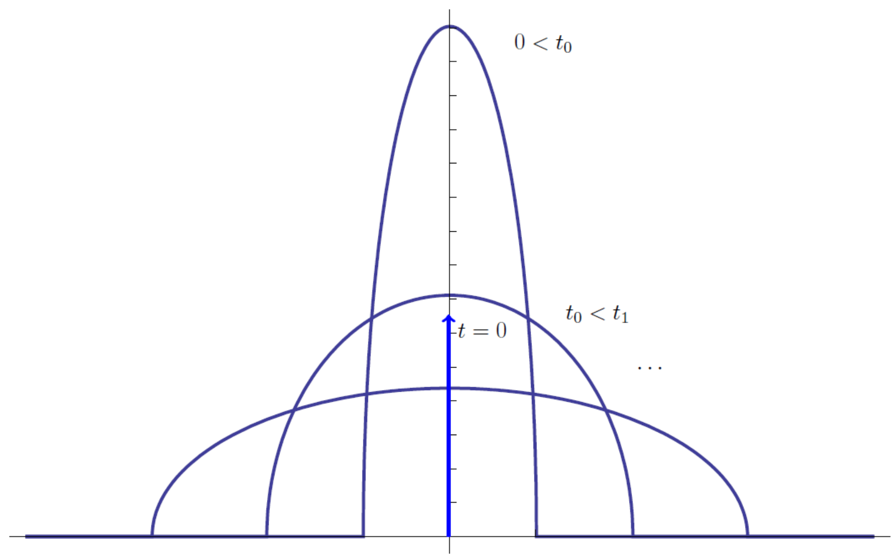

Figure 1.1: Instantaneous point source solution of the nonlinear diffusion equation

are discontinuous on the boundary of support $\{|x| = \zeta_0 t^{\frac{1}{2+N\sigma}}\}$. Therefore, to justify the Barenblatt solution one needs to generalize the notion of the solution of the nonlinear diffusion equation.

In the next definition we introduce the notion of weak solution. Let $D \subset \mathbb{R}^N$ be open domain.

Definition 1.1.1. (Weak Solution) *We say that a function $u = u(x,t)$ is a weak solution of the Dirichlet problem for the Nonlinear Diffusion Equation,*

$$\begin{cases} u_t = \Delta u^m, \quad \text{in} \;\; \mathrm{Q_T} = D \times (0, \mathrm{T}) \\ u(x,0) = u_0(x), \quad \text{in} \;\; D \\ u(x,t) = 0, \quad \text{on} \;\; \mathrm{S_T} = \partial D \times (0, \mathrm{T}) \end{cases}$$

if $u \geq 0$, $u^m \in L_2\left(0,T;H_0^1(D)\right)$ *and* u *satisfies the integral identity*

$$\iint\limits_{Q_T} (\nabla u^m . \nabla \phi - u\phi_t) dx dt = \int_D u_0(x)\phi(x,0)dx$$

for any $\phi \in C^1(\overline{Q}_T)$ *satisfying* $\phi(x,T) = \phi|_{S_T} = 0$.

In fact, instantaneous point source solution is a solution of the nonlinear diffusion equation in the sense of Definition 1.1.1.

1.2 Historical Review

In this dissertation we consider the nonlinear parabolic equation

$$u_t - a(u^m)_{xx} + bu^\beta + c(u^p)_x = 0, \tag{1.3}$$

with $u = u(x,t)$, $a > 0$, b, $c \in \mathbb{R}^1$, $m > 0$, $\beta > 0$, $p > 0$. Equation (1.3) is usually called a reaction-diffusion-convection equation. It is a simple and widely used model for various physical, chemical and biological problems involving diffusion with a source or absorption, and accompanied with additional convective flow as for instance in modeling filtration in porous media, transport of thermal energy in a plasma, flow of a chemically reacting fluid from a flat surface, evolution of populations etc.

The mathematical theory of degenerate parabolic equations begins with paper [94], where the first existence, uniqueness and regularity results, as well as some qualitative properties of solutions of different initial and boundary value problems for general diffusion equation (including as a particular case an equation (1.3) with $b = 0, c = 0,\ m > 1$), have been established. There has been a considerable amount of published work on this subject during the last five decades. For a general list of references we can refer to books [47, 107], and various survey articles such as [25, 99, 75, 108, 104] etc. General theory of the nonlinear degenerate parabolic equations in non-cylindrical domains were developed in a serious of works in [1, 2, 3, 4, 5, 6]. In particular, existence, uniqueness, regularity and comparison results of the boundary value problems for the reaction-diffusion equation (1.3) with $c = 0$ was presented in [1, 2]. The goal of this dissertation is to extend the results to the case of reaction-diffusion-convection equation (1.3) with $c \in \mathbb{R}^1, p > 0$. To review the boundary-value problems in non-cylindrical domains with non-smooth boundaries consider a Dirichlet problem for the heat equation

$$u_t = u_{xx} \text{ in } E = \Big\{(x,t):\ \phi_1(t) < x < \phi_2(t),\ \ 0 < t \le T\Big\}, \quad (1.4)$$

$$u(x,0) = u_0(x), \quad \phi_1(0) \le x \le \phi_2(0), \quad (1.5)$$

$$u(\phi_i(t),t) = \psi_i(t), \quad 0 \le t \le T, \quad (1.6)$$

where $0 < T \le +\infty$, $\phi_i, \psi_i \in C[0;T]$, $\phi_1(t) < \phi_2(t)$ for $t \in [0;T]$ and $u_0 \in C\left([\phi_1(0);\phi_2(0)]\right)$ and $u_0(\phi_i(0)) = \psi_i(0)$, $i = 1,2$. In [60] it is proved that there exists a classical solution to the problem (1.4)–(1.6) if $\phi_i(t)$ satisfies a Hölder condition with Hölder exponent more than $\frac{1}{2}$. Optimal result is proved in [100]: there exists a classical solution to the problem (1.4)–(1.6) (which is continuous in $\bar{E}$) if for each $t_0 > 0$ there exists a function $p(h)$ such that p is defined for all negative h with sufficiently small absolute value, p is positive and monotonically convergent to 0 as $h \to -0$, for sufficiently small $|h|$

$$\phi_1(t_0) - \phi_1(t) \leq 2(t_0 - t)^{1/2}\Big(-\log p(t - t_0)\Big)^{1/2}, \qquad t \in [t_0 - |h|; t_0] \quad (1.7)$$

$$\phi_2(t_0) - \phi_2(t) \geq -2(t_0 - t)^{1/2}\Big(-\log p(t - t_0)\Big)^{1/2}, \qquad t \in [t_0 - |h|; t_0] \quad (1.8)$$

and

$$\lim_{\varepsilon \to 0-} \int_c^{\varepsilon} \frac{p(h)|\log p(h)|^{\frac{1}{2}}}{h}\, dh = -\infty,$$

where c is a suitable negative constant. In [100] also a necessary condition was derived which is close to the sufficient one but still differs slightly.

Let $\phi \in C[0; T]$ and for any fixed $t_0 > 0$ define the functions

$$\omega_{t_0}^{-}(\phi; \delta) = \max\Big(\phi(t_0) - \phi(t) : \ t_0 - \delta \leq t \leq t_0\Big)$$

$$\omega_{t_0}^{+}(\phi; \delta) = \min\Big(\phi(t_0) - \phi(t) : \ t_0 - \delta \leq t \leq t_0\Big)$$

For sufficiently small $\delta > 0$ these functions are well defined and converge to zero as $\delta \to 0+$. The function $\omega_{t_0}^{-}(\phi; \cdot)$ will be called the left modulus of lower semicontinuity of the function ϕ at the point t_0 and accordingly the function $\omega_{t_0}^{+}(\phi; \cdot)$ will be called the left modulus of upper semicontinuity of the function ϕ at the point t_0.

Hence, the conditions (1.7),(1.8) consist of the upper (respectively lower) estimation for left modulus of lower (respectively upper) semicontinuity of the left (respectively right) boundary curve at each $t_0 > 0$.

In particular, if at some point $t_0 > 0$

$$\omega^-_{t_0}(\phi_1;\delta) \le \kappa\delta^\alpha \text{ or } \ \omega^+_{t_0}(\phi_2;\delta) \ge -\kappa\delta^\alpha$$

for sufficiently small $\delta > 0$ and with $\kappa > 0,\ 0 < \alpha < 1/2$, then the nonexistence of a classical solution to problem (1.4)–(1.6) is possible.

In [88], a necessary and sufficient condition for regularity of a boundary point in the Dirichlet problem for the heat equation in arbitrary spatial dimension has been announced. Wiener type necessary and sufficient condition which is a geometric characterization for a boundary point of an arbitrary bounded open subset of $\mathbb{R}^{N+1}$ to be regular for the heat equation has been established in [55]. A similar criterion for linear parabolic operators with smooth, variable coefficients was established in [59]. Sufficient conditions for boundary regularity in the case of general quasilinear nondegenerate parabolic equations were found in [58, 112]. Sharp geometric condition for the regularity of the boundary point for the heat equation is presented in [5]. Multidimensional Kolmogorov-Petrovski test for the boundary regularity or irregularity of the characteristic boundary point for the heat equation is proved in [100, 7].

1.3 Formulation of the Open Problems and Preview of Main Results

In this dissertation we are interested in Cauchy-Dirichlet and Dirichlet problems to equation (1.3). Let us formulate the problems:

I. The Cauchy-Dirichlet problem (CDP): find a solution of equation (1.3) in

$$D = \Big\{(x,t): \ s(t) < x < +\infty, \ \ 0 < t \leq T\Big\},$$

with conditions

$$u(s(t),t) = \psi(t), \qquad 0 \leq t \leq T, \tag{1.9}$$

$$u(x,0) = u_0(x), \qquad s(0) \leq x < +\infty, \tag{1.10}$$

where $s \in C[0;T]$, $\psi \in C[0;T]$, $\psi \geq 0$ for $t \in [0;T]$ and $\sup \psi < +\infty$, $u_0 \in C\Big([s(0);+\infty)\Big)$, $u_0 \geq 0$ for $x \in [s(0);+\infty)$, $u_0(s(0)) = \psi(0)$ and $\sup u_0 < +\infty$.

II. The Dirichlet problem (DP): find a solution of equation (1.3) in E with conditions (1.5), (1.6), where u_0, ϕ_i, ψ_i satisfy the same conditions as in (1.5), (1.6) and also $u_0 \geq 0$, $\psi_i \geq 0$ and $\sup \psi_i < +\infty$.

Obviously, in general the equation (1.3) degenerates at points (x,t) where $u = 0$ and we cannot expect the considered problems to have classical solutions. We shall follow the following notions of generalized solutions.

Definition 1.1 We shall say that the function $u(x,t)$ is a solution of CDP in D if

(a) u is non-negative and continuous in $\bar{D}$, satisifying (1.9), (1.10) and $u \in L_\infty\Big(D \cap (t \leq T_1)\Big)$ for any finite $T_1 \in (0;T]$.

(b) for any finite t_0, t_1 such that $0 \leq t_0 < t_1 \leq T$ and for any C^∞ functions $\mu_i(t), t_0 \leq t \leq t_1, \ \ i = 1,2,$ such that $s(t) < \mu_1(t) < \mu_2(t)$ for $t \in [t_0; t_1]$, the following integral identity holds

$$I(u,f,D_1) = \int_{t_0}^{t_1} \int_{\mu_1(t)}^{\mu_2(t)} \Big(uf_t + au^m f_{xx} - bu^\beta f + cu^p f_x\Big)\, dx\ dt \ - \int_{\mu_1(t)}^{\mu_2(t)} uf\Big|_{t=t_0}^{t=t_1} dx \ - \int_{t_0}^{t_1} au^m f_x \Big|_{x=\mu_1(t)}^{x=\mu_2(t)} dt = 0, \tag{1.11}$$

where

$$D_1 = \Big\{(x,t): \ \mu_1(t) < x < \mu_2(t), \ \ t_0 < t < t_1\Big\}$$

and $f \in C^{2,1}_{x,t}(\bar{D}_1)$ is an arbitrary function that equals zero when $x = \mu_i(t), \ t_0 \leq t \leq t_1, \ i = 1,2.$

Definition 1.2 We shall say that the function $u(x,t)$ is a solution of DP in E if

(a) u is non-negative and continuous in $\bar{E}$, satisfying (1.5), (1.6),

(b) for any finite t_0, t_1 such that $0 \leq t_0 < t_1 \leq T$ and for any C^∞ functions $\mu_i(t), \ t_0 \leq t \leq t_1, \ \ i = 1,2$ such that $\phi_1(t) < \mu_1(t) < \mu_2(t) < \phi_2(t)$ for $t \in [t_0; t_1]$, the integral identity (1.11) holds, where $f \in C^{2,1}_{x,t}(\bar{D}_1)$ is an arbitrary function that equals zero when $x = \mu_i(t), \ t_0 \leq t \leq t_1, \ \ i = 1,2.$

Furthermore, for both problems, we assume that $0 < T \le +\infty$ if $b \ge 0$ or $b < 0$ and $0 < \beta \le 1$, and $T \in (0; T^*)$ if $b < 0$ and $\beta > 1$, where $T^* = M^{1-\beta} \Big/ (b(1-\beta))$ and $M = \max(\sup u_0,\ \sup \psi) + \varepsilon$ in CDP and $M = \max(\sup u_0,\ \sup \psi_1,\ \sup \psi_2) + \varepsilon$ in DP and $\varepsilon > 0$ is an arbitrary sufficiently small number.

In Chapter 2 we consider a CDP. In Section 2.1 (Theorem 2.1.1) we prove that there exists a solution of CDP if for each $t_0 > 0$ there exists a function $F(\delta)$ such that F is defined for all positive and sufficiently small δ, F is positive and convergent to 0 as $\delta \to +0$ and

$$\omega^-_{t_0}(s;\delta) \le \delta^{\frac{1}{2}} F(\delta). \tag{1.12}$$

Furthermore, this assumption will be called assumption (L). In particular, the assumption (L) is satisfied if at every fixed point $t_0 > 0$, s is a left-lower-Hölder continuous with Hölder exponent $\nu > \frac{1}{2}$ i.e. for each $t_0 > 0$ there exists a $\kappa > 0$ and $\nu > \frac{1}{2}$ such that $\omega^-_{t_0}(s;\delta) \le \kappa\delta^{\nu}$ for sufficiently small positive δ. Then we prove in Section 2.2 (Theorem 2.2.1) the uniqueness of the solution of the CDP if $a > 0$, $m > 0$, $c \in \mathbb{R}$, $p > 0$ and either $b \ge 0$, $\beta > 0$ or $b < 0$, $\beta \ge 1$, and if s satisfies the assumption (L) and for each compact subsegment $[0; T_1] \subset [0; T]$ there exists a positive constant M_0 such that

$$s(t) - s(\tau) \ge -M_0(t-\tau) \quad \text{for} \quad 0 \le \tau \le t \le T_1. \tag{1.13}$$

If the initial and boundary data have a positive infimum under the assumption (L) on the curve s, there is also uniqueness in the case when $a > 0$, $m > 0$, $c \in \mathbb{R}$,

$+p > 0,\ b < 0,\ 0 < \beta < 1$ (see Remark 2.1 and Theorem 2.2.2). In Section 2.3 we prove the comparison theorem under the same conditions as in the case of uniqueness (see Theorem 2.3.1 and Remark 2.2).

In Chapter 3 we consider DP. In Section 3.1 (Theorem 3.1.1) we prove the existence of a solution of the DP, if ϕ_1 satisfies assumption (L) and ϕ_2 satisfies assumption (R), that is to say, for each $t_0 > 0$ there exists a function $F(\delta)$ as before, such that

$$\omega^{+}_{t_0}(\phi_2;\delta) \geq -\delta^{\frac{1}{2}} F(\delta), \tag{1.14}$$

for sufficiently small positive δ. In particular, the assumption (R) is satisfied if at every fixed point $t_0 > 0$, ϕ_2 is a left-upper-Hölder continuous with Hölder exponent $\nu > \frac{1}{2}$, i.e. for each $t_0 > 0$ there exists a $\kappa > 0$ and $\nu > \frac{1}{2}$ such that $\omega^{+}_{t_0}(\phi_2;\delta) \geq -\kappa\delta^{\nu}$, for all sufficiently small positive δ. Then in Section 3.2 (Theorem 3.2.1) we prove the uniqueness of the solution of the DP if

(a) $a > 0,\ m > 0,\ c \in \mathbb{R},\ p > 0$ and either $b \geq 0,\ \beta > 0$ or $b < 0,\ \beta \geq 1$,

(b) ϕ_1 satisfies assumption (L), ϕ_2 satisfies assumption (R)

(c) for each compact subsegment $[\delta; T_1] \subset (0; T]$ there exists a positive constant M_0 such that

$$\phi_1(t) - \phi_1(\tau) \geq -M_0(t-\tau) \quad \text{for} \quad 0 < \delta \leq \tau \leq t \leq T_1, \tag{1.15a}$$

$$\phi_2(t) - \phi_2(\tau) \leq M_0(t-\tau) \quad \text{for} \quad 0 < \delta \leq \tau \leq t \leq T_1. \tag{1.15b}$$

From these results it easily follows that under the conditions (a) and (b), the solution of the DP is unique even if there exists a finite number of points $t_i, \ i = 1, \ldots, k$ such that $t_0 = 0 < t_1 < \cdots < t_k < t_{k+1} = T$ and for arbitrary compact subsegment $[\delta; T_1] \subset \Big(t_i, t_{i+1}\Big), i = 0, 1, \ldots k$, there exists a positive constant M_0 such that (1.13) is valid. If $T = +\infty$ then uniqueness is still the case if (1.13) violates on a numerate number of points $t_k, \ k = 1, 2, \ldots$ with $t_k \uparrow +\infty$ as $k \to +\infty$ (see Corollary 3.1). If the initial and boundary data have a positive infimum under the assumptions (L) and (R), there is also uniqueness in the case when $a > 0, \ m > 0, \ b < 0, \ 0 < \beta < 1$ (see Theorem 3.2.2 and Remark 2.1). Finally we present the comparison theorem under the same conditions as in the case of uniqueness (see Theorem 3.2.3 and Remark 3.1).

We apply the methods developed in [1] based on parabolic regularization, Holmgrem's method, Bernstein techniques for the interior Hölder regularity estimates, construction of barriers and boundary regularity. The most difficult step is the proof of continuity of the constructed limit solution to CDP or DP up to the nonsmooth boundary. This step is proved by extending a new techniques developed in [1] using the classical method of barriers and a limiting process.

A particular motivation for this work arises from the problem about the evolution of interfaces, and the local behaviour of solutions near the interface, in problems for equation (1.3). Using the results of the paper [1], a full description of the evolution of interfaces and of the local solution near the interface for the reaction-diffusion equation (1.3) with $c = 0$, for all relevant values of parameters is presented in [9] for the slow diffusion case ($m > 1$), and in [10] for the fast diffusion case $0 < m < 1$. Similar classification for the reaction-diffusion equation with p-Laplacian type diffusion term is presented in a recent paper [11].

In Chapter 4, we apply the results of general theory for reaction-diffusion-convection equation (1.3) to pursue the problem on the evolution of interfaces and local solution near the interfaces.

Chapter 2

Cauchy-Dirichlet Problem

2.1 Existence

In this section we shall suppose that $a > 0,\ m > 0,\ b \in \mathbb{R}^1,\ \beta > 0,\ c \in \mathbb{R}^1,\ p > 0.$ Our purpose is to prove the following theorem.

Theorem 2.1.1. *If s satisfies the assumption (L) then there exists a solution of the CDP.*

Proof. Let $\{\varepsilon_n\}$ be an arbitrary real monotonic sequence with $\varepsilon_n \to +0$ and $\{r_n\}$ an arbitrary real monotonic sequence with $r_n \to +\infty$.

Let

$$T_n \equiv T, \quad n = 1, 2, \dots \quad \text{if} \quad T < +\infty$$

and $\{T_n\}$ be a monotonic positive sequence such that

$$T_n \to +\infty \ \text{ as } \ n \to +\infty \ \text{ if } \ T = +\infty.$$

Suppose that $\{s_n\}$ is an arbitrary sequence of functions such that $s_n \in C^\infty[0;T_n]$ and

$$\lim_{n\to+\infty} \max_{0\le t\le T_n} |s_n(t) - s(t)| = 0.$$

For simplicity, suppose that $s(0) = 0$ and let $s_n(0) = s_{0n} \ge 0, \quad n = 1, 2, \ldots$. Some restriction on the sequence of numbers $\{s_{0n}\}$ will be formulated below. Let $\gamma_b = 1$ if $b < 0$, and γ_b be an arbitrary number such that

$$\gamma_b > \max\left(m^{-1}; \beta^{-1}; p^{-1}; 1\right) \quad \text{if} \ \ b \ge 0.$$

Henceforth, we shall write γ instead of γ_b. Without loss of generality we may suppose that $\varepsilon_1^\gamma < M$. Take two functional sequences $\{\psi_n\}$ and $\{u_{0n}\}$ and a sequence of numbers $\{s_{0n}\}$ $(s_n(0) = s_{0n})$ such that

1. $s_{0n} \ge 0, \quad n = 1, 2, \ldots, \quad \lim_{n\to\infty} s_{0n} = 0$,
2. $u_0(0) - \chi(\varepsilon_n)/2 \le u_0(s_{0n}) \le \left(u_0^m(0) + (\chi(\varepsilon_n)/2)^m\right)^{1/m}, \quad n = 1, 2, \ldots$,
3. $\varepsilon_n^\gamma \le u_{0n}(x), \ \psi_n(t) \le M \ \text{ for } \ (x,t) \in [0; r_n] \times [0; T_n]$,
4. $u_{0n} \in C^\infty[0; r_n], \quad \psi_n \in C^\infty[0; T_n], \quad n = 1, 2, \ldots$,
5. $u_{0n}(s_{0n}) = \psi_n(0), \quad a(u_{0n}^m)''(s_{0n}) - s_n'(0)u_{0n}'(s_{0n}) - bu_{0n}^\beta(s_{0n}) - c(u_{0n}^p)'(s_{0n}) + b\theta_b\varepsilon_n^{\beta\gamma} = \psi_n'(0), \quad \theta_b = (1, \text{ if } b > 0; \ 0, \text{ if } b \le 0)$,

6. $u_{0n}(r_n) = M, \quad a\,(u_{0n}^m)''\,(r_n) - c(u_{0n}^p)'(r_n) - b\theta_b M^\beta + b\theta_b \varepsilon_n^{\beta\gamma} = 0$,

7. $0 \le u_{0n}(x) - u_0(x) \le \chi(\varepsilon_n) \quad \text{for} \;\; 0 \le x \le r_n - 1$,

8. $0 \le \psi_n^m(t) - \psi^m(t) \le \chi^m(\varepsilon_n) \quad \text{for} \;\; 0 \le t \le T_n$,

where $\chi(x) = Kx^\gamma$ for $x \ge 0$ and $K > 1$ is a fixed constant. If the initial and boundary data have a positive infimum, then we may assume that $\chi(x),\ x > 0$, is an arbitrary continuous positive monotonic function with $\lim_{x\to 0^+} \chi(x) = 0$. Obviously, it is possible to construct such sequences. Consider an auxiliary problem

$$u_t = a(u^m)_{xx} - c(u^p)_x - bu^\beta + b\theta_b\varepsilon_n^{\beta\gamma} \;\text{ in } D_n, \tag{2.1}$$

$$u(x,0) = u_{0n}(x), \quad s_{0n} \le x \le r_n, \tag{2.2}$$

$$u(s_n(t),t) = \psi_n(t), \quad u(r_n,t) = \psi^1(t), \quad 0 \le t \le T_n, \tag{2.3}$$

where $D_n = \Big\{(x,t) : s_n(t) < x < r_n, \quad 0 < t \le T_n\Big\}$ and

$$\psi^1(t) = \begin{cases} \left[M^{1-\beta} - b(1-\theta_b)(1-\beta)t\right]^{1/(1-\beta)}, \text{ if } \;\beta \ne 1, \\ M\exp\left(-b(1-\theta_b)t\right), \text{ if } \;\beta \ne 1. \end{cases}$$

Without loss of generality, we may suppose that

$$r_n > V_n \equiv 1 + \max_{[0;T_n]} |s_n(t)|, \quad n = 1,2,\dots\;, \quad r_nV_n^{-1} \to +\infty \quad \text{as} \;\; n \to +\infty.$$

If we introduce new variables

$$r_n\,(x - s_n(t))\,(r_n - s_n(t))^{-1} \to y, \quad t \to t,$$

then (2.1)–(2.3) will be transformed to the problem

$$\begin{aligned} \text{Ł}_n v \quad \equiv \quad & v_t - ar_n^2 \left(r_n - s_n(t)\right)^{-2} (v^m)_{yy} \; + cr_n(r_n - s_n(t))^{-1}(v^p)_y + \\ & (r_n - y)\left(r_n - s_n(t)\right)^{-1} s_n'(t) v_y + bv^\beta - b\theta_b \, \varepsilon_n^{\beta\gamma} = 0 \text{ in } D_n' \qquad (2.4) \\ v(y,0) = \quad & u_{0n}\left(s_{0n} + r_n^{-1}(r_n - s_{0n})y\right), \quad 0 \le y \le r_n, \qquad (2.5) \\ v(0,t) \; = \quad & \psi_n(t), \quad v(r_n,t) = \psi^1(t), \qquad 0 \le t \le T_n, \qquad (2.6) \end{aligned}$$

where

$$D_n' = \Big\{(y,t) : 0 < y < r_n, \quad 0 < t \le T_n\Big\}.$$

From [90] (Theorem 6.1, §6, ch.5) it easily follows that there exists a unique classical solution $v = v_n(y,t)$ of the problem (2.4)-(2.6) such that $v_n \in C_{x,t}^{2+\mu,1+\mu/2}\left(\bar{D}_n'\right)$ with some $\mu > 0$. Maximum principle imples

$$\varepsilon_n^\gamma \le v_n(y,t) \le \psi^1(t) \;\text{ in }\; \bar{D}_n'. \qquad (2.7)$$

Therefore, the function

$$u_n(x,t) = v_n\left(r_n(x - s_n(t))(r_n - s_n(t))^{-1}, t\right)$$

is the classical solution from $C_{x,t}^{2+\mu,1+\mu/2}(\bar{D}_n)$ of the problem (2.1)–(2.3) and

$$\varepsilon_n^\gamma \le u_n(x,t) \le \psi^1(t) \quad \text{for} \;\; (x,t) \in \bar{D}_n. \qquad (2.8)$$

From [57] (Theorem 10,§5,ch.3) it follows that $(u_n)_x \in C_{x,t}^{2+\mu_1,1+\mu_1/2}(D_n)$ for some $\mu_1 > 0$.

The next step consists in proving the uniform Hölder continuity of the sequence

u_n on every compact subset G of D (obviously u_n is defined on G for $n \geq N$ if N is chosen large enough). Consider a sequence of compacts

$$D^{(k)} = \Big\{(x,t) : s^{(k)}(t) \leq x \leq \ell_k,\ k^{-1} \leq t \leq T_k\Big\},\ k = 1, 2, \ldots$$

where ℓ_k is a monotonic sequence such that $\ell_k \to +\infty$ as $k \to +\infty$; $T_k \equiv T$ if $T < +\infty$ and $\{T_k\}$ is a monotonic sequence such that $T_k \to +\infty$ as $k \to +\infty$ if $T = +\infty$; $\big\{s^{(k)}\big\}$ is a sequence of functions such that $s^{(k)} \in C^{\infty}[0; T_k]$,

$$s^{(k)}(t) > s^{(k+1)}(t) > s(t) \quad \text{for} \quad 0 \leq t \leq T_k,\ k = 1, 2, \ldots$$

and

$$\lim_{k \to +\infty} \max_{0 \leq t \leq T_k} \left| s^{(k)}(t) - s(t) \right| = 0.$$

Hence, we have

$$D = \bigcup_{k=1}^{\infty} D^{(k)}, \quad D^{(k)} \subset D^{(k+1)}, \quad k = 1, 2, \ldots\ . \tag{2.9}$$

Obviously, for each fixed k there exists a number $n(k)$ such that

$$D^{(k)} \subset D_n \ \text{ for } \ n \geq n(k).$$

The sequence $\{u_n\}$, $n \geq n(k)$ satisfies the following inequality

$$\left| (u_n^m)_x \right| \leq M_1(k) \ \text{ in } \ D^{(k)}, \tag{2.10}$$

where $M_1(k)$ is a constant which depends on k and does not depend on n. The estimation (2.10) may be proved by Bernstein's method, for example in the form given in [104]. It implies that

$$|u_n(x,t) - u_n(y,t)| \leq M_2(k)\,|x-y|^{\alpha} \quad \text{in} \;\; D^{(k)}, \tag{2.11}$$

where $\alpha = \min(1; m^{-1})$. It is well-known from (2.11) that the Hölder estimate follows with respect to time variable as well. As a matter of fact the following Hölder estimate may be proved exactly as it is proved in [104].

$$|u_n(x,t) - u_n(y,\tau)| \leq M_3(k)\left(|x-y|^{\alpha} + |t-\tau|^{\alpha/1+\alpha}\right) \;\; \text{in} \;\; D^{(k)}. \tag{2.12}$$

Thus $\{u_n\}$, $n \geq n(k)$ is uniformly bounded and equicontinuous in $D^{(k)}$. It should be pointed out that the equicontinuity of the sequence $\{u_n\}$ in $D^{(k)}$ may be established by using more general results of [48]. From (2.12), (2.9), by a diagonalisation argument and the Arzela-Ascoli theorem, we may find a subsequence n' and a limit function $\tilde{u} \in C(D)$ such that $u_{n'} \to \tilde{u}$ as $n' \to \infty$, pointwise in D, and the convergence is uniform on compact subsets of D. Obviously, $\tilde{u} \in L_\infty(D)$ if $b \geq 0$ or $b < 0$ and $\beta > 1$ and $\tilde{u} \in L_\infty(D \cap (t \leq T_1))$ for any finite $T_1 > 0$ if $b < 0$ and $0 < \beta \leq 1$. Now consider a function $u(x,t)$ such that

$$\begin{cases} u(x,t) = \tilde{u}(x,t) \quad \text{for} \quad (x,t) \in D, \\ u(x,0) = u_0(x) \quad \text{for} \quad s(0) \leq x < +\infty, \\ u(s(t),t) = \psi(t) \quad \text{for} \quad 0 \leq t \leq T. \end{cases}$$

Obviously the function $u(x,t)$ satisfies the integral identity (1.11). The continuity of u at the points $(x_0, 0)$, $x_0 > s(0)$ of the line $t = 0$ may easily be established. If u_0^m

is locally Lipschitz continuous, it follows from the estimations (2.11),(2.12) which may be proved up to the line $t = 0$. In general, the continuity on the line $t = 0$ may be established by constructing barriers.

It remains only to prove the continuity of $u(x,t)$ at the points $(s(t),t),\ t \geq 0$. For that, first consider a function

$$v(y,t) = u(y + s(t), t) \quad \text{in} \quad \bar{D}',$$

where

$$D' = \Big\{(y,t) : 0 < y < +\infty, \quad 0 < t \leq T\Big\}.$$

Obviously

$$v \in C(D') \cap L^{\infty}(D') \quad \text{if} \quad b \geq 0 \ \text{ or } \ b < 0 \ \text{ and } \beta > 1$$

$$v \in C(D') \cap L^{\infty}(D' \cap (t \leq T_1)) \quad \text{if} \quad b < 0, \quad 0 < \beta \leq 1$$

and T_1 is an arbitrary finite number from $(0;T]$.

The sequence $\{v_{n'}\}$ converges to v as $n' \to +\infty$ pointwise in $\bar{D}'$ and the convergence is uniform on compact subsets of D'. Continuity of the function $u(x,t)$ at the points $(s(t),t),\ t \geq 0$ is equivalent to continuity of the function $v(y,t)$ at the points $(0,t),\ t \geq 0$.

If $t_0 \geq 0$ and $\psi(t_0) > 0$, we shall prove that for arbitrary sufficiently small $\varepsilon > 0$

the following two inequalities are valid

$$\liminf v(y,t) \geq \psi(t_0) - \varepsilon \quad \text{as} \quad (y,t) \to (0,t_0), \quad (y,t) \in D', \tag{2.13}$$

$$\limsup v(y,t) \leq \psi(t_0) + \varepsilon \quad \text{as} \quad (y,t) \to (0,t_0), \quad (y,t) \in D'. \tag{2.14}$$

As $\varepsilon > 0$ is arbitrary, the continuity of $v(y,t)$ at the boundary point $(0,t_0)$ follows from (2.13), (2.14). If $\psi(t_0) = 0$, then it is sufficient to prove (2.14), since (2.13) (with $\varepsilon = 0$ in the right-hand side) directly follows from the fact that v is non-negative in $\bar{D}'$.

Let $\psi(t_0) > 0$. Take an arbitrary $\varepsilon \in (0;\psi(t_0))$ and prove the inequality (2.13). Consider a function

$$\omega_n(y,t) = f\Big(h(\mu) - (1 - r_n^{-1}s_n(t))y + \mu(t-t_0) + s_n(t_0) - s_n(t)\Big), \quad \mu > 0, \quad h > 0,$$

where

$$f(\zeta) = M_1(\zeta/h(\mu))^{\alpha}, \ M_1 = \psi(t_0) - \epsilon$$

Assume that either $c \geq 0$ or $c < 0, \ p \geq 1$.

Then if $b \leq 0$, we take the two cases:

(a) $\alpha > m^{-1}$, if $0 < m \leq 1$ and (b) $m^{-1} < \alpha < (m-1)^{-1}$ if $m > 1$.

If $b > 0$ we take six different cases (as shown in Figure 2.1).

I $\quad m^{-1} < \alpha \leq \min\Big((m-1)^{-1};(1-\beta)^{-1}\Big), \quad$ if $m > 1, \quad 0 < \beta < 1,$

II $\quad m^{-1} < \alpha \leq (m-1)^{-1}, \quad$ if $m > 1, \quad \beta \geq 1,$

III $\quad \alpha > m^{-1}, \quad$ if $0 < m \leq 1, \quad \beta \geq 1,$

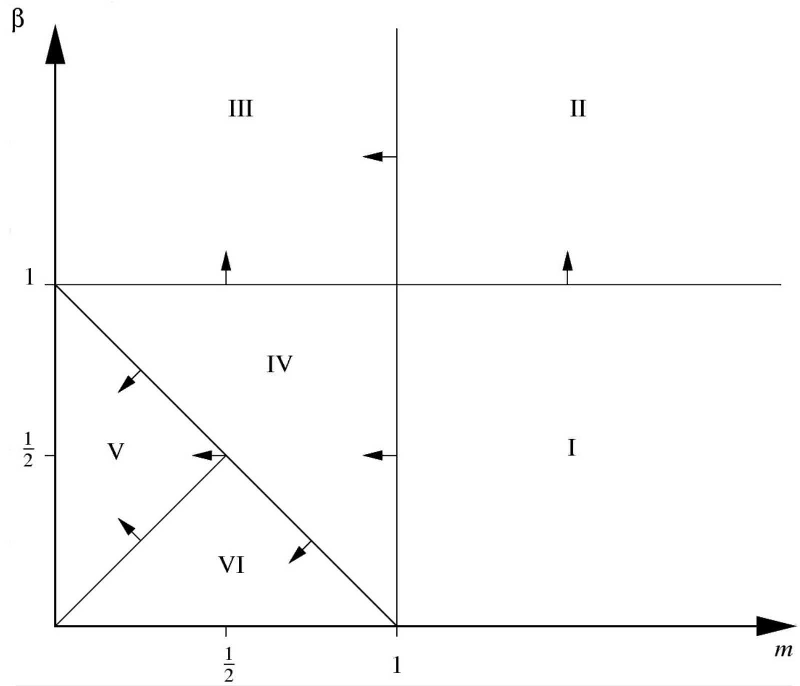

Figure 2.1: Domain of the boundary regularity barrier in the parameter space (m, β)

IV $\quad m^{-1} < \alpha \le (1-\beta)^{-1}$, $\qquad$ if $0<\beta<1, \;\; 1-\beta < m \le 1$,

V $\quad \alpha > m^{-1}$, $\qquad$ if $0 < m \le \frac{1}{2}, \;\; m \le \beta \le 1-m$,

VI $\quad m^{-1} < \alpha \le 2/(m-\beta)$, $\qquad$ if $0 < \beta < \frac{1}{2}, \;\; \beta < m \le 1-\beta$.

If $c<0, \; 0<p<1$, then we have four different cases

(A) $\quad \max\left(m^{-1}, (p-\beta)^{-1}\right) < \alpha$, $\qquad$ if $\; 0 < m \le \beta < p$,

(B) $\quad m^{-1} < \alpha \le 2/(m-\beta)$, $\qquad$ if $\; 0 < \beta < \min\left(p; m\right)$,

(C) $\quad \alpha > m^{-1}$, $\qquad$ if $\; 0 < m \le p \le \beta$,

(D) $\quad m^{-1} < \alpha \le (m-p)^{-1}$, $\qquad$ if $\; 0 < p < m, \; \beta \ge p$.

If $t_0 > 0$ then we choose

$$h(\mu) = M_3\mu^{-1}F(\mu^{-2}),\ M_3 = \Big(\big(M_2M_1^{-1}\big)^{1/\alpha} - 1\Big)^{-1},\quad M_2 = \psi(t_0) - \varepsilon/2,\ \mu \geq \mu_0,$$

where $\mu_0 = \delta_0^{-1/2}$ and we assume that the curve s satisfies condition (1.12) at the point t_0 for $\delta \in (0;\delta_0]$. If $t_0 = 0$ we choose $h(\mu) = \mu^{-2}, \quad \mu \geq \mu_0 = 1$. We consider a function $f(\zeta)$ for $\zeta \in [0; h(\mu)(1+\lambda)]$, where λ is a positive number such that $\lambda \geq \big(M_2M_1^{-1}\big)^{1/\alpha} - 1$. It may easily be checked that

$$L_n\omega_n = \mu f' - a(f^m)'' - c(f^p)' + bf^\beta - b\theta_b\ \varepsilon_n^{\beta\gamma}. \tag{2.15}$$

Assume that $c \geq 0$ or $c < 0,\ p \geq 1$. If either $b \leq 0$ or $b > 0$ and m, β belong to one of the regions I-IV (Figure 1), then from (2.15) it follows that

$$L_n\omega_n \leq f^{\frac{\alpha-1}{\alpha}}\Big\{\alpha M_1^{1/\alpha}\mu h^{-1}(\mu) - h^{-2}(\mu)aM_1^{2/\alpha}m\alpha(m\alpha-1)M_4^{((m-1)\alpha-1)/\alpha}$$
$$- c\,(1-\theta_c)\,M_1^{1/\alpha}\alpha p h^{-1}(\mu)M_4^{p-1} + b\theta_b M_4^{\beta-1+1/\alpha}\Big\}, \tag{2.16}$$

where $M_4 = M_1(1+\lambda)^\alpha$.

Since

$$\mu h(\mu) \to 0 \ \text{ as } \ \mu \to \infty,$$

we can choose and fix $\mu_1 \geq \mu_0$ so large that if $\mu \geq \mu_1$,

$$L_n\omega_n \leq 0 \quad \text{for} \quad 0 \leq \zeta \leq h(\mu)(1+\lambda). \tag{2.17}$$

If however $b > 0$ and m, β belong to one of the regions V, VI (Figure 1), then from (2.15) it follows that

$$L_n\omega_n \leq f^\beta\Big\{\mu h^{-1}(\mu)\alpha M_1^{1/\alpha}M_4^{1-\beta-1/\alpha} - h^{-2}(\mu)aM_1^{2/\alpha}m\alpha(m\alpha-1)M_4^{m-\beta-2/\alpha}$$

$$- c\,(1-\theta_c)\,M_1^{1/\alpha}\alpha p h^{-1}(\mu) M_4^{p-\beta-1/\alpha} + b\Big\}. \quad (2.18)$$

As before, from (2.18), it follows that we can choose and fix $\mu_1 \geq \mu_0$ so large that if $\mu \geq \mu_1$ then (2.17) is valid.

Now assume that $c < 0,\ 0 < p < 1$. If we have one of the two cases (A) and (B), then we can estimate $L_n \omega_n$ similar as in (2.18) (with b replaces with $\theta_b b$) and derive the same conclusion.

Finally, if $c < 0,\ 0 < p < 1$ and we have one of the two cases (C) and (D) , we estimate $L_n \omega_n$ as follows

$$L_n \omega_n \leq f^{\frac{\alpha p-1}{\alpha}} \Big\{ \alpha M_1^{1/\alpha} \mu h^{-1}(\mu) M_4^{1-p} - a M_1^{2/\alpha} \alpha m(m\alpha - 1) h^{-2}(\mu) M_4^{m-p-1/\alpha}$$
$$- c M_1^{1/\alpha} \alpha p h^{-1}(\mu) + \theta_b b M_4^{\beta-p+1/\alpha} \Big\}. \quad (2.19)$$

As before, from (2.19), it follows that we can choose and fix $\mu_1 \geq \mu_0$ so large that if $\mu \geq \mu_1$ then (2.17) is valid.

Let $t_0 > 0$. Since $\psi(t)$ is continuous there exist the numbers $\mu_2 \geq \mu_1$ and δ_1 such that

$$\psi(t) > \psi(t_0) - \varepsilon/2 \quad \text{for} \;\; t_0 - \mu_2^{-2} \leq t \leq t_0 + \delta_1,$$

where, if $t_0 = T$ (and T is finite), we choose $\delta_1 = 0$ and if $t_0 < T$ then $\delta_1 = \delta_1(\varepsilon) > 0$ is such that $t_0 + \delta_1 < T$. If $t_0 = 0$ then we choose $\delta_1 = \delta_1(\varepsilon) > 0$ such that

$$\psi(t) > \psi(0) - \varepsilon/2 \quad \text{for} \;\; 0 \leq t \leq \delta_1.$$

Let us now estimate $\omega_n(0,t)$ in the neighbourhood of t_0. Since $\omega_n(0,t_0) = \psi(t_0) - \varepsilon$ and f is continuous and the sequence $\{s_n\}$ uniformly converges to a continuous function s as $n \to +\infty$, for $\forall$ fixed $\mu \geq \mu_2$ there exists a number $\delta_2 = \delta_2(\mu, \varepsilon) \leq \delta_1$ which does not depend on n and a number $N_1 = N_1(\mu, \varepsilon)$ such that for arbitrary $n \geq N_1$

$$\omega_n(0,t) < \psi(t_0) - \varepsilon/2 \quad \text{for} \;\; t_0 \leq t \leq t_0 + \delta_2$$

(we choose $\delta_2 = 0$ if $t_0 = T$ and $\delta_2 > 0$ if $t_0 < T$). Now suppose that $t_0 > 0$ and consider $\omega_n(0,t)$ for $t_0 - \mu^{-2} \leq t \leq t_0 + \delta_2$, $\mu \geq \mu_2$ and $n \geq N_1$. Since the sequence $\{s_n\}$ uniformly converges to s as $n \to +\infty$, we may suppose without loss of generality that $\omega_{t_0}^{-}(s_n;\delta)$ satisfies (1.12) for $\delta \in (0;\delta_0]$ uniformly with respect to $n \geq N_1$. If $t \in [t_0 - \mu^{-2}; t_0]$ then we have

$$\omega_n(0,t) \leq f\Big(h(\mu) + s_n(t_0) - s_n(t)\Big) \leq f\Big((M_3^{-1}+1)h(\mu)\Big) = \psi(t_0) - \varepsilon/2.$$

If $t_0 = 0$ we choose and fix $\mu_2 \geq \mu_1$ and $N_2 \geq N_1$ so large that if $\mu \geq \mu_2$ and $n \geq N_2$ then

$$u_0\Big((1 - s_{0n} r_n^{-1})y + s_{0n}\Big) \geq u_0(0) - \varepsilon/2 = \psi(0) - \varepsilon/2 \;\; \text{for} \;\; 0 \leq y \leq r_n(r_n - s_{0n})^{-1} h(\mu).$$

Now let $N_3 \geq N_2$ be chosen so large that $\varepsilon_n^{\gamma} < \psi(t_0) - \varepsilon$ for $n \geq N_3$.

Let

$$\eta_n = \left[M_1^{-1}\, \varepsilon_n^{\gamma}\right]^{1/\alpha} h(\mu), n \geq N_3.$$

Obviously, $\eta_n \to 0$ as $n \to \infty$ uniformly with respect to $\mu \geq \mu_2$. Then we set

$$\begin{aligned}
\Omega_n &= \Big\{(y,t): \ t_0 - d_{t_0}(\mu) < t \leq t_0 + \delta_2, \quad 0 < y < \xi_n(t)\Big\}, \\
\Lambda_n &= \Big\{(y,t): \ t_0 - d_{t_0}(\mu) < t \leq t_0 + \delta_2, \quad y = \xi_n(t)\Big\}, \\
\xi_n(t) &= r_n \left(r_n - s_n(t)\right)^{-1} \left(h(\mu) + \mu(t-t_0) + s_n(t_0) - s_n(t) - \eta_n\right), \\
d_{t_0}(\mu) &= (0, \text{ if } t_0 = 0; \ \mu^{-2}, \text{ if } t_0 > 0)\,.
\end{aligned}$$

If $t_0 > 0$ then, since

$$\xi_n(t_0 - \mu^{-2}) \leq r_n \left(r_n - s_n(t_0 - \mu^{-2})\right)^{-1} \left((1 + M_3^{-1})h(\mu) - \mu^{-1}\right), \tag{2.20}$$

we may choose and fix $\mu_3 \geq \mu_2$ so large that for arbitrary $\mu \geq \mu_3$

$$\xi_n(t_0 - \mu^{-2}) < 0 \quad \text{for} \quad n \geq N_3. \tag{2.21}$$

Without loss of generality we may suppose that if $T = +\infty$ then for arbitrary fixed $\mu \geq \mu_3$

$$t_0 + \delta_2 \leq T_n \quad \text{for} \quad n \geq N_3. \tag{2.22}$$

Let us now compare $\omega_n(y,t)$ and $v_n(y,t)$ in Ω_n for fixed $\mu \geq \mu_3$ and for $n \geq N_3(\mu, \varepsilon)$:

$$\omega_n = f(\eta_n) = \varepsilon_n^{\gamma} \leq v_n \quad \text{for} \quad (y,t) \in \Lambda_n$$

$$\omega_n(0,t) \leq \psi(t_0) - \varepsilon/2 < \psi(t) \leq \psi_n(t) = v_n(0,t) \quad \text{for} \quad t_0 - d_{t_0}(\mu) \leq t \leq t_0 + \delta_2.$$

If $t_0 = 0$ we also have

$$\begin{aligned}
\omega_n(y,0) &\leq f(h(\mu)) = u_0(0) - \varepsilon \leq u_0\Big((1 - r_n^{-1}s_{0n})y + s_{0n}\Big) \\
&\leq v_n(y,0) \quad \text{for} \quad 0 \leq y \leq r_n(r_n - s_{0n})^{-1}\left(h(\mu) - \eta_n\right).
\end{aligned}$$

We can now apply the maximum principle. Obviously, ω_n is a smooth function in $\bar{\Omega}_n$. Moreover, ω_n is bounded away from zero in $\bar{\Omega}_n$ by ε_n^γ. Consider a function

$$z(y,t) = v_n(y,t) - \omega_n(y,t).$$

Since $z \geq 0$ on the parabolic boundary of Ω_n, by applying maximum principle it follows that $z \geq 0$ in $\bar{\Omega}_n$. Let

$$P = \Big\{(y,t) : 0 < y < y_0, \quad 0 < t \leq t_0 + \delta_2\Big\},$$

where $0 < y_0 < r_n$ and $\Omega_n \subset P \subset D'_n$ for $\mu \geq \mu_3$ and $n \geq N_3$. Let

$$\bar{\omega}_n(y,t) = \Big\{\omega_n(y,t) \text{ in } \bar{\Omega}_n;\ \varepsilon_n^\gamma \text{ in } \bar{P}\backslash\bar{\Omega}_n\Big\}.$$

Since $v_n \geq \varepsilon_n^\gamma$ in $\bar{P}$, we have $\bar{\omega}_n(y,t) \leq v_n(y,t)$ in $\bar{P}$. In the limit $n' \to +\infty$, we have

$$\omega(y,t) \leq v(y,t) \quad \text{in} \quad \bar{P} \tag{2.23}$$

where

$$\omega(y,t) = \begin{cases} f(h(\mu) - y + \mu(t - t_0) + s(t_0) - s(t)), & (y,t) \in \bar{\Omega} \\ 0, & (y,t) \in \bar{P}\backslash\bar{\Omega} \end{cases}$$

and

$$\Omega = \Big\{(y,t) : t_0 - d_{t_0}(\mu) < t \leq t_0 + \delta_2,\ 0 < y < h(\mu) + \mu(t - t_0) + s(t_0) - s(t)\Big\}.$$

Obviously, we have

$$\lim_{\substack{(y,t)\to(0,t_0)\\(y,t)\in\bar{P}}} \omega(y,t) = \lim_{\substack{(y,t)\to(0,t_0)\\(y,t)\in\bar{\Omega}}} \omega(y,t) = \psi(t_0) - \varepsilon.$$

Hence, from (2.23), (2.13) follows.

Let us now prove (2.14). Let $\bar{M} = \psi^1(t_0 + \bar{\delta})$, where $\bar{\delta} > 0$, is so small that the function ψ^1 is defined and continuous at the point $t_0 + \bar{\delta}$. Take an arbitrary $\varepsilon > 0$ such that $\psi(t_0) + \varepsilon < \bar{M}$. As before, consider a function

$$\omega_n(y,t) = f_1\left(h_1(\mu) - \left(1 - r_n^{-1} s_n(t)\right) y + \mu(t - t_0) + s_n(t_0) - s_n(t)\right), \;\; \mu > 0, \;\; h_1 > 0,$$

where

$$f_1(\zeta) = \left[\bar{M}^{1/\alpha} + \zeta h_1^{-1}(\mu)\left(M_5^{1/\alpha} - \bar{M}^{1/\alpha}\right)\right]^{\alpha}, \quad M_5 = \psi(t_0) + \varepsilon,$$

and α is an arbitrary number such that $0 < \alpha < \min(m^{-1}; p^{-1})$. If $t_0 > 0$ then we choose

$$h_1(\mu) = M_7 \mu^{-1} F(\mu^{-2}), \quad M_7 = \left(\bar{M}^{1/\alpha} - M_5^{1/\alpha}\right)\left(M_5^{1/\alpha} - M_6^{1/\alpha}\right)^{-1},$$

$$M_6 = \psi(t_0) + \varepsilon/2, \quad \mu \geq \mu_0,$$

where $\mu_0 = \delta_0^{-1/2}$ and as before, we assume that the curve s satisfies the condition (1.12) at the point t_0 for $\delta \in (0; \delta_0]$. If $t_0 = 0$ we choose

$$h_1(\mu) = \mu^{-2}, \quad \mu \geq \mu_0 = 1.$$

We consider a function $f_1(\zeta)$ for $\zeta \in [0; h_1(\mu)(1 + \lambda_1)]$, where λ_1 is a positive number such that

$$\left(M_5^{1/\alpha} - M_6^{1/\alpha}\right)\left(\bar{M}^{1/\alpha} - M_5^{1/\alpha}\right)^{-1} \leq \lambda_1 < M_5^{1/\alpha}\left(\bar{M}^{1/\alpha} - M_5^{1/\alpha}\right)^{-1}.$$

Let us transform $L_n\omega_n$:

$$L_n\omega_n = \mu f_1' - a\left(f_1^m\right)'' - c(f_1^p)' + bf_1^\beta - b\theta_b\varepsilon_n^{\beta\gamma} \geq \mu h_1^{-1}(\mu)\alpha\left(M_5^{1/\alpha} - \bar{M}^{1/\alpha}\right)M_{10}$$

$$+ am\alpha(1-m\alpha)h_1^{-2}(\mu)\left(\bar{M}^{1/\alpha} - M_5^{1/\alpha}\right)^2 M_9 + b(1-\theta_b)\bar{M}^\beta - b\theta_b\varepsilon_n^{\beta\gamma}$$

$$+ch_1^{-1}(\mu)\alpha p(\bar{M}^{1/\alpha} - M_5^{1/\alpha})M_{11}^{(\alpha p-1)/\alpha},$$

where $M_{10} = \bar{M}^{(\alpha-1)/\alpha}$ if $\alpha \geq 1$, $M_{10} = M_8^{(\alpha-1)/\alpha}$ if $\alpha < 1$, and

$$M_8 = \left[M_5^{1/\alpha} - \lambda_1\left(\bar{M}^{1/\alpha} - M_5^{1/\alpha}\right)\right]^\alpha > 0,\ M_9 = \bar{M}^{(m\alpha-2)/\alpha},$$

$$M_{11} = \bar{M}^{(\alpha p-1)/\alpha} \text{ if } c \geq 0,\ M_{11} = M_8^{(\alpha p-1)/\alpha} \text{ if } c < 0.$$

Since $\mu h_1(\mu) \to 0$ as $\mu \to \infty$, we can choose and fix $\mu_1 \geq \mu_0$ so large that if $\mu \geq \mu_1$ then

$$L_n\omega_n > 0 \text{ for } 0 \leq \zeta \leq h_1(\mu)(1+\lambda_1).$$

Let $t_0 > 0$. Since $\psi(t)$ is continuous, there exists a number $\mu_2 \geq \mu_1$ and δ_1 such that

$$\psi(t) < \psi(t_0) + \varepsilon/2 \text{ for } t_0 - \mu_2^{-2} \leq t \leq t_0 + \delta_1,$$

where, if $t_0 = T$ (and T is finite), we choose $\delta_1 = 0$ and if $t_0 < T$ then $\delta_1 = \delta_1(\varepsilon) \in (0; \bar{\delta}]$ is some number such that $t_0 + \delta_1 < T$. If $t_0 = 0$ then we choose $\delta_1 = \delta_1(\varepsilon) > 0$ such that $\psi(t) < \psi(0) + \varepsilon/2$ for $0 \leq t \leq \delta_1$. Let us now estimate $\omega_n(0,t)$ in the neighbourhood of t_0. Since $\omega_n(0,t_0) = \psi(t_0) + \varepsilon$ and f is continuous and the sequence $\{s_n\}$ uniformly converges to a continuous function s as $n \to +\infty$, there exists a number $0 \leq \delta_2 = \delta_2(\mu,\varepsilon) \leq \delta_1$ which does not depend on n and a number $N_1 = N_1(\mu,\varepsilon)$ such that for arbitrary $n \geq N_1$

$$\omega_n(0,t) > \psi(t_0) + \varepsilon/2 \text{ for } t_0 \leq t \leq t_0 + \delta_2$$

(we choose $\delta_2 = 0$ if $t_0 = T$ and $\delta_2 > 0$ if $t_0 < T$.) Now suppose that $t_0 > 0$ and consider $\omega_n(0,t)$ for $t_0 - \mu^{-2} \leq t \leq t_0 + \delta_2$, $\mu \geq \mu_2$ and $n \geq N_1$. As before, we may suppose that $\omega_{t_0}^-(s_n;\delta)$ satisfies (1.12) for $\delta \in (0;\delta_0]$ uniformly with respect to $n \geq N_1$. If $t_0 \in [t_0 - \mu^{-2}; t_0]$ then we have

$$\omega_n(0,t) \geq f_1\Big(h_1(\mu) + s_n(t_0) - s_n(t)\Big) \geq f_1\Big(\big(M_7^{-1} + 1\big)\, h_1(\mu)\Big) = \psi(t_0) + \varepsilon/2.$$

Now we can choose $N_2 = N_2(\mu,\varepsilon) \geq N_1$ so large that for $n \geq N_2$

$$\psi(t) \leq \psi_n(t) < \psi(t_0) + \varepsilon/2 \quad \text{for} \quad t_0 - \mu^{-2} \leq t \leq t_0 + \delta_2.$$

If $t_0 = 0$ we choose $\mu_2 \geq \mu_1$ and $N_2 \geq N_1$ so large that if $\mu \geq \mu_2$ and $n \geq N_2$, then

$$\begin{aligned}
&\psi(t) \leq \psi_n(t) < \psi(0) + \varepsilon/2 \quad \text{for} \quad 0 \leq t \leq \delta_2 \\
&u_{0n}\Big((1 - s_{0n}r_n^{-1})y + s_{0n}\Big) \leq u_0\Big((1 - s_{0n}r_n^{-1})y + s_{0n}\Big) + K\varepsilon_n^{\gamma} < u_0(0) + \varepsilon \\
&\qquad \text{for} \quad 0 \leq y \leq r_n(r_n - s_{0n})^{-1} h_1(\mu).
\end{aligned}$$

Then we set $\Omega_n, \Lambda_n, \xi_n$ as before, by replacing h and η_n with h_1 and 0 respectively. We then derive (2.19)-(2.21), replacing M_3 and N_3 with M_7 and N_2 respectively.

Let us now compare $\omega_n(y,t)$ and $v_n(y,t)$ in Ω_n for fixed $\mu \geq \mu_3$ and for $n \geq N_2$. We have

$$\begin{aligned}
&\omega_n(0,t) > v_n(0,t) \quad \text{for} \quad t_0 - d_{t_0}(\mu) \leq t \leq t_0 + \delta_2 \\
&\omega_n = \bar{M} = \psi^1(t_0 + \bar{\delta}) \geq \psi^1(t_0 + \delta_2) \geq v_n \quad \text{for} \quad (x,t) \in \bar{\Lambda}_n.
\end{aligned}$$

If $t_0 = 0$ then

$$\omega_n(y,0) \geq f_1(\mu^{-2}) = u_0(0) + \varepsilon > u_{0n}\Big((1 - s_{0n}r_n^{-1})y + s_{0n}\Big) = v_n(y,0)$$
$$\text{for} \quad 0 \leq y \leq r_n(r_n - s_{0n})^{-1}\mu^{-2}.$$

Consider a function $z(y,t) = v_n(y,t) - \omega_n(y,t)$. Since $z \leq 0$ on the parabolic boundary of Ω_n, by applying maximum principle it follows that $z \leq 0$ in $\bar{\Omega}_n$. As before, consider a rectangular P, where $0 < y_0 < r_n$ and $\Omega_n \subset P \subset D'_n$ for $\mu \geq \mu_3$ and $n \geq N_2$. Let

$$\bar{\omega}_n(y,t) = \Big\{ \omega_n(y,t) \text{ in } \bar{\Omega}_n;\ \bar{M} \ \text{ in } \ \bar{P} \backslash \bar{\Omega}_n \Big\}.$$

Since $v_n(y,t) \leq \bar{M}$ in $\bar{P}$, we have $\bar{\omega}_n \geq v_n$ in $\bar{P}$. In the limit as $n' \to \infty$, we have

$$\omega(y,t) \geq v(y,t) \quad \text{in} \ \ \bar{P}, \tag{2.24}$$

where

$$\omega(y,t) = \Big\{ f_1\Big(h_1(\mu) + \mu(t - t_0) + s(t_0) - s(t) - y \Big) \text{ in } \bar{\Omega};\ \bar{M} \text{ in } \bar{P} \backslash \bar{\Omega} \Big\}$$

and Ω is defined as before with h being replaced by h_1. Obviously

$$\lim_{\substack{(y,t) \to (0,t_0) \\ (y,t) \in \bar{P}}} \omega(y,t) = \lim_{\substack{(y,t) \to (0,t_0) \\ (y,t) \in \bar{\Omega}}} \omega(y,t) = \psi(t_0) + \varepsilon.$$

Hence, from (2.24),(2.14) follows and we have completed the proof of continuity of $v(y,t)$ on the line $y = 0$, that is to say the continuity of $u(x,t)$ on the curve $x = s(t),\ t \geq 0$. The theorem is proved. □

Remark 2.1 *It should be noted that since we construct the solution as a limit of a sequence of classical solutions to non-degenerate parabolic problems, by using a generalization of the Nash theorem [90] (Theorem 10.1, ch.III) and apriori interior estimations [57] (10, Theorem 10, ch.III), one may show by standard methods that* *the generalized solution is a classical solution in a neighbourhood of any interior point* (x_0, t_0), *where* $u(x_0, t_0) > 0$. *If in particular, constructed solution has a positive infimum, then it is a classical solution and uniqueness of this solution immediately follow from the existence theorem (which includes continuity up to the boundaries) and from the classical maximum principle. This observation is of a general nature and it relates to both problems considered in this dissertation (see Theorems 2.2.2 and 3.2.2 below).*

2.2 Uniqueness

Throughout this section we shall suppose that the boundary curve s satisfies the assumption (L).

Theorem 2.2.1. *Let* $a > 0,\ m > 0,\ c \in \mathbb{R}^1,\ p > 0$ *and either* $b \geq 0,\ \beta > 0$ *or* $b < 0,\ \beta \geq 1$. *If* s *satisfies (1.13) then the solution of the CDP is unique.*

Proof. Suppose that g_1 and g_2 are two solutions of CDP. Let $\bar{t} \in (0; T]$ be an arbitrary finite number. We shall prove uniqueness by proving that for some limit solution $u = \lim u_n$ the following inequalities are valid

$$\int_{s(t)}^{+\infty} \Big(u(x,t) - g_i(x,t)\Big)\omega(x)\ dx \leq 0, \quad i = 1, 2, \tag{2.25}$$

for every $t \in (0; \bar{t}]$ and for every $\omega \in C_0^\infty\Big((s(t); +\infty)\Big)$ such that $|\omega| \leq 1$. Obviously, from (2.25) it follows that

$$g_1 = u = g_2 \quad \text{for} \quad s(t) \le x < +\infty, \quad 0 \le t \le \bar{t},$$

which implies uniqueness in view of the arbitrariness of $\bar{t}$.

Let $t \in (0; \bar{t}]$ be fixed and let $\omega \in C_0^\infty\Big((s(t); +\infty)\Big)$ be an arbitrary function such that $|\omega| \le 1$. Assume that

$$\text{supp } \omega = (q_1; q_2), \quad s(t) < q_1 < q_2 < +\infty. \tag{2.26}$$

Let $\chi(x) = Kx^\gamma$ for $x \ge 0$ (see the proof of the Theorem 2.1.1). Suppose also that the sequences $\{r_n\}$, $\{s_n\}$ satisfy, in addition to the conditions from Theorem 2.1.1, the following:

$$s_n'(\tau) \ge -M_0 \quad \text{for} \quad 0 \le \tau \le \bar{t}, \qquad n = 1, 2, \dots ,$$

(the possibility of which follows from (1.13)

$$s_n(\tau) > s_{n+1}(\tau) > s(\tau) \quad \text{for} \quad 0 \le \tau \le \bar{t}, \qquad n = 1, 2, \dots ,$$

$$\max_{0 \le \tau \le \bar{t}} \Big| g_i^m(s_n(\tau), \tau) - g_i^m(s(\tau), \tau) \Big| \le \chi^m(\varepsilon_n), \qquad n = 1, 2, \dots ; \quad i = 1, 2,$$

$$r_n \ge \max(n c_n^{-1}; \quad \alpha_n + d_n \bar{t} + \varepsilon_n^{-1}), \qquad n = 1, 2, \dots ,$$

where

$$c_n = \frac{-\bar{D}+\left[\bar{D}^2+4A_0(\varepsilon_n^{\gamma-1}+b\varepsilon_n^{\beta\gamma-1})\right]^{\frac{1}{2}}}{2A_0},\ d_n = (A_0+D_0)\varepsilon_n^{1-\gamma} \qquad \text{if} \quad b \geq 0,$$

$$c_n = \frac{-|c|\varepsilon_n^{p-1}+\left[c^2\varepsilon_n^{2(p-1)}+4a\varepsilon_n^{m-1}\right]^{\frac{1}{2}}}{2a\varepsilon_n^{m-1}},\ d_n = a\ \varepsilon_n^{m-1}+|c|\varepsilon_n^{p-1} \qquad \text{if} \quad b<0, \quad 0<m<1,$$

$$c_n = \frac{-|c|\varepsilon_n^{p-1}+\left[c^2\varepsilon_n^{2(p-1)}+4\bar{A}\right]^{\frac{1}{2}}}{2\bar{A}},\ d_n = A_0+D_0 \qquad \text{if} \quad b<0, \quad m \geq 1,$$

$$\alpha_n = \max\Big(\ln 2;\ (2a\varepsilon_n^{m\gamma-1})^{-1}\Big[D_0+(D_0^2+4aB_0\varepsilon_n^{m\gamma-1})^{\frac{1}{2}}\Big]\Big)$$

if $b \geq 0$, or $b<0$ and $m \geq 1$,

$$\alpha_n = \max\Big(\ln 2; (2\Delta)^{-1}\Big[|c|\varepsilon_n^{p-1}+(4B_0\Delta+c^2\varepsilon_n^{2(p-1)})^{\frac{1}{2}}\Big]\Big) \qquad \text{if} \quad b<0, \quad 0<m<1,$$

and $A_0,\ B_0,\ \Delta_0$ are positive constants (defined below). Without loss of generality we may assume that

$$s_n(t) < q_1, \quad r_n \geq q_2+1, \quad n=1,2,\ldots$$

Since the proof of (2.25) is similar for each i, we shall henceforth let $g = g_i$. Let

$$R_n = \Big\{(x,\tau) : s_n(\tau) < x < r_n, \quad 0 \leq \tau < t\Big\}.$$

Take any function $f \in C^{2,1}_{x,t}(\bar{R}_n)$ such that

$$f = 0 \ \text{ for } \ x = s_n(\tau) \ \text{ and } \ x = r_n,\ 0 \leq \tau \leq t.$$

Let $u = \lim u_n$ be a limit solution of CDP constructed in the proof of Theorem 2.1.

We have

$$I(u_n, f, R_n) + b\theta_b\varepsilon_n^{\beta\gamma}\int_0^t \int_{s_n(\tau)}^{r_n} f(x,\tau)\ dx\ d\tau - I(g, f, R_n) = 0, \tag{2.27}$$

If $b \geq 0$ then we transform (2.27) as follows.

$$
\begin{aligned}
&\int_{s_n(t)}^{r_n} (u_n(x,t) - g(x,t)) f(x,t)\ dx = \\
&\int_{s_{0n}}^{r_n} (u_{0n}(x) - u_0(x)) f(x,0)\ dx + \\
&a\int_0^t \Big(\psi_n^m(\tau) - g^m(s_n(\tau),\tau)\Big) f_x(s_n(\tau),\tau)\ d\tau - \\
&a\int_0^t \Big(u_n^m(r_n,\tau) - g^m(r_n,\tau)\Big) f_x(r_n,\tau)\ d\tau \ \ + \\
&\int_0^t \int_{s_n(\tau)}^{r_n} \big(C_n^k f_\tau + A_n^k f_{xx} - B_n^k f + D_n^k f_x\big) \Big(u_n^{1/\gamma} - g^{1/\gamma}\Big)\ dx\ d\tau + \\
&b\ \varepsilon_n^{\beta\gamma} \int_0^t \int_{s_n(\tau)}^{r_n} f\ dx\ d\tau \ \ + \int_0^t \int_{s_n(\tau)}^{r_n} \Big(\big(C_n - C_n^k\big) f_\tau + \big(A_n - A_n^k\big) f_{xx} - \\
&\qquad \big(B_n - B_n^k\big) f + (D_n - D_n^k) f_x \Big) \Big(u_n^{1/\gamma} - g^{1/\gamma}\Big)\ dx\ d\tau,
\end{aligned}
\tag{2.28}
$$

where

$$
\begin{aligned}
A_n(x,t) &= am\gamma \int_0^1 \Big(\theta u_n^{1/\gamma} + (1-\theta) g^{1/\gamma}\Big)^{m\gamma - 1}\ d\theta, \\
B_n(x,t) &= b\beta\gamma \int_0^1 \Big(\theta u_n^{1/\gamma} + (1-\theta) g^{1/\gamma}\Big)^{\beta\gamma - 1}\ d\theta, \\
C_n(x,t) &= \gamma \int_0^1 \Big(\theta u_n^{1/\gamma} + (1-\theta) g^{1/\gamma}\Big)^{\gamma - 1}\ d\theta, \\
D_n(x,t) &= cp\gamma \int_0^1 \Big(\theta u_n^{1/\gamma} + (1-\theta) g^{1/\gamma}\Big)^{p\gamma - 1}\ d\theta,
\end{aligned}
$$

and $A_n^k,\ B_n^k,\ C_n^k,\ D_n^k,\ \ k = 1, 2, \ldots$ are C^∞ approximations of $A_n,\ B_n,\ C_n,\ D_n$, respectively, in $\bar{R}_n$. We assume that

$$
\max\ L_n^k \leq \max\ L_n, \ \ \min\ L_n^k \geq \min\ L_n \quad \text{in} \quad \bar{R}_n, \tag{2.29}
$$

where L stands for A to C, respectively. If $b < 0$ then we transform (2.27) as follows.

$$\int_{s_n(t)}^{r_n} (u_n(x,t) - g(x,t))f(x,t)\,dx = \int_{s_{0n}}^{r_n} (u_{0n}(x) - u_0(x))f(x,0)\,dx +$$
$$a\int_0^t \Big(\psi_n^m(\tau) - g^m(s_n(\tau),\tau)\Big) f_x(s_n(\tau),\tau)\,d\tau -$$
$$a\int_0^t \Big(u_n^m(r_n,\tau) - g^m(r_n,\tau)\Big) f_x(r_n,\tau)\,d\tau \;+$$
$$\int_0^t \int_{s_n(\tau)}^{r_n} \left(f_\tau + A_n^k f_{xx} - B_n^k f + D_n^k f_x\right)(u_n - g)\,dx\,d\tau \;+$$
$$\int_0^t \int_{s_n(\tau)}^{r_n} \Big((A_n - A_n^k)f_{xx} - (B_n - B_n^k)f + (D_n - D_n^k)f_x\Big)(u_n - g)\,dx\,d\tau \tag{2.30}$$

where $A_n,\ B_n,\ D_n,\ A_n^k,\ B_n^k,\ D_n^k$ are the same as before (note that $\gamma = 1$ if $b < 0$).

Since $\varepsilon_n^\gamma \le u_n(x,\tau) \le \psi^1(t)$, in $\bar{R}_n$, we have

$$\begin{array}{lll}
a\,\varepsilon_n^{m\gamma-1} \le A_n, A_n^k \le \bar{A} & \text{in } \bar{R}_n \text{ if } b \ge 0 \text{ or } b < 0 \text{ and } m \ge 1, \\
0 < \Delta \le A_n, A_n^k \le a\,\varepsilon_n^{m-1} & \text{in } \bar{R}_n, \text{ if } b < 0 \text{ and } 0 < m < 1, \\
|b|\,\varepsilon_n^{\beta\gamma-1} \le |B_n|,\ \left|B_n^k\right| \le \bar{B} & \text{in } \bar{R}_n, \\
\varepsilon_n^{\gamma-1} \le C_n,\ C_n^k \le \bar{C}, & \text{in } \bar{R}_n, \text{if } b \ge 0, \\
|c|\,\varepsilon_n^{p\gamma-1} \le |D_n|,\ \left|D_n^k\right| \le \bar{D} & \text{in } \bar{R}_n, \text{if } b \ge 0 \text{ or } b < 0 \text{ and } m \ge 1, \\
|c|\,\bar{D} \le |D_n|,\ \left|D_n^k\right| \le |c|\,\varepsilon_n^{p-1} & \text{in } \bar{R}_n \text{ if } b < 0, 0 < m < 1.
\end{array} \tag{2.31}$$

where $\bar{A},\ \bar{B},\ \bar{C},\ \bar{D},\ \Delta$ are some positive constants which do not depend on n, k. Furthermore, we shall suppose that $A_0 = \bar{A},\ C_0 = \bar{C},\ D_0 = \bar{D},\ \Delta_0 = \Delta$ and

$B_0 > \bar{B}$. Then consider the problem

$$\mathcal{L}_1 f = E_n^k f_\tau + A_n^k f_{xx} - B_n^k f + D_n^k f_x = 0 \quad \text{in } R_n, \tag{2.32a}$$

$$f(x,t) = \omega(x), \qquad s_n(t) \le x \le r_n, \tag{2.32b}$$

$$f(s_n(\tau),\tau) = f(r_n,\tau) = 0. \qquad 0 \le \tau \le t. \tag{2.32c}$$

where $E_n^k = C_n^k$ if $b \ge 0$, and $E_n^k \equiv 1$ if $b < 0$. The existence and uniqueness of the classical solution to (2.32a)-(2.32c) follows from [57].

The solution $f = f(x,\tau)$ has the following properties

I $|f| \le \exp(\sigma_b B_0(t-\tau)), \qquad (x,\tau) \in \bar{R}_n, \qquad \sigma_b = (1 \text{ if } b < 0; \; 0 \text{ if } b \ge 0),$

II $|f| \le \exp[c_n(q_2 - x) + (1 + \sigma_b B_0)(t-\tau)], \qquad (x,\tau) \in \bar{R}_n,$

III $|f| \le \exp[q_2 - x + (d_n + \sigma_b B_0)(t-\tau)], \qquad (x,\tau) \in \bar{R}_n,$

IV $|f_x(r_n,\tau)| = O(\alpha_n \exp(-\varepsilon_n^{-1})) \quad \text{as } n \to +\infty \quad \text{for } \tau \in [0;t],$

V $|f_x(s_n(\tau),\tau)| = O(\varepsilon_n^s) \quad \text{as } n \to +\infty \quad \text{for } \tau \in [0;t],$

where $s = (1 - m\gamma$, if $b \ge 0$; $1 - m$, if $b < 0$ and $m \ge 1$; 0, if $b < 0$ and $0 < m < 1)$.

VI $\|f\|_{W_q^{2,1}(R_n)} \le M_*(n), \qquad q > 1,$

where the constant M_* does not depend on k.

If $b > 0$ then the property I directly follows from maximum principle. If b=0 consider a function

$$f_1 = (1 \pm f)\exp(\tau)$$

Obviousily, $f_1 \geq 0$ in $\bar{R}_n \backslash R_n$. Moreover, we have

$$\mathcal{L}_1 f_1 = E_n^k f_1 \quad \text{for} \quad (x, \tau) \in R_n \tag{2.33}$$

If f_1 attains its negative minimum in $\bar{R}_n$ at some points of R_n, then at the minimum point we have

$$E_n^k f_{1\tau} + A_n^k f_{1xx} + D_n^k f_{1x} \geq 0 \tag{2.34}$$

which is a contradiction of (2.33). Hence $f_1 \geq 0$ in $\bar{R}_n$, which implies I if $b = 0$. If $b < 0$ then consider

$$f_1 = \exp(B_0(\tau - t))f \tag{2.35}$$

which satisfies

$$\mathcal{L}_2 f_1 = f_{1\tau} + A_n^k f_{1xx} - (B_n^k + B_0) f_1 + D_n^k f_x = 0 \quad \text{in } R_n, \tag{2.36a}$$

$$f_1(x, t) = \omega(x), \qquad s_n(t) \leq x \leq r_n, \tag{2.36b}$$

$$f_1 = 0 \qquad \text{for} \quad x = s_n(t),\ x = r_n,\ 0 \leq \tau \leq t. \tag{2.36c}$$

Since $b < 0$, we have

$$B_n^k + B_0 \geq B_0 - \bar{B} > 0 \quad \text{in} \quad \bar{R}_n.$$

Suppose now that $f_1 < -1$ at some point of R_n. Then f_1 attains its negative minimum at some point of R_n , at which point (2.34) (with $E_n^k = 1$) will be valid. This is contradiction of (2.36a). similarly, the assumption $f_1 > 1$ leads to a contradiction of (2.36a). Hence $|f_1| \leq 1$ in $\bar{R}_n$, which implies I, if $b < 0$.

To prove the property II , in the case when $b > 0$, consider a function

$$f_1(x,\tau) = \exp\{c_n(q_2 - x) + t - \tau\} \pm f(x,\tau).$$

Obviously, $f_1 \geq 0$ in $\bar{R}_n \backslash R_n$. Moreover, we have

$$\mathcal{L}_1 f_1(x,\tau) = \exp\{c_n(q_2 - x) + t - \tau\} \left[-C_n^k + c_n^2 A_n^k - B_n^k - c_n D_n^k\right]$$

$$\leq \exp\{c_n(q_2 - x) + t - \tau\} \left[A_0 c_n^2 - \varepsilon_n^{\gamma-1} - b\varepsilon_n^{\beta\gamma-1} + c_n \bar{D}\right] = 0 \text{ in } R_n. \qquad (2.37)$$

If f_1 attains its negative minimum at some point of R_n , at the minimum point of (2.34) is valid. Hence, at the minimum point we have a contradiction of (2.32a),(2.37). Therefore, $f_1 \geq 0$ in $\bar{R}_n$, which implies II in the case when $b > 0$. If $b = 0$ then consider a function

$$f_1(x,\tau) = \exp(\tau)[\exp\{c_n(q_2 - x) + t - \tau\} \pm f(x,\tau)].$$

Obviously, $f_1 \geq 0$ in $\bar{R}_n \backslash R_n$. Moreover, we have

$$\mathcal{L}_1 f_1(x,\tau) = C_n^k f_{1\tau} + A_n^k f_{1xx} = \exp\{c_n(q_2 - x) + t - \tau\} \left[A_n^k c_n^2 - C_n^k - D_n^k c_n\right] + C_n^k f_1$$

$$\leq \exp\{c_n(q_2 - x) + t\} \left[A_0 c_n^2 - \varepsilon_n^{\gamma-1} + c_n \bar{D}\right] + C_n^k f_1 = C_n^k f_1 \text{ in } R_n \qquad (2.38)$$

As before, from (2.38) it follows that $f_1 \geq 0$ in $\bar{R}_n$, which implies II in the case when $b = 0$.

If $b < 0$, then consider a function f_1 from (2.35) which satisfies (2.36a)-(2.36c). Then consider a function

$$f_2(x,\tau) = \exp\left(c_n(q_2 - x) + t - \tau\right) \pm f_1.$$

Obviously, $f_2 \geq 0$ in $\bar{R}_n \backslash R_n$. Moreover, we have

$$\mathcal{L}_2 f_2(x,\tau) = \left[-1 + c_n^2 A_n^k - B_n^k - B_0 - c_n D_n^k\right] \exp\{c_n(q_2 - x) + t - \tau\}$$

$$\leq \left[c_n^2 A_n^k - 1 + c_n|c|\varepsilon_n^{p-1}\right] \exp\{c_n(q_2 - x) + t - \tau\} \leq 0 \quad \text{in} \quad R_n. \tag{2.39}$$

As before, from (2.36a), (2.39) it follows that

$$f_2 \geq 0 \quad \text{in} \quad \bar{R}_n,$$

which implies the property II in the case of $b < 0$.

To prove the property III , in the case when $b > 0$ consider a function

$$f_1(x,\tau) = \exp\Big(q_2 - x + d_n(t - \tau)\Big) \pm f(x,\tau).$$

Obviously, $f_1 \geq 0$ in $\bar{R}_n \backslash R_n$. Moreover, we have

$$\mathcal{L}_1 f_1(x,\tau) = \exp\Big(q_2 - x + d_n(t - \tau)\Big)\left[-C_n^k d_n + A_n^k - B_n^k - D_n^k\right]$$

$$\leq \exp\Big(q - x + d_n(t - \tau)\Big)\left[A_0 - \varepsilon_n^{\gamma-1} d_n + D_0\right] = 0 \quad \text{in} \quad R_n. \tag{2.40}$$

As before, from (2.40) the property III follows. If $b = 0$ then consider a function

$$f_1(x,\tau) = \exp(\tau)[\exp\Big(q - x + d_n(t - \tau)\Big) \pm f(x,\tau)].$$

Obviously, $f_1 \geq 0$ in $\bar{R}_n \backslash R_n$. Moreover, we have

$$\mathcal{L}_1 f_1(x,\tau) = C_n^k f_{1\tau} + A_n^k f_{1xx} + D_n^k f_{1x} = \exp\Big(q_2 - x + d_n(t-\tau) + \tau\Big)\left[A_n^k - d_n C_n^k - D_n^k\right] + C_n^k f_1$$

$$\leq \exp\Big(q_2 - x + d_n(t-\tau) + \tau\Big)\left[A_0 - d_n \varepsilon_n^{\gamma-1} + D_0\right] + C_n^k f_1 = C_n^k f_1 \text{ in } R_n \quad (2.41)$$

As before, from (2.41) it follows that $f_1 \geq 0$ in R_n which implies property III in the case of $b = 0$. If $b < 0$ then consider a function

$$f_2(x,\tau) = \exp\Big(q_2 - x + d_n(t-\tau)\Big) \pm f_1(x,\tau).$$

Obviously, $f_2 \geq 0$ in $\bar{R}_n \backslash R_n$. Moreover, we have

$$\mathcal{L}_2 f_2(x,\tau) = \left[-d_n + A_n^k - B_n^k - B_0 - D_n^k\right] \exp\Big(q_2 - x + d_n(t-\tau)\Big)$$

$$\leq \left[-d_n + A_n^k + |D_n^k|\right] \exp\Big(q - x + d_n(t-\tau)\Big) \leq 0 \quad \text{in} \quad R_n. \quad (2.42)$$

As before, from (2.36a), (2.42) it follows that $f_2 \geq 0 \quad \text{in} \quad \bar{R}_n$, which implies III in the case of $b < 0$.

To prove the property IV , in the case when $b > 0$ consider a function

$$f_1(x,\tau) = \exp\Big(q_2 + d_n t - r_n + 1 + \alpha_n(x - r_n + 1)\Big) \pm f(x,\tau) \quad \text{in} \quad R_{1n}.$$

where

$$R_{1n} = \Big\{(x,\tau) : r_n - 1 < x < r_n,\ 0 \leq \tau < t\Big\}$$

Obviously, $f_1 \geq 0$ in $\bar{R}_{1n} \backslash R_{1n}$. Moreover, in R_{1n} , we have

$$\mathcal{L}_1 f_1 = \exp\Big(q_2 + d_n t - r_n + 1 + \alpha_n(x - r_n + 1)\Big)\left[\alpha_n^2 A_n^k - B_n^k + \alpha_n D_n^k\right]$$

$$\geq \exp\Big(q_2 + d_n t - r_n + 1 + \alpha_n(x - r_n + 1)\Big)\left[\alpha_n^2 a\varepsilon_n^{m\gamma-1} - B_0 - \alpha_n \bar{D}\right] \geq 0. \quad (2.43)$$

If f_1 attains its positive maximum in $\bar{R}_{1n}$ at some point of R_{1n}, then at the maximum point we have

$$\mathcal{L}_1 f_1 < 0$$

which contradicts (2.43). Moreover, since $\alpha_n \geq \ln 2$ from property III it follows that

$$f_1(r_n - 1, \tau) \leq f_1(r_n, \tau), \qquad \text{for } \ 0 \leq \tau \leq t. \qquad (2.44)$$

Hence f_1 attains its positive maximum in $\bar{R}_{1n}$ on the line $x = r_n$.

Hence $f_{1x}(r_n, \tau) \geq 0$ for $0 \leq \tau \leq t$ which implies IV in the case of $b > 0$. If $b = 0$ then consider a function

$$f_1(x, \tau) = \exp(\tau)\Big[\exp\Big(q_2 + d_n t - r_n + 1 + \alpha_n(x - r_n + 1)\Big) \pm f(x, \tau)\Big].$$

Obviously, $f_1 \geq 0$ in $\bar{R}_{1n} \backslash R_{1n}$. Moreover, we have

$$\mathcal{L}_1 f_1 = C_n^k f_{1\tau} + A_n^k f_{1xx} + D_n^k f_{1x} = C_n^k f_1 + (a\varepsilon_n^{m\gamma-1}\alpha_n^2 - D_0\alpha_n)$$

$$\times \exp\Big(q + d_n t - r_n + 1 + \alpha_n(x - r_n + 1) + \tau\Big) \geq C_n^k f_1 \text{ in } R_{1n}. \quad (2.45)$$

As before, from (2.45) it follows that f_1 cannot attain its positive in $\bar{R}_{1n}$ at the points of R_{1n}. Since $\alpha_n \geq \ln 2$ the inequality (2.44) is valid. Hence f_1 attains its maximum on the whole line $x = r_n$, which again implies IV if $b = 0$.

If $b < 0$ then consider a function f_1 from (2.35) which satisfies equation (2.36a)-(2.36c) in R_{1n}. Then consider a function

$$f_2(x, \tau) = \exp\Big(q_2 + d_n t - r_n + 1 + \alpha_n(x - r_n + 1)\Big) \pm f_1(x, \tau).$$

From the estimation III it follows that, $f_2 \geq 0$ in $\bar{R}_{1n} \backslash R_{1n}$. Moreover, in R_{1n}, we have

$$\mathcal{L}_2 f_2(x,\tau) = \left[A_n^k \alpha_n^2 - B_n^k - B_0 + D_n^k \alpha_n\right] \exp\Big(q + d_n t - r_n + 1 + \alpha_n(x - r_n + 1)\Big)$$

$$\geq \left[\Delta \alpha_n^2 - B_0 - |c| \alpha_n \varepsilon_n^{p-1}\right] \exp\Big(q_2 + d_n t - r_n + 1 + \alpha_n(x - r_n + 1)\Big) \geq 0. \quad (2.46)$$

If f_2 attains its positive maximum in $\bar{R}_{1n}$ at some point of R_{1n}, then at the maximum point we have

$$\mathcal{L}_2 f_2 < 0.$$

which contradicts (2.46). Moreover, since $\alpha_n \geq \ln 2$ from property III, (2.44) for f_2 follows. Hence f_2 attains its positive maximum in $\bar{R}_{1n}$ on the line $x = r_n$. Therefore,

$$f_{2x}(r_n, \tau) \geq 0 \text{ for } 0 \leq \tau \leq t$$

or

$$f_{1x}(r_n, \tau) \leq \alpha_n \exp\Big(q_2 + 1 + d_n t + \alpha_n - r_n\Big) \text{ for } 0 \leq \tau \leq t,$$

which implies IV in the case of $b < 0$.

Let us prove property V. Assume that

$$e_n = a(2(C_0 M_0 + D_0))^{-1} \varepsilon_n^{m\gamma - 1} \qquad \text{if } \; b \geq 0,$$

$$e_n = \bar{e} \varepsilon_n^{m-1}, \quad \bar{e} = \min\Big(2\tilde{\delta}; a(2M_0)^{-1}; (a/B_0)^{1/2}\Big), \qquad \text{if } \; b < 0, \; m \geq 1,$$

$$e_n = \bar{e} = \min\Big(2\tilde{\delta}; B_0^{-1}\Big(((M_0 + D_0)^2 + 2B_0\Delta)^{1/2} - (M_0 + D_0)\Big)\Big), \qquad \text{if } \; b < 0, \; 0 < m < 1$$

where $\tilde{\delta} > 0$ is chosen such that $s_n(t) + \tilde{\delta} < p, \quad n = 1, 2, \ldots$.

Let

$$R_{2n} = \Big\{(x, \tau) : s_n(\tau) < x < s_n(\tau) + e_n/2, \quad 0 \le \tau < t\Big\}.$$

Obviously, $s_n(t) + e_n/2 < p$, and $s_n(\tau) + e_n/2 < r_n$ for $0 \le \tau \le t$, if n is chosen large enough. If $b > 0$ consider a function

$$f_1(x, \tau) = 4e_n^{-2}(s_n(\tau) - x + e_n)^2 \pm f(x, \tau).$$

Obviously $f_1 \ge 0$ in $\bar{R}_{2n}/R_{2n}$. Moreover, we have

$$\mathcal{L}_1 f_1 \ge -8(C_0 M_0 + D_0)e_n^{-1} + 8a\varepsilon_n^{m\gamma-1}e_n^{-2} - 4B_0 \ge 0 \quad \text{in} \quad R_{2n},$$

if n is chosen large enough. It follows that f_1 cannot attain its maximum in $\bar{R}_{2n}$ at some point of R_{2n}. Since

$$\begin{aligned} & f_1\Big(s_n(\tau) + e_n/2, \tau\Big) \le f_1\Big(s_n(\tau), \tau\Big) && \text{for} \;\; 0 \le \tau \le t \\ & f_1(x, t) \le f_1(s_n(t), t) && \text{for} \;\; s_n(t) \le x \le s_n(t) + e_n/2, \end{aligned} \tag{2.47}$$

the function f_1 attains its maximum in $\bar{R}_{2n}$ on the whole curve $x = s_n(\tau),\ 0 \le \tau \le t$. That is to say

$$f_{1x}\Big(s_n(\tau), \tau\Big) \le 0 \quad \text{for} \;\; 0 \le \tau \le t$$

or

$$\Big|f_x\Big(s_n(\tau), \tau\Big)\Big| \le 8e_n^{-1} \quad \text{for} \;\; 0 \le \tau \le t,$$

which implies V if $b > 0$.

If $b = 0$, then consider the function

$$f_1(x,\tau) = \exp(\tau)\Big[4e_n^{-2}\Big(s_n(\tau) - x + e_n\Big)^2 \pm f(x,\tau)\Big].$$

Obviously, $f_1 \geq 0$ in $\bar{R}_{2n}/R_{2n}$. Moreover, we have

$$\mathcal{L}_1 f_1 = C_n^k f_{1\tau} + A_n^k f_{1xx} + D_n^k f_{1x} = C_n^k f_1 +$$

$$e^\tau\Big[8C_n^k e_n^{-2} s_n'(\tau)\Big(s_n(\tau) - x + e_n\Big) + 8A_n^k e_n^{-2} - 8D_n^k e_n^{-2}(s_n(\tau) - x + e_n)\Big]$$

$$\geq C_n^k f_1 + e^\tau\Big[-8(C_0 M_0 + D_0)e_n^{-1} + 8a\varepsilon_n^{m\gamma-1} e_n^{-2}\Big] \geq C_n^k f_1 \text{ in } R_{2n}. \qquad (2.48)$$

from which it follows that f_1 attains its maximum in $\bar{R}_{2n}$ at some point of $\bar{R}_{2n}/R_{2n}$. Since f_1 also satisfies (2.47), the maximum is attained on the whole curve $x = s_n(\tau)$. As before, this implies V in the case of $b = 0$. If $b < 0$ then we first consider a function f_1 from (2.35) which satisfies (2.36a)-(2.36c). Then consider a function

$$f_2(x,\tau) = 4e_n^{-2}(s_n(\tau) - x + e_n)^2 \pm f_1(x,\tau).$$

Obviously, $f_2 \geq 0$ in $\bar{R}_{2n}/R_{2n}$. Moreover, we have

$$\mathcal{L}_2 f_2 = 8e_n^{-2} s_n'(\tau)(s_n(\tau) - x + e_n) + 8A_n^k e_n^{-2} - 4(B_n^k + B_0)e_n^{-2}(s_n(\tau) - x + e_n)^2$$

$$-8D_n^k e_n^{-2}(s_n(\tau) - x + e_n) \geq -8(M_0 + D_0)e_n^{-1} + 8\Delta e_n^{-2} - 4B_0 = 0 \text{ in } R_{2n}. \quad (2.49)$$

It may be easily checked that f_2 satisfies (2.47). As before, it follows that f_2 attains its maximum in $\bar{R}_{2n}$ on the whole curve

$$x = s_n(\tau), \quad 0 \leq \tau \leq t.$$

That is to say

$$f_{2x}(s_n(\tau),\tau) \le 0 \quad \text{for} \quad 0 \le \tau \le t$$

or

$$|f_x(s_n(\tau),\tau)| \le 8exp(B_0(t-\tau))e_n^{-1} \;\; \text{for} \quad 0 \le \tau \le t$$

which implies V in the case of $b < 0$.

The property VI follows from [90] (Theorem 9.1, §9, ch.IV).

Now consider (2.28) (resp. (2.30)) with $f = f(x,\tau)$, which is a solution of the problem (2.32a)-(2.32c). Then we have

$$\int_{s_n(t)}^{r_n} \Big(u_n(x,t) - g(x,t)\Big)\omega(x)\ dx = \int_{s_{0n}}^{r_n} \Big(u_{0n}(x) - u_0(x)\Big) f(x,0)\ dx \ +$$
$$a\int_0^t \Big(\psi_n^m(\tau) - g^m(s_n(\tau),\tau)\Big)\ f_x(s_n(\tau),\tau)d\tau -$$
$$a\int_0^t \Big(u_n^m(r_n,\tau) - g^m(r_n,\tau)\Big)\ f_x(r_n,\tau)\ d\tau \ +$$
$$\int_0^t\int_{s_n(\tau)}^{r_n} \Big((1-\sigma_b)(C_n - C_n^k)f_\tau + (A_n - A_n^k)f_{xx}$$
$$-(B_n - B_n^k)f + (D_n - D_n^k)f_x\Big)\left(u_n^{1/\gamma} - g^{1/\gamma}\right)\ dxd\tau \ +$$
$$b\theta_b\varepsilon_n^{\beta\gamma}\int_0^t\int_{s_n(\tau)}^{r_n} f\ dxd\tau = \sum_{i=1}^{5} J_i. \tag{2.50}$$

By using the properties I-VI we estimate the right-hand side of (2.50) as follows:

$$|J_1| \le \int_{s_{0n}}^{r_n-1} |u_{0n}(x) - u_0(x)|\ |f(x,0)|\ dx +$$
$$\int_{r_n-1}^{r_n} |u_{0n}(x) - u_0(x)|\ |f(x,0)|\ dx \le$$

$$K\varepsilon_n^\gamma \int_0^{+\infty} \exp\Big[c_n(q-x)+(1+\sigma_b B_0)t\Big]\ dx+$$

$$2M\exp\Big[c_n(q-r_n+1)+(1+\sigma_b B_0)t\Big] \leq$$

$$K\exp\Big[c_n q+(1+\sigma_b B_0)t\Big]\ \varepsilon_n^\gamma c_n^{-1}+$$

$$2M\exp\Big(c_n(q+1)+(1+\sigma_b B_0)t-n\Big)=o(1), \quad n\to+\infty,$$

$$|J_2| \leq a\int_0^t \Big(\Big|\psi_n^m(\tau)-\psi^m(\tau)\Big|+\Big|g^m(s_n(\tau),\tau)-g^m(s(\tau),\tau)\Big|\Big)\ \Big|f_x(s_n(\tau),\tau)\Big|\ d\tau$$

$$= O\left(\varepsilon_n^{m\gamma+s}\right) \quad \text{as} \ \ n\to+\infty,$$

$$|J_3| \leq a\int_0^t \Big|u_n^m(r_n,\tau)-g^m(r_n,\tau)\Big|\ \Big|f_x(r_n,\tau)\Big|\ d\tau = O\Big(\alpha_n\exp\left(-\varepsilon_n^{-1}\right)\Big), \quad \text{as} \ \ n\to+\infty.$$

In view of VI we have

$$\lim_{k\to+\infty}\ J_4=0.$$

If $b\leq 0$, then $J_5=0$, but if $b>0$ then we have

$$|J_5| \leq b\varepsilon_n^{\beta\gamma} \int_0^t\int_{s_n(\tau)}^{r_n} \exp\Big[c_n(q-x)+t-\tau\Big]\ dx\ d\tau$$

$$\leq b\Big[\exp(t)-1\Big]\exp\Big(c_n(q-\tilde{s})\Big)\varepsilon_n^{\beta\gamma}c_n^{-1}=o(1) \quad \text{as} \ \ n\to+\infty,$$

where $\tilde{s}$ is an arbitrary number such that $\tilde{s}\leq \min\limits_{0\leq\tau\leq t}\ s_n(\tau),\ n=1,2,\dots$. By using these estimates in (2.50) and passing to the limit first with respect to $k\to+\infty$ and then with respect to $n\to+\infty$ from (2.50), (2.25) follows. The theorem is proved. □

Theorem 2.2.2. Let $a > 0,\ b < 0,\ c \in \mathbb{R},\ m > 0,\ 0 < \beta < 1,\ p > 0$. Then if

$$u_0(x),\ \psi(t) \geq \delta > 0 \quad \text{for} \quad (x,t) \in [0;+\infty) \times [0;T], \tag{2.51}$$

then the CDP has a unique solution.

Proof. As in the proof of Theorem 2.2.1 we shall prove uniqueness by proving that for some limit solution $u = \lim u_n$ constructed in the proof of the Theorem 2.1.1, the inequalities (2.25) are valid. As before, we suppose that arbitrary finite $\bar{t} \in (0;T]$ is fixed. Let $t \in (0;\bar{t}]$ and $\omega \in C_0^\infty\Big((s(t);+\infty)\Big)$ be an arbitrary function such that $|\omega| \leq 1$ and

$$\text{supp}\ \omega = (q_1;q_2), \quad s(t) < q_1 < q_2 < +\infty.$$

Suppose that the function $\chi(x)$ and the sequences $\{r_n\},\ \{s_n\}$ in addition satisfy the following conditions:

$$\chi(x) = O(exp(-x^{-1}))\ \ x \to 0^+,$$

$$s_n'(\tau) \geq -M_0 \ \text{ for } \ 0 \leq \tau \leq \bar{t}, \qquad n = 1,2,\dots\ ,$$

$$s_n(\tau) > s_{n+1}(\tau) > s(\tau) \ \text{ for } \ 0 \leq \tau \leq \bar{t}, \qquad n = 1,2,\dots\ ,$$

$$\max_{0 \leq \tau \leq \bar{t}} \Big| g_i^m(s_n(\tau),\tau) - g_i^m(s(\tau),\tau) \Big| \leq \chi^m(\varepsilon_n), \qquad n = 1,2,\dots\ ; \quad i = 1,2,$$

$$r_n \geq \max\Big(c_n^{-1}(n - b_0\varepsilon_n^{\beta-1}\bar{t});\quad \alpha_n + d_n\bar{t} + \varepsilon_n^{-1} - b_0\varepsilon_n^{\beta-1}\bar{t}\Big),$$

where $b_0 < b$ is an arbitrary number, the sequences $\{c_n\},\ \{d_n\}$ are the same as in the proof of Theorem 2.2.1 (when $b < 0$) and

$$\alpha_n = \max\Big(\ln 2;\ (2a\varepsilon_n^{m-1})^{-1}\left[D_0 + (D_0^2 - 4ab_0\varepsilon_n^{m+\beta-2})^{\frac{1}{2}}\right]\Big) \qquad \text{if} \;\; m \geq 1,$$

$$\alpha_n = \max\Big(\ln 2; (2\Delta)^{-1}\left[|c|\varepsilon_n^{p-1} + (-4b_0\varepsilon^{\beta-1}\Delta + c^2\varepsilon_n^{2(p-1)})^{\frac{1}{2}}\right]\Big) \quad \text{if} \;\; 0 < m < 1,$$

As mentioned before, we can choose the function χ arbitrary in view of the condition (2.51). As in the proof of the previous theorem , we then derive (2.30), where $A_n, B_n, D_n, A_n^k, B_n^k, D_n^k$ are the same as before . They satisfy the following conditions.

$$\begin{array}{lcll}
a\ \varepsilon_n^{m-1} & \leq & A_n, A_n^k \leq \bar{A} & \text{in } \bar{R}_n \quad \text{if} \;\; m \geq 1, \\
0 < \Delta_0 & \leq & A_n, A_n^k \leq a\ \varepsilon_n^{m-1} & \text{in } \bar{R}_n, \quad \text{if} \;\; 0 < m < 1, \\
b\ \varepsilon_n^{\beta-1} & \leq & B_n,\ B_n^k \leq -\bar{B} & \text{in } \bar{R}_n. \\
|c|\ \varepsilon_n^{p-1} & \leq & |D_n|\,,\ \left|D_n^k\right| \leq \bar{D} & \text{in } \bar{R}_n, \;\; \text{if} \quad m \geq 1, \\
|c|\ \bar{D} & \leq & |D_n|\,,\ \left|D_n^k\right| \leq |c|\ \varepsilon_n^{p-1} & \text{in } \bar{R}_n \text{ if} \quad 0 < m < 1.
\end{array}$$

Then we consider the problem (2.32a)-(2.32c). There exists a unique classical solution of this problem, which has the following properties

I $\;|f| \leq \exp\left(-b_0\varepsilon_n^{\beta-1}(t-\tau)\right), \qquad (x,\tau) \in \bar{R}_n,$

II $\;|f| \leq \exp\left[c_n(q_2 - x) + (1 - b_0\varepsilon_n^{\beta-1})(t-\tau)\right], \qquad (x,\tau) \in \bar{R}_n,$

III $\;|f| \leq \exp\left[q_2 - x + (d_n - b_0\varepsilon_n^{\beta-1})(t-\tau)\right], \qquad (x,\tau) \in \bar{R}_n,$

IV $\;|f_x(r_n,\tau)| = O\left(\alpha_n \exp\left(-\varepsilon_n^{-1}\right)\right) \quad \text{as } n \to +\infty \quad \text{for } \tau \in [0;t],$

V $\;|f_x(s_n(\tau),\tau)| = O\left(e_n^{-1}\exp\left(-b_0\varepsilon^{\beta-1}\bar{t}\right)\right) \quad \text{as} \;\; n \to +\infty \quad \text{for} \;\; \tau \in [0;t],$

where

$$e_n = \bar{e}\varepsilon_n^{m-1}, \quad \bar{e} = \min\left(2\tilde{\delta}; a(2M_0)^{-1}; (-ab_0^{-1})^{\frac{1}{2}}\varepsilon_n^{\frac{1-\beta}{2}}\right), \quad \text{if} \;\; m \geq 1,$$

$$e_n = \bar{e} = \min\left(2\tilde{\delta}; -b_0^{-1}\varepsilon_n^{1-\beta}\Big(((M_0 + D_0)^2 - 2b_0\varepsilon_n^{\beta-1}\Delta)^{1/2} - (M_0 + D_0)\Big)\right),$$

if $0 < m < 1$,

VI $\|f\|_{W_q^{2,1}(R_n)} \leq M_*(n), \qquad q > 1,$

where the constant M_* does not depend on k.

The proof of properties I-VI coincides with the proof given in that of Theorem 2.2 for $b < 0$ (the only difference being that B_0 must be replaced by $-b_0\varepsilon_n^{\beta-1}$). Finally we consider (2.30) with $f = f(x, \tau)$ which is the solution of the problem (2.36a)-(2.36c). By using the properties I-IV the right-hand side of (2.30) may be estimated as in the proof of Theorem 2.2.1 The theorem is proved. □

2.3 Comparison Theorem

In this section we shall prove the comparison theorem for solution of the CDP.

Definition 2.3.1. *We shall say that the function $g(x, t)$ is a supersolution (respectively subsolution) of equation (1.3) in D if*

(a) *g is non-negative and continuous in $\bar{D}$ and $g \in L_\infty(D \cap (t \leq T_1))$ for any finite $T_1 \in (0; T]$,*

(b) *for any finite* t_0, t_1 *such that* $0 \le t_0 < t_1 \le T$ *and for any* C^∞ *functions* $\mu_i(t),\ t_0 \le t \le t_1,\ i = 1, 2$ *such that* $s(t) < \mu_1(t) < \mu_2(t)$ *for* $t \in [t_0; t_1]$ *(see the Definition 1.1), the integral inequality*

$$I(g, f, D_1) \le 0 \qquad (\ge 0) \tag{2.52}$$

holds where $f \in C^{2,1}_{x,t}(\bar{D}_1)$ *is an arbitrary non-negative function such that*

$$f(\mu_i(t), t) = 0 \ \text{ for } \ t_0 \le t \le t_1, \quad i = 1, 2.$$

The next lemma gives a sufficient condition for super- or subsolutions.

Lemma 2.3.1. *Let* g *be a non-negative and continuous function in* $\bar{D}$ *belonging to* $C^{2,1}_{x,t}$ *in* D *outside a finite number of curves* $x = \eta_i(t)$*, which divide* D *into a finite number of subdomains* D^j*, where* $\eta_i \in C[0; T]$*; for arbitrary* $\delta > 0$ *and finite* $\Delta \in (\delta; T]$ *the function* η_i *is absolutely continuous in* $[\delta; \Delta]$*. Let* g *satisfy the inequality*

$$Lg = g_t - a(g^m)_{xx} + bg^\beta + c(g^p)_x \ge 0 \quad (\le 0)$$

at the points of D*, where* $g \in C^{2,1}_{x,t}$*. Assume also that the function* $(g^m)_x$ *is continuous in* D *and*

$$g \in \mathrm{L}_\infty(D \cap (t \le T_1)) \ \text{ for any finite } \ T_1 \in (0; T].$$

Then g *is a supersolution (subsolution) of equation (1.3) in* D*.*

Proof. Let D_1 be given and take non-negative $f \in C^{2,1}_{x,t}(\bar{D})$ such that

$$f(\mu_i(t), t) = 0 \ \text{ for } \ t_0 \le t \le t_1, \quad i = 1, 2.$$

Let $\delta_n \equiv t_0,\ n = 1, 2, \ldots$ if $t_0 > 0$, whilst if $t_0 = 0$, then δ_n is a positive monotone sequence such that

$$\lim \delta_n = 0, \quad 0 < \delta_{n+1} < \delta_n < t_1, \quad n = 1, 2, \ldots$$

Integrating by parts the expression $(-Lg)f$ in all regions

$$D^j \cap \Big((x,t) : t \geq \delta_n \Big),$$

then summing and taking the limit as $n \to +\infty$ yields (2.52).

The lemma is proved. □

Theorem 2.3.1. (Comparison). *Let the conditions of Theorem 2.2.1 be satisfied. Let u be a solution of the CDP and g be a supersolution (respectively, subsolution) of equation (1.3) in D and*

$$u_0(x) \leq\ (\geq) \quad g(x,0) \qquad \text{for} \ \ s(0) \leq x < +\infty, \tag{2.53a}$$

$$\psi(t) \ \leq \ (\geq) \quad g(s(t),t) \qquad \text{for} \ \ 0 \leq t \leq T. \tag{2.53b}$$

Then

$$u(x,t) \leq \ (\geq)\ g(x,t) \quad \text{in} \ \ \bar{D}.$$

Proof. First, prove the theorem for supersolutions. The proof is similar to the proof of uniqueness. Suppose on the contrary that

$$g(x_*,t) < u(x_*,t) \ \ \text{for some} \ \ (x_*,t) \in D.$$

The continuity of g and u implies that

$$g(x,t) < u(x,t) \text{ for } x \in [x_* - \mu; x_* + \mu]$$

, where $\mu > 0$ is chosen such that $s(t) < x_* - \mu$. Then we take an arbitrary function $\omega \in C_0^\infty\Big((s(t); +\infty)\Big)$ such that

$$0 \le \omega \le 1 \quad \text{for } s(t) \le x < +\infty,$$

$$\omega > 0 \quad \text{for } |x - x_*| < \mu; \quad \omega = 0 \text{ for } |x - x_*| \ge \mu.$$

Our goal will be achieved if we prove the inequality

$$\int_{s(t)}^{+\infty} \Big(u(x,t) - g(x,t)\Big)\omega(x)\, dx \le 0, \tag{2.54}$$

which is a contradiction of our assumption. To prove (2.54), first we construct a sequence $\{u_n\}$ as in the Theorems 2.1.1 and 2.2.1 Since u is a unique solution of CDP we have $u = \lim u_n$. Since g is a supersolution of equation (1.3) in R_n and u_n is a solution of equation (2.1) in R_n, we have instead of (2.27)

$$I(u_n, f, R_n) + b\theta_b \varepsilon_n^{\beta\gamma} \int_0^t \int_{s_n(\tau)}^{r_n} f(x,\tau)\, dx d\tau - I(g, f, R_n) \ge 0. \tag{2.55}$$

or

$$\int_{s_n(t)}^{r_n} (u_n(x,t) - g(x,t)) f(x,t)\, dx \le \int_{s_{0n}}^{r_n} (u_{0n}(x) - g(x,0)) f(x,0)\, dx +$$
$$a\int_0^t \Big(\psi_n^m(\tau) - g^m(s_n(\tau),\tau)\Big) f_x(s_n(\tau),\tau)\, d\tau -$$
$$a\int_0^t \Big(u_n^m(r_n,\tau) - g^m(r_n,\tau)\Big) f_x(r_n,\tau)\, d\tau \;\; +$$

$$\int_0^t \int_{s_n(\tau)}^{r_n} \left(C_n^k f_\tau + A_n^k f_{xx} - B_n^k f + D_n^k f_x\right)\left(u_n^{1/\gamma} - g^{1/\gamma}\right)\, dx\, d\tau +$$

$$b\, \varepsilon_n^{\beta\gamma} \int_0^t \int_{s_n(\tau)}^{r_n} f\, dx\, d\tau \; + \int_0^t \int_{s_n(\tau)}^{r_n} \Big(\left(C_n - C_n^k\right) f_\tau + \left(A_n - A_n^k\right) f_{xx} - \tag{2.56}$$

$$\left(B_n - B_n^k\right) f + (D_n - D_n^k) f_x\Big)\left(u_n^{1/\gamma} - g^{1/\gamma}\right)\, dx\, d\tau,$$

Note that if $b < 0$ then we have (2.56) with $\gamma = 1, C_n = C_n^k \equiv 1$. Then instead of f in (2.56) we take the classical solution of the problem (2.32a)-(2.32c). Since ω is a non-negative function, from the maximum principle it follows that

$$f \geq 0, \quad \text{for} \;\; (x,\tau) \in \bar{R}_n$$

and hence

$$f_x(s_n(\tau),\tau) \geq 0 \quad \text{for} \;\; 0 \leq \tau \leq t.$$

By using this, from (2.56), (2.53a), (2.53b) it follows that

$$\int_{s_n(t)}^{r_n} \Big(u_n(x,t) - g(x,t)\Big)\omega(x)\, dx \leq \int_{s_{0n}}^{r_n} \Big(u_{0n}(x) - u_0(x)\Big) f(x,0)\, dx \; +$$

$$a \int_0^t \Big(\psi_n^m(\tau) - \psi^m(\tau) + g^m(s(\tau),\tau) - g^m(s_n(\tau),\tau)\Big)\, f_x(s_n(\tau),\tau) d\tau -$$

$$a \int_0^t \Big(u_n^m(r_n,\tau) - g^m(r_n,\tau)\Big)\, f_x(r_n,\tau)\, d\tau \quad + b\theta_b \varepsilon_n^{\beta\gamma} \int_0^t \int_{s_n(\tau)}^{r_n} f\, dx d\tau +$$

$$\int_0^t \int_{s_n(\tau)}^{r_n} \Big((1-\sigma_b)(C_n - C_n^k) f_\tau + (A_n - A_n^k) f_{xx} - (B_n - B_n^k) f +$$

$$+ (D_n - D_n^k) f_x\Big)\left(u_n^{1/\gamma} - g^{1/\gamma}\right)\, dx d\tau \tag{2.57}$$

The rest of the proof coincides with the proof given in the uniqueness Theorem 2.2.1. The proof for the subsolution is similar. The theorem is proved. □

Remark 2.2 *If the conditions of the Theorem 2.2.2 are satisfied, then the Comparison Theorem 2.3.1 is valid if we require g to be a classical smooth supersolution (respectively, subsolution) of equation (1.3) in D and satisfying (2.53a),(2.53b) (see Remark 2.1 and the proof of the Theorem 2.2.2). If in addition to the conditions of the Theorem 2.2.2, the boundary curve s satisfies also (1.13), then the assertion of Theorem 2.3.1 is valid. The proof is similar to that of the Theorem 2.3.1.*

Remark 2.3 *It should be noted that the definition of super- and subsolutions and the sufficient condition for super- and subsolutions in the case of DP coincide with the definition and Lemma 2.3.1 given in this section. The only difference is that the domain D should be replaced by the domain E respectively.*

Chapter 3

Dirichlet Problem

3.1 Existence

In this section we shall suppose that $a > 0,\ m > 0,\ b \in \mathbb{R}^1,\ c \in \mathbb{R}^1,\ p > 0$ and $\beta > 0$.

Theorem 3.1.1. *If ϕ_1 satisfies the assumption (L) and ϕ_2 satisfies the assumption (R) then there exists a solution of DP.*

Proof. Let the sequences $\{\varepsilon_n\},\ \{T_n\}$ be chosen as in the proof of Theorem 2.1.1. Suppose that $\{\phi_{in}\},\ i = 1, 2$ are arbitrary sequences of functions such that

$$\phi_{in} \in C^\infty[0; T_n], \quad \phi_{1n}(t) < \phi_{2n}(t) \ \text{ for } \ t \in [0; T_n]$$

$$\lim_{n\to+\infty} \max_{0\le t\le T_n} \Big|\phi_{in}(t) - \phi_i(t)\Big| = 0$$

For simplicity, suppose that $\phi_1(0) = 0,\ \phi_2(0) = H > 0$ and let

$$\phi_{1n}(0) = \phi_{1n}^0,\ \phi_{2n}(0) = \phi_{2n}^0, n = 1, 2, \ldots$$

Some additional restrictions on the sequence of numbers $\{\phi^0_{in}\},\ i=1,2$ will be formulated below. Let γ_b be defined as in the proof of Theorem 2.1 and as before we will write γ instead of γ_b. Without loss of generality we may suppose that $\varepsilon_1^\gamma < M$.

Now take functional sequences $\{u_{0n}\},\ \{\psi_{in}\},\ i=1,2$ and a sequence of numbers $\{\phi^0_{in}\}$,

$i=1,2$ such that

1. $\phi^0_{1n} \in [0;H/4],\quad \phi^0_{2n} \in [(3/4)H;H],\ n=1,2,\dots,\ \lim\limits_{n\to\infty}\phi^0_{1n}=0,\ \lim\limits_{n\to\infty}\phi^0_{2n}=H,$

2. $u_0(0)-\chi(\varepsilon_n)/2 \le u_0(\phi^0_{1n}) \le \Big(u_0^m(0)+(\chi(\varepsilon_n)/2)^m\Big)^{1/m},\ n=1,2,\dots,$

3. $u_0(H)-\chi(\varepsilon_n)/2 \le u_0(\phi^0_{2n}) \le \Big(u_0^m(H)+(\chi(\varepsilon_n)/2)^m\Big)^{1/m},\ n=1,2,\dots,$

4. $\varepsilon_n^\gamma \le u_{0n}(x),\quad \psi_{in}(t) \le M\quad \text{for}\quad (x,t)\in[0;H]\times[0;T_n],$

5. $u_{0n}\in C^\infty[0;H],\quad \psi_{in}\in C^\infty[0;T_n],\ i=1,2,\ n=1,2,\dots,$

6. $u_{0n}(\phi^0_{1n})=\psi_{1n}(0),\quad a\left(u_{0n}^m\right)''(\phi^0_{1n})+\phi'_{1n}(0)u'_{0n}(\phi^0_{1n})-c(u_{0n}^p)'(\phi^0_{1n})-bu_{0n}^\beta(\phi^0_{1n})+$ $b\theta_b\varepsilon_n^{\beta\gamma}=\psi'_{1n}(0),$

7. $u_{0n}(\phi^0_{2n})=\psi_{2n}(0),\quad a\left(u_{0n}^m\right)''(\phi^0_{2n})+\phi'_{2n}(0)u'_{0n}(\phi^0_{2n})-c(u_{0n}^p)'(\phi^0_{2n})-bu_{0n}^\beta(\phi^0_{2n})+$ $b\theta_b\varepsilon_n^{\beta\gamma}=\psi'_{2n}(0),$

8. $0\le u_{0n}(x)-u_0(x)\le\chi(\varepsilon_n)\quad \text{for}\quad 0\le x\le H,$

9. $0\le \psi_{in}^m(t)-\psi_i^m(t)\le\chi^m(\varepsilon_n)\quad \text{for}\quad 0\le t\le T_n,\ i=1,2,$

where the constant θ_b and the function χ are the same as in the proof of Theorem 2.1.1. Consider the auxiliary problem

$$u_t = a(u^m)_{xx} - c(u^p)_x - bu^\beta + b\theta_b\varepsilon_n^{\beta\gamma} \text{ in } E_n, \tag{3.1}$$

$$u(x,0) = u_{0n}(x), \quad \phi_{1n}^0 \le x \le \phi_{2n}^0, \tag{3.2}$$

$$u(\phi_{in}(t),t) = \psi_{in}(t), \quad 0 \le t \le T_n, \quad i = 1,2, \tag{3.3}$$

where

$$E_n = \Big\{(x,t) : \phi_{1n}(t) < x < \phi_{2n}(t), \quad 0 < t \le T_n\Big\}.$$

If we introduce new variables

$$H\Big(x - \phi_{1n}(t)\Big)\Big(\phi_{2n}(t) - \phi_{1n}(t)\Big)^{-1} \to y, \quad t \to t,$$

then (3.1)–(3.3) will be transformed to the problem

$$v_t = aH^2\Big(\phi_{2n}(t) - \phi_{1n}(t)\Big)^{-2}(v^m)_{yy} + \Big[H\phi_{1n}'(t) + \Big(\phi_{2n}'(t) - \phi_{1n}'(t)\Big)y\Big] \times$$

$$\Big(\phi_{2n}(t) - \phi_{1n}(t)\Big)^{-1}v_y - cH(\phi_{2n}(t) - \phi_{1n}(t))^{-1}(v^p)_y - b(v^\beta - \theta_b\varepsilon_n^{\beta\gamma}) \text{ in } E_n' \tag{3.4}$$

$$v(y,0) = u_{0n}\Big(\phi_{1n}^0 + H^{-1}\left(\phi_{2n}^0 - \phi_{1n}^0\right)y\Big) \text{ for } 0 \le y \le H, \tag{3.5}$$

$$v(0,t) = \psi_{1n}(t), \quad v(H,T) = \psi_{2n}(t) \text{ for } 0 \le t \le T_n, \tag{3.6}$$

where

$$E_n' = \Big\{(y,t) : 0 < y < H, \ \ 0 < t \le T_n\Big\}.$$

From [90] (Theorem 6.1, §6, ch.5) it follows that there exists a unique classical solution $v = v_n(y,t)$ of the problem (3.4)–(3.6) such that $v_n \in C_{x,t}^{2+\mu,1+\mu/2}(\bar{E}_n')$ for some $\mu > 0$. Maximum principle implies (2.7) in E'_n. Therefore, the function

$$u_n(x,t) = v_n\Big(H\left(x - \phi_{1n}(t)\right)\ \left(\phi_{2n}(t) - \phi_{1n}(t)\right)^{-1}, t\Big)$$

is the classical solution from $C^{2+\mu,1+\mu/2}_{x,t}(\bar{E}_n)$ of the problem (3.1)–(3.3) and (2.8) is valid in $\bar{E}_n$.

The sequence $\{u_n\}$ is uniformly bounded and equicontinuous on every compact subset of E. The proof completely coincides with the proof given in that of Theorem 2.1.1. As before, by a diagonalization argument and the Arzela-Ascoli theorem we may find a sub-sequence n' and a limit function $\tilde{u} \in C(E)$ such that $u_{n'} \to \tilde{u}$ as $n' \to +\infty$, pointwise in E, and the convergence is uniform on compact subsets of E. Obviously, $\tilde{u} \in L_\infty(E)$ if $b \geq 0$ or $b < 0$ and $\beta > 1$ and $\tilde{u} \in L_\infty(E \bigcap (t \leq T_1))$ for any finite $T_1 > 0$, if $b < 0$ and $0 < \beta \leq 1$.

Now consider a function $u(x,t)$ such that $u = \tilde{u}$ for $(x,t) \in E,\ u(x,0) = u_0(x)$ for $0 \leq x \leq H$ and $u(\phi_i(t),t) = \psi_i(t)$ for $0 \leq t \leq T,\ i = 1,2$. The function $u(x,t)$ satisfies the integral identity (1.11) in the sense of Definition 1.2. The continuity of u at points $(x_0,0),\ 0 < x_0 < H$, of the line $t = 0$, may be established as is mentioned in the proof of Theorem 2.1.1.

It remains only to prove the continuity of $u(x,t)$ at the points $(\phi_i(t),t),\ t \geq 0$. For that, first consider a function

$$v(y,t) = u\Big(H^{-1}(\phi_2(t) - \phi_1(t))y + \phi_1(t), t\Big), \quad (y,t) \in \bar{E}',$$

where

$$E' = \Big\{(y,t) : 0 < y < H, \quad 0 < t \leq T\Big\}.$$

Obviously

$$v \in C(E') \cap \mathrm{L}_\infty(E') \;\text{ if }\; b \geq 0 \;\text{ or }\; b < 0 \;\text{ and }\; \beta > 1$$

and

$$v \in C(E') \cap \mathrm{L}_\infty\Big(E' \cap (t \leq T_1)\Big) \;\text{ if }\; b < 0 \;\text{ and }\; 0 < \beta \leq 1,$$

and T_1 is an arbitrary finite number from $(0; T]$. The sequence $\{v_{n'}\}$ converges to v as $n' \to +\infty$ pointwise in $\bar{E}'$ and convergence is uniform on compact subsets of E'. Continuity of the function $u(x,t)$ at the points $(\phi_i(t), t)\ t \geq 0,\ i = 1, 2$ is equivalent to continuity of the function $v(y,t)$ at the points $(0,t),\ (H,t),\ t \geq 0$. First, prove the continuity at the points $(H,t),\ t \geq 0$.

If $t_0 \geq 0$ and $\psi_2(t_0) > 0$, it is enough to show that for arbitrary sufficiently small $\varepsilon > 0$ the following two inequalities are valid

$$\liminf\ v(y,t) \;\geq\; \psi_2(t_0) - \varepsilon \quad \text{as} \;\; (y,t) \to (H, t_0), \tag{3.7}$$

$$\limsup v(y,t) \;\leq\; \psi_2(t_0) + \varepsilon \quad \text{as} \;\; (y,t) \to (H, t_0). \tag{3.8}$$

Because $\varepsilon > 0$ is arbitrary, from (3.7) and (3.8), the continuity of v at the boundary points (H, t_0) follows. If $\psi_2(t_0) = 0$, however, then it is sufficient to prove (3.8), since (3.7) (with $\varepsilon = 0$ in the right-hand side) directly follows from the fact that v is non-negative in $\bar{E}'$.

Let $\psi_2(t_0) > 0$. Take an arbitrary $\varepsilon \in (0; \psi_2(t_0))$ and prove the inequality (3.7). The proof is similar to that of (2.13). Consider a function

$$\omega_n(y,t) = f\Big(h(\mu)+H^{-1}\Big(\phi_{2n}(t)-\phi_{1n}(t)\Big)y+\mu(t-t_0)-\phi_{2n}(t_0)+\phi_{1n}(t)\Big),\ \ \mu > 0,\ h > 0,$$

where

$$f(\xi) = M_1\Big(\xi\Big/h(\mu)\Big)^{\alpha}, \quad M_1 = \psi_2(t_0) - \varepsilon,$$

and by choosing the value of α appropriately we divide the analysis into different cases, as in the proof of Theorem 2.1.1 (see also Figure 1 if $b > 0$). We then choose $h, M_i,\ i = \overline{1,3}$, as in the proof of (2.13) (replacing ψ by ψ_2), and similar analysis leads to the following estimation

$$\bar{\omega}_n(y,t) \leq v_n(y,t) \quad \text{in} \quad \bar{E}'_n$$

where

$$\begin{aligned}
\bar{\omega}_n &= \Big\{\omega_n \ \text{ in } \ \bar{\Omega}_n;\ \varepsilon_n^{\gamma} \ \text{ in } \ \bar{E}'_n \backslash \bar{\Omega}_n\Big\}, \\
\Omega_n &= \Big\{(y,t):\ t_0 - d_{t_0}(\mu) < t \leq t_0 + \delta_1, \ \ \zeta_n(t) < y < H\Big\}, \\
\zeta_n(t) &= H\Big(\phi_{2n}(t) - \phi_{1n}(t)\Big)^{-1}\Big[-h(\mu) - \mu(t-t_0) + \phi_{2n}(t_0) - \phi_{1n}(t) + \eta_n\Big]
\end{aligned}$$

and $d_{t_0}(\mu),\ \eta_n$ are defined as before. Since ϕ_2 satisfies (1.14), for arbitrary μ fixed and large enough, there exists $N = N(\mu)$ such that

$$\zeta_n(t_0 - \mu^{-2}) > H \quad \text{for} \quad n \geq N.$$

In the final limit as $n' \to +\infty$, we have

$$\omega(y,t) \leq v(y,t) \ \text{ in } \ \bar{E}, \tag{3.9}$$

where

$$\omega(y,t) = \begin{cases} f\Big(h(\mu) + H^{-1}\Big(\phi_2(t) - \phi_1(t)\Big)y + \mu(t-t_0) - \phi_2(t_0) + \phi_1(t)\Big), & (y,t) \in \bar{\Omega} \\ 0, & (y,t) \in \bar{E}'\backslash\bar{\Omega} \end{cases}$$

and

$$\begin{aligned} \Omega &= \Big\{(y,t) : t_0 - d_{t_0}(\mu) < t \le t_0 + \delta_3, \quad \zeta(t) < y < H\Big\}, \\ \zeta(t) &= H\Big(\phi_2(t) - \phi_1(t)\Big)^{-1}\Big[-h(\mu) - \mu(t-t_0) + \phi_2(t_0) - \phi_1(t)\Big]. \end{aligned}$$

Obviously, we have

$$\lim_{\substack{(y,t)\to(H,t_0)\\(y,t)\in\bar{E}'}} \omega(y,t) = \lim_{\substack{(y,t)\to(H,t_0)\\(y,t)\in\bar{\Omega}}} \omega(y,t) = \psi_2(t_0) - \varepsilon.$$

Hence, from (3.9), (3.7) follows. The proof of (3.8) is similar to the given proof of (3.7) and to that of (2.14), therefore we omit it. Thus we have proved the continuity of the limit solution at the boundary points $(\phi_2(t), t),\ t \ge 0$. The proof of continuity of the limit solution at the points $(\phi_1(t), t),\ t \ge 0$ is similar to the given proof and to the proof of continuity of the limit solution to CDP on the boundary curve $(s(t), t),\ t \ge 0$. The theorem is proved.

□

3.2 Uniqueness and Comparison Results

In this section we shall suppose that the boundary curve ϕ_1 (respectively ϕ_2) satisfies the assumption (L) (respectively assumption (R)).

Theorem 3.2.1. *Let* $a > 0,\ m > 0,\ c \in \mathbb{R}^1$ *and either* $b \geq 0,\ \beta > 0$ *or* $b < 0,\ \beta \geq 1$. *Suppose that for each compact subsegment* $[\delta; T_1] \subset (0; T]$ *there exists a positive constant* M_0 *such that the conditions (1.15a),(1.15b) are satisfied. Then the solution of the DP is unique.*

Proof. The proof is similar to the proof given in the case of CDP (Theorem 2.2.1). Suppose that g_1 and g_2 are two solutions of DP. Let $\bar{t} \in (0; T]$ be an arbitrary finite number. As before, uniqueness will be proved by confirming that for some limit solution $u = \lim u_n$ the following inequalities are valid

$$\int_{\phi_1(t)}^{\phi_2(t)} \Big(u(x,t) - g_i(x,t)\Big)\omega(x)\ dx \leq 0, \quad i = 1, 2, \tag{3.10}$$

for every $t \in (0; \bar{t}\,]$ and for an arbitrary function

$$\omega \in C_0^\infty\Big(\big(\phi_1(t); \phi_2(t)\big)\Big) \ \text{ for } \ |\omega| \leq 1.$$

Let (2.25) be valid with $\phi_1(t) < q_1 < q_2 < \phi_2(t)$.

Suppose that $\chi(x) = Kx^\gamma$ for $x \geq 0$ (see the proof of Theorem 3.1.1). Take an arbitrary sequence of real numbers $\{\delta_\ell\}$ such that

$$0 < \delta_{\ell+1} < \delta_\ell < \bar{t}, \quad \ell = 1, 2, \dots\ ; \quad \delta_\ell \to 0+ \quad \text{as} \quad \ell \to +\infty.$$

Suppose also that the sequences $\{\phi_{jn}\}$, $j = 1, 2$, in addition to conditions from the proof of Theorem 3.1.1, satisfy the following properties:

$$\phi'_{1n}(\tau) \geq -M_0(\ell), \quad \phi'_{2n}(\tau) \leq M_0(\ell), \quad \text{for } \delta_\ell \leq \tau \leq \bar{t}, \quad n = 1, 2, \dots ,$$

(the possibility of which follows from (1.15a),(1.15b))

$$\phi_1(\tau) < \phi_{1,n+1}(\tau) < \phi_{1n}(\tau) < \phi_{2n}(\tau) < \phi_{2,n+1}(\tau) < \phi_2(\tau)$$

$$\text{for } 0 \leq \tau \leq \bar{t}, \quad n = 1, 2, \dots$$

$$\max_{0 \leq \tau \leq \bar{t}} |g_i^m(\phi_{jn}(\tau), \tau) - g_i^m(\phi_j(\tau), \tau)| \leq \chi^m(\varepsilon_n), \quad n = 1, 2, \dots; \quad j = 1, 2.$$

Without loss of generality we may assume that

$$\phi_{1n}(t) < q_1 < q_2 < \phi_{2n}(t), \ n = 1, 2, \dots ; \delta_\ell < t, \ \ell = 1, 2, \dots .$$

Since the proof of (3.10) is similar for each i, we shall henceforth write $g = g_i$. Let

$$E_{1n}^\ell = \Big\{(x, \tau) : \phi_{1n}(\tau) < x < \phi_{2n}(\tau), \quad \delta_\ell \leq \tau < t\Big\}.$$

Take any function $f \in C_{x,t}^{2,1}\left(\bar{E}_{1n}^\ell\right)$ such that $f = 0$ for $x = \phi_{in}(\tau)$, $\delta_\ell \leq \tau \leq t$, $i = 1, 2$. Let $u = \lim u_n$ be the limit solution of DP constructed in the proof of Theorem 3.1.1. We have

$$I\left(u_n, f, E_{1n}^\ell\right) + b\theta_b \varepsilon_n^{\beta\gamma} \int_{\delta_\ell}^{t} \int_{\phi_{1n}(\tau)}^{\phi_{2n}(\tau)} f(x, \tau)\, dx\, d\tau - I\left(g, f, E_{1n}^\ell\right) = 0, \tag{3.11}$$

If $b \geq 0$ we transform (3.11) as follows

$$\int_{\phi_{1n}(t)}^{\phi_{2n}(t)} \Big(u_n(x,t) - g(x,t)\Big) f(x,t)\ dx = \int_{\phi_{1n}(\delta_\ell)}^{\phi_{2n}(\delta_\ell)} \Big(u_n(x,\delta_\ell) - g(x,\delta_\ell)\Big) f(x,\delta_\ell)\ dx\ +$$

$$a\int_{\delta_\ell}^{t} \Big(\psi_{1n}^m(\tau) - g^m\Big(\phi_{1n}(\tau),\tau\Big)\Big) f_x\Big(\phi_{1n}(\tau),\tau\Big)\ d\tau\ -$$

$$a\int_{\delta_\ell}^{t} \Big(\psi_{2n}^m(\tau) - g^m\Big(\phi_{2n}(\tau),\tau\Big)\Big) f_x(\phi_{2n}(\tau),\tau)\ d\tau\ +$$

$$\int_{\delta_\ell}^{t} \int_{\phi_{1n}(\tau)}^{\phi_{2n}(\tau)} \Big(C_n^k f_\tau + A_n^k f_{xx} - B_n^k f + D_n^k f_x\Big)\Big(u_n^{1/\gamma} - g^{1/\gamma}\Big)\ dx\ d\tau\ +$$

$$\int_{\delta_\ell}^{t} \int_{\phi_{1n}(\tau)}^{\phi_{2n}(\tau)} \Big(\left(C_n - C_n^k\right) f_\tau + \left(A_n - A_n^k\right) f_{xx} - \left(B_n - B_n^k\right) f + \left(D_n - D_n^k\right) f_x\Big) \times$$

$$\Big(u_n^{1/\gamma} - g^{1/\gamma}\Big)\ dx\ d\tau + b\varepsilon_n^{\beta\gamma} \int_{\delta_\ell}^{t} \int_{\phi_{1n}(\tau)}^{\phi_{2n}(\tau)} f\ dx\ d\tau \qquad (3.12a)$$

where $A_n,\ B_n,\ C_n,\ D_n$ are the same as in the proof of Theorem 2.2.1. As before, assume that $A_n^k,\ B_n^k,\ C_n^k,\ D_n^k, k = 1, 2, \ldots$ are C^∞ approximations of $A_n,\ B_n,\ C_n,\ D_n$, respectively, in $\bar{E}_{1n}^\ell$ and that they satisfy (2.28). If $b < 0$ then we transform (3.11) as follows

$$\int_{\phi_{1n}(t)}^{\phi_{2n}(t)} \Big(u_n(x,t) - g(x,t)\Big) f(x,t)\ dx = \int_{\phi_{1n}(\delta_\ell)}^{\phi_{2n}(\delta_\ell)} \Big(u_n(x,\delta_\ell) - g(x,\delta_\ell)\Big) f(x,\delta_\ell)\ dx\ +$$

$$a\int_{\delta_\ell}^{t} \Big(\psi_{1n}^m(\tau) - g^m\left(\phi_{1n}(\tau),\tau\right)\Big) f_x\left(\phi_{1n}(\tau),\tau\right)\ d\tau\ -$$

$$a\int_{\delta_\ell}^{t} \Big(\psi_{2n}^m(\tau) - g^m\left(\phi_{2n}(\tau),\tau\right)\Big) f_x\left(\phi_{2n}(\tau),\tau\right)\ d\tau\ +$$

$$\int_{\delta_\ell}^{t} \int_{\phi_{1n}(\tau)}^{\phi_{2n}(\tau)} \Big(f_\tau + A_n^k f_{xx} - B_n^k f + D_n^k f_x\Big)(u_n - g)\ dx\ d\tau\ +$$

$$\int_{\delta_\ell}^{t} \int_{\phi_{1n}(\tau)}^{\phi_{2n}(\tau)} \Big(\left(A_n - A_n^k\right) f_{xx} - \left(B_n - B_n^k\right) f + \left(D_n - D_n^k\right) f_x \Big) (u_n - g)\, dx\, d\tau, \quad (3.12b)$$

where $A_n,\ B_n,\ A_n^k,\ B_n^k$ are the same as before. Since u_n satisfies (2.8), the estimations (2.30) are valid in this case as well (note that $\gamma = 1$ if $b < 0$). Then consider a problem

$$\mathcal{L}_1 f \qquad = E_n^k f_\tau + A_n^k f_{xx} - B_n^k f + D_n^k f_x = 0 \quad \text{in} \ \ E_{1n}^\ell, \qquad (3.13a)$$

$$f(x,t) \qquad = \omega(x), \quad \phi_{1n}(t) \le x \le \phi_{2n}(t), \qquad (3.13b)$$

$$f(\phi_{1n}(\tau),\tau) \quad = f(\phi_{2n}(\tau),\tau) = 0 \quad \delta_\ell \le \tau \le t. \qquad (3.13c)$$

where E_n^k is defined as in (2.31). There exists a unique classical solution to problem (3.13a)-3.13c) [57]. The solution $f = f(x,\tau)$ has the following properties:

I $|f| \le \exp\Big(\sigma_b B_0 (t-\tau)\Big), \quad (x,\tau) \in \bar{E}_{1n}^\ell,$

II $|f_x(\phi_{in}(\tau),\tau)| = O\left(\varepsilon_n^s\right)$ as $n \to +\infty$ for $\tau \in [\delta_\ell; t]\,, \quad i = 1,2,$

where $s = \Big(1 - m\gamma$ if $b \ge 0; 1 - m$ if $b < 0$ and $m \ge 1; 0,$ if $b < 0$ and $0 < m < 1\Big),$

III $\|f\|_{W_q^{2,1}\left(E_{1n}^\ell\right)} \le M_*(n), \quad q > 1,$

where the constant $M_*(n)$ does not depend on k and ℓ.

The proof is similar to that given in the case of problem (2.31).

Let us consider (3.12a) (or (3.12b)) with $f = f(x,\tau)$, which is a solution of the problem (3.13a)-(3.13c).

Then we have

$$\int\limits_{\phi_{1n}(t)}^{\phi_{2n}(t)} \Big(u_n(x,t) - g(x,t)\Big)\omega(x)\ dx =$$
$$\int\limits_{\phi_{1n}(\delta_\ell)}^{\phi_{2n}(\delta_\ell)} \Big(u_n(x,\delta_\ell) - g(x,\delta_\ell)\Big) f(x,\delta_\ell)\ dx \ +$$
$$a\int_{\delta_\ell}^{t} \Big(\psi_{1n}^m(\tau) - g^m\left(\phi_{1n}(\tau),\tau\right)\Big) f_x\left(\phi_{1n}(\tau),\tau\right)\ d\tau \ -$$
$$a\int_{\delta_\ell}^{t} \Big(\psi_{2n}^m(\tau) - g^m\left(\phi_{2n}(\tau),\tau\right)\Big) f_x\left(\phi_{2n}(\tau),\tau\right)\ d\tau \ +$$
$$b\theta_b\varepsilon_n^{\beta\gamma}\int_{\delta_\ell}^{t}\int\limits_{\phi_{1n}(\tau)}^{\phi_{2n}(\tau)} f\ dx\ d\tau + \int\limits_{\delta_\ell}^{t}\int\limits_{\phi_{1n}(\tau)}^{\phi_{2n}(\tau)} \Big[(1-\sigma_b)\left(C_n - C_n^k\right) f_\tau + \left(A_n - A_n^k\right) f_{xx} -$$
$$\left(B_n - B_n^k\right) f + \left(D_n - D_n^k\right) f_x\Big] \left(u_n^{1/\gamma} - g^{1/\gamma}\right)\ dx\ d\tau = \sum_{i=1}^{5} J_i. \tag{3.14}$$

To estimate the right-hand side of (3.14) we can now use properties I-III. Obviously from I and III it follows that

$$\lim_{n\to+\infty}\ J_4 = 0, \qquad \lim_{k\to+\infty}\ J_5 = 0.$$

In view of the property II we have

$$|J_{i+1}| \le a\int_{\delta_\ell}^{t}\Big(|\psi_{in}^m(\tau) - \psi_i^m(\tau)| + |g^m\left(\phi_{in}(\tau),\tau\right) - g^m\left(\phi_i(\tau),\tau\right)|\Big)\times$$
$$|f_x\left(\phi_{in}(\tau),\tau\right)|\ d\tau = O\left(\varepsilon_n^{m\gamma+s}\right) \quad \text{as} \ \ n\to+\infty, \ \ i=1,2.$$

To estimate J_1, first introduce a function

$$\omega(y,t) = \begin{cases} u_n(x,\delta_\ell), & if\ \phi_{1n}(\delta_\ell) \le x \le \phi_{2n}(\delta_\ell) \\ \psi_{1n}(\delta_\ell), & if\ x < \phi_{1n}(\delta_\ell), \\ \psi_{2n}(\delta_\ell), & if\ x > \phi_{2n}(\delta_\ell). \end{cases}$$

Obviously $u_n^\ell,\ x \in \mathbb{R}^1$ is bounded uniformly with respect to n, ℓ. By using property I we have

$$|J_1| \le \exp(\sigma_b B_0 t) J_1^1, \qquad J_1^1 = \int_{\phi_1(\delta_\ell)}^{\phi_2(\delta_\ell)} \Big| u_n^\ell(x) - g(x,\delta_\ell) \Big|\ dx.$$

From Lebesgue's theorem it follows that

$$\lim_{n\to+\infty} J_1^1 = J_1^2, \qquad J_1^2 = \int_{\phi_1(\delta_\ell)}^{\phi_2(\delta_\ell)} \Big| u(x,\delta_\ell) - g(x,\delta_\ell) \Big|\ dx.$$

By using these estimations in (3.14) and passing to the limit first with respect to $k \to +\infty$ then with respect to $n \to +\infty$, from (3.14) it follows that

$$\int_{\phi_1(t)}^{\phi_2(t)} \Big(u(x,t) - g(x,t) \Big) \omega(x) dx \le \exp(\sigma_b B_0 t) J_1^2. \tag{3.15}$$

Let

$$u_\ell(x) = \begin{cases} u(x,\delta_\ell) - g(x,\delta_\ell), & if\ \phi_1(\delta_\ell) \le x \le \phi_2(\delta_\ell), \\ 0, & if\ x \notin \Big[\phi_1(\delta_\ell); \phi_2(\delta_\ell)\Big]. \end{cases}$$

Obviously, $u_\ell,\ x \in \mathbb{R}^1$ is bounded uniformly with respect to ℓ. Hence, we have (noting that $\phi_1(0) = 0,\ \phi_2(0) = H$)

$$J_1^2 \le C\Big(\Big|\phi_1(\delta_\ell)\Big| + \Big|H - \phi_2(\delta_\ell)\Big| \Big) + J_1^3, \qquad J_1^3 = \int_0^H |u_\ell|\ dx,$$

where the constant C does not depend on ℓ. From Lebesgue's theorem it follows that

$\lim_{\ell\to\infty} J_1^3 = 0$. By using these estimations in (3.15) and passing to the limit $\ell \to +\infty$, from (3.15), (3.10) follows. The theorem is proved. □

From the Theorems 3.1.1 and 3.2.1 the following corollary easily follows.

Corollary 3.1 *Let $a > 0,\ m > 0,\ c \in \mathbb{R}^1$ and either $b \geq 0,\ \beta > 0$ or $b < 0,\ \beta \geq 1$. The solution of the DP is unique if there exists a finite number of points $t_i,\ i = 1, \dots, k$ such that $t_0 = 0 < t_1 < \cdots < t_k < t_{k+1} = T$ and for the arbitrary compact subsegment $[\delta_1; \delta_2] \subset (t_i; t_{i+1}),\ \ i = 0, 1, \dots, k$ there exists a positive constant M_0 such that (1.15a)–(1.15b) is satisfied for $0 < \delta_1 \leq \tau \leq t \leq \delta_2$. If $T = +\infty$ the uniqueness is still the case even if $k = +\infty$ and $t_i \uparrow +\infty$ as $i \to +\infty$.*

Theorem 3.2.2. *Let $a > 0,\ m > 0,\ b < 0,\ 0 < \beta < 1,\ c \in \mathbb{R}^1,\ p > 0$. Then, if*

$$u_0(x),\ \psi_1(t),\ \psi_2(t) \geq \bar{\delta} > 0, \quad for\ (x,t) \in [0; H] \times [0; T], \tag{3.16}$$

the DP has a unique solution.

Proof. The proof is similar to that of the previous theorem by using the same arguments as in the proof of Theorem 2.2.2. As before, uniqueness will be proved by showing that, for the two arbitrary solutions g_1 and g_2 of the DP, there exists a limit solution $u = \lim u_n$ of the DP such that the inequalities (3.10) are valid. Assume that an arbitrary finite number $\bar{t} \in (0; T]$ is fixed.

Let

$$\omega \in C_0^\infty\Big((\phi_1(t);\phi_2(t))\Big), \qquad t \in (0;\bar{t}\,]$$

be an arbitrary function such that $|\omega| \le 1$ and (2.26) is valid.

Assume that $\chi(x),\ x > 0$ is an arbitrary positive monotone function such that

$$\chi(x) = O(exp(-x^{-1})) \ \ x \to 0^+. \tag{3.17}$$

Assume also that the sequences $\{\phi_{in}\},\ i = 1,2$ satisfy the same conditions as in the proof of Theorem 2.2.2 and 3.2.1. The only difference is that we now choose $\phi_\ell \equiv 0$ and the constant M_0 relates to whole segment $[0;\bar{t}]$. As before, we shall denote $g = g_i$.

Let

$$E_{1n} = \Big\{(x,\tau) : \phi_{1n}(\tau) < x < \phi_{2n}(\tau), \quad 0 \le \tau < t\Big\} \tag{3.18}$$

Take any function $f \in C^{2,1}_{x,t}\left(\bar{E}^\ell_{1n}\right)$ such that $f = 0$ for $x = \phi_{in}(\tau),\ 0 \le \tau \le t,\ i = 1,2$. Let $u = \lim u_n$ be the limit solution of DP constructed in the proof of Theorem 3.2.1. As mentioned in the proof of Theorem 2.1.1 the function χ may be chosen arbitrary in view of condition (3.16).

We have

$$I\left(u_n, f, E_{1n}\right) - I\left(g, f, E_{1n}\right) = 0.$$

As in the proof of the previous theorem , we then derive (3.12b) (with $\delta_\ell = 0$), where $A_n, B_n, D_n, A^k_n, B^k_n, D^k_n$ are the same as before. They satisfy the conditions (2.29).

Then we consider a problem (3.13a)-(3.13c) in E_{1n} (with $\delta_\ell = 0$). There exists a unique classical solution of this problem which has the following properties:

I $\quad |f| \leq \exp\left(-b_0 \varepsilon_n^{\beta-1}(t-\tau)\right), \qquad (x,\tau) \in \bar{E}_{1n},$

where $b_0 < b$ is an arbitrary number.

II $\quad |f_x(s_n(\tau),\tau)| = O\left(e_n^{-1}\exp\left(-b_0\varepsilon_n^{\beta-1}\right)\right) \quad \text{as} \quad n \to +\infty \quad \text{for} \quad \tau \in [0;t],$

where

$$e_n = \bar{e}\varepsilon_n^{m-1}, \quad \bar{e} = \min\left(2\tilde{\delta}; a(2M_0)^{-1}; (-ab_0^{-1})^{\frac{1}{2}}\varepsilon_n^{\frac{1-\beta}{2}}\right), \quad \text{if} \quad m \geq 1$$

$$e_n = \bar{e} = \min\left(2\tilde{\delta}; -b_0^{-1}\varepsilon_n^{1-\beta}\left(((M_0+D_0)^2 - 2b_0\varepsilon_n^{\beta-1}\Delta)^{1/2} - (M_0+D_0)\right)\right),$$

if $\quad 0 < m < 1$

III $\quad \|f\|_{W_q^{2,1}(R_n)} \leq M_*(n), \qquad q > 1,$

where the constant M_* does not depend on k.

By using the same arguments as in the proof of Theorem 2.2.2, the properties I-III may be proved as in the proof of Theorem 2.2.1 ,3.2.1. Then we consider (3.12b) with $f = f(x,\tau)$, which leads to

$$\int_{\phi_{1n}(t)}^{\phi_{2n}(t)} \Big(u_n(x,t) - g(x,t)\Big)\omega(x)\ dx = \int_{\phi_{1n}}^{\phi_{2n}} \Big(u_{0n}(x) - u_0(x)\Big) f(x,0)\ dx \ +$$
$$a\int_0^t \Big(\psi_{1n}^m(\tau) - g^m\left(\phi_{1n}(\tau),\tau\right)\Big) f_x\left(\phi_{1n}(\tau),\tau\right)\ d\tau \ -$$
$$a\int_0^t \Big(\psi_{2n}^m(\tau) - g^m\left(\phi_{2n}(\tau),\tau\right)\Big) f_x\left(\phi_{2n}(\tau),\tau\right)\ d\tau \ +$$
$$\int_0^t \int_{\phi_{1n}(\tau)}^{\phi_{2n}(\tau)} \Big[\left(A_n - A_n^k\right) f_{xx} - \left(B_n - B_n^k\right) f + \left(D_n - D_n^k\right) f_x\Big] (u_n - g)\ dx\ d\tau$$

$$= \sum_{i=1}^{4} J_i. \tag{3.19}$$

By using properties I-III the right hand side of (3.19) may be estimated as follows:

$$|J_1| \le H\chi(\varepsilon_n) \exp\left(-b_0 \varepsilon_n^{\beta-1} t\right) = o(1) \quad \text{as} \;\; n \to +\infty, \quad i = 1, 2.$$

$$|J_{i+1}| \le a \int_0^t \Big(|\psi_{in}^m(\tau) - \psi_i^m(\tau)| + |g^m\left(\phi_{in}(\tau), \tau\right) - g^m\left(\phi_i(\tau), \tau\right)| \Big) \left| f_x\left(\phi_{in}(\tau), \tau\right)\right| \, d\tau$$

$$\le 2a\chi^m(\varepsilon_n) \int_0^t \left| f_x\left(\phi_{in}(\tau), \tau\right)\right| \, d\tau = o\left(1\right) \quad \text{as} \;\; n \to +\infty, \quad i = 1, 2.$$

$\lim\limits_{n\to+\infty} J_4 = 0$. By using these estimations in (3.19), (3.10) follows. The theorem is proved. □

Finally, we present the following comparison result (see Remark 2.3 in Section 2.3).

Theorem 3.2.3. *Let the conditions of Theorem 3.2.1 (see also Corollary 3.1) be satisfied. Let u be a solution of DP and g be a supersolution (respectively subsolution) of equation (1.3) in E and $u \le (\ge) g$ in $\bar{E} \backslash E$. Then $u \le (\ge) g$ in $\bar{E}$.*

As in the case of Theorem 3.2.3, the proof is similar to that of the uniqueness Theorem 3.2.1.

Remark 3.1 *If the conditions of the Theorem 3.2.2 are satisfied, then the Comparison Theorem 3.2.3 is valid if we require g to be a classical smooth supersolution (respectively subsolution) of equation (1.3) in E (see Remark 2.1). Suppose that in addition to the conditions of the Theorem 3.2.2, for each compact subsegment*

$[0;T_1] \subset [0;T]$ *there exist a positive constant* M_0 *such that (1.15a)–(1.15a) is satisfied with* $\delta = 0$. *Then the assertion of Theorem 3.2.3 is valid.*

Chapter 4

Evolution of Interfaces for Reaction-Diffusion-Convection Equations

4.1 Description of the Problem

We consider the Cauchy problem(CP)

$$u_t - (u^m)_{xx} + bu^\beta + c(u^p)_x = 0, \tag{4.1}$$

$$u(x,0) = u_0(x), \quad x \in \mathbb{R}, \tag{4.2}$$

with $m > 1$, $c < 0$, $b > 0$, $p > 0$, $\beta > 0$, $0 < T \leq +\infty$, and u_0 is non-negative and continuous. The goal of this Chapter is to apply the results of the general theory developed in Chapters 2 & 3 to analyze the behavior of interfaces separating the regions where $u = 0$ and where $u > 0$. Due to invariance of (4.1) with respect to translation, without loss of generality we will investigate the case when $\eta(0) = 0$, where

$$\eta(t) = \sup\,\{x : u(x,t) > 0\}.$$

is a so called interface function. We are interested in the short-time behavior of the interface function $\eta(t)$ and local solution near the interface. We shall assume that

$$u_0 \sim C(-x)_+^{\alpha} \ \text{ as } x \to 0- \quad \text{for some} \quad C > 0,\ \alpha > 0. \tag{4.3}$$

The direction of the movement of the interface and its asymptotic is an outcome of the competition between the diffusion, convection and reaction terms and depends on the parameters $b,\ c,\ \beta,\ p,\ C$ and α. Since the main results are local in nature, without loss of generality we may suppose that u_0 either is bounded or satisfies some restriction on its growth rate as $x \to -\infty$ which is suitable for existence, uniqueness, and comparison results. The special global case

$$u_0(x) = C(-x)_+^{\alpha}, \quad x \in \mathbb{R}, \tag{4.4}$$

will be considered when the solution to the problem (4.1), (4.4) is of self-similar form. Our estimations are global in time in these special cases.

If $c = 0$, then the initial development of interfaces and structure of local solution near the interfaces is very well understood in the case of the reaction-diffusion equations:

$$u_t - (u^m)_{xx} + bu^{\beta} = 0 \ \ x \in \mathbb{R}, \quad 0 < t < T. \tag{4.5}$$

Full classification of the evolution of interfaces and the local behavior of solutions near the interfaces in CP (4.5), (4.2), (4.3) was presented in [9] for the case of slow diffusion ($m > 1$) case (see Figure 4.1), and in [10] for the fast diffusion ($0 < m < 1$) case. The major obstacle in solving interface development problem for nonlinear degenerate parabolic equations is a problem of non-uniform asymptotics in the sense of singular perturbations theory, namely that the dominant balance as $t \to 0+$ between the terms in (4.1), (4.5) on curves which approach the boundary of the support on the initial line depending on how they do so. Comparison theorems proved in [1] were essential tool in developing rigorous proof method in [9, 10] for solving interface problem for the reaction-diffusion equation (4.5). The rigorous proof method developed in [9, 10] is based on a barrier technique using special comparison theorems in irregular domains with characteristic boundary curves. In this chapter we apply the general theory developed in Chapters 2 & 3 to solve the interface problem in the cases when either diffusion or absorption are dominating factors. The methods used are rescaling and blow-up techniques for the identification of the asymptotics of the solution along the class of interface type curves, construction of the barriers and application of the comparison theorem in non-cylindrical domains with characteristic boundary curves, as they are developed in papers [9, 10].

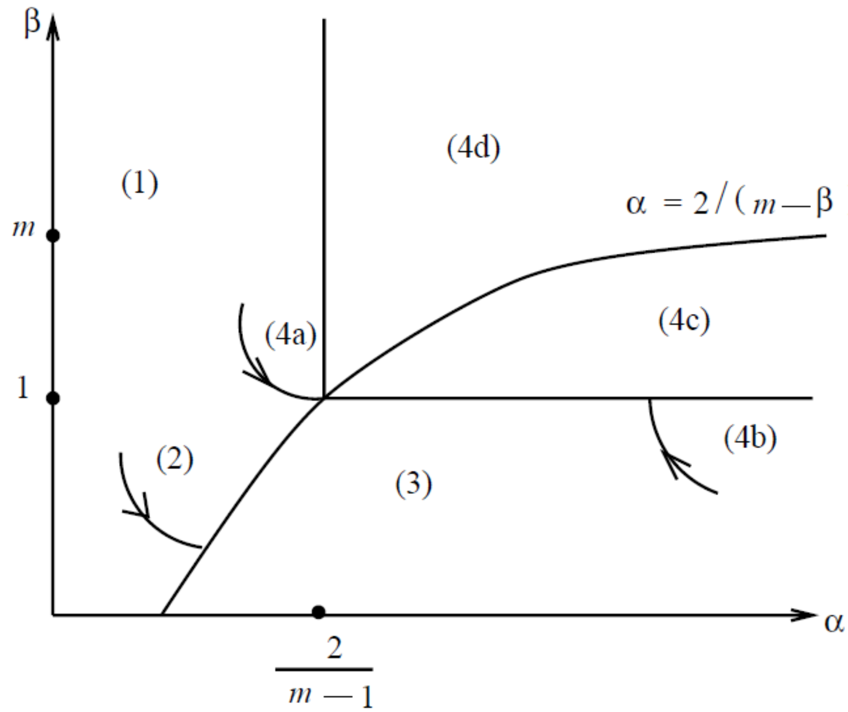

Figure 4.1: Classification of the initial interface development in the (α, β) plane for the Cauchy Problem (4.1)-(4.3) with $c = 0$.

4.2 Main Results

In the next theorem we identify the range of parameters when diffusion dominates over reaction and convection, and accordingly interface expands.

Theorem 4.2.1. *Let* $m > 1,\ b > 0,\ \beta > 0,\ c < 0,\ p > 0,$ *and*

$$\alpha < min\Big(2/(m - \min\{1, \beta\});\ 1/(m - \min\{(m+1)/2, p\}\Big). \tag{4.6}$$

Then the interface initially expands and

$$\eta(t) \sim \xi_* t^{1/(2-\alpha(m-1))} \quad as\ t \to 0+, \tag{4.7}$$

where

$$\xi_* = C^{\frac{m-1}{2-\alpha(m-1)}} \xi'_* \tag{4.8}$$

and $\xi'_* > 0$ *depends on* m *and* α *only (see Lemma 4.3.1). For arbitrary* $\rho < \xi_*$ *there exists* $f(\rho) > 0$ *depending on* $C, m,$ *and* α *such that*

$$u(x,t) \sim f(\rho) t^{\frac{\alpha}{2-\alpha(m-1)}} \quad \text{as} \quad t \to 0+ \tag{4.9}$$

along the curve $x = \xi_\rho(t) = \rho t^{\frac{1}{2-\alpha(m-1)}}$.

A function f is a shape function of the self-similar solution of (1.3),(4.4) with $b = 0, c = 0$ (see Lemma 4.3.1):

$$u_*(x,t) = t^{\frac{\alpha}{2-\alpha(m-1)}} f(\xi), \qquad \xi = x t^{-\frac{1}{2-\alpha(m-1)}}, \tag{4.10}$$

In fact, f is a unique solution of the following nonlinear ODE problem:

$$\begin{cases} \frac{d^2 f^m}{d\xi^2} + \frac{1}{2-\alpha(m-1)} \xi \frac{df}{d\xi} - \frac{\alpha}{2-\alpha(m-1)} f = 0, \ -\infty < \xi < \xi_* \\ f(-\infty) \sim C(-\xi)^\alpha, f(\xi_*) = 0, f(\xi) \equiv 0, \ \xi \geq \xi_* \end{cases} \tag{4.11}$$

Its dependence on C is given through the following relation:

$$f(\rho) = C^{\frac{2}{2+\alpha(1-m)}} f_0(C^{\frac{m-1}{\alpha(m-1)-2}} \rho), \tag{4.12a}$$

$$f_0(\rho) = w(\rho, 1), \qquad \xi'_* = \sup\{\rho : f_0(\rho) > 0\} > 0 \tag{4.12b}$$

where w is a solution of (4.1), (4.4) with $b = 0, c = 0, C = 1$. According to [9] we also have that

$$\xi'_* = A_0^{\frac{m-1}{2}} \left[\frac{(m(2-\alpha(m-1))}{m-1} \right]^{\frac{1}{2}} \xi''_*, \tag{4.13}$$

where $A_0 = w(0,1)$ and ξ_*'' is some number in $[\xi_1, \xi_2]$, where

$$\xi_1 = \Big(\alpha(m-1)\Big)^{-\frac{1}{2}}, \quad \xi_2 = 1 \qquad \text{if} \quad (m-1)^{-1} \le \alpha < 2(m-1)^{-1},$$

$$\xi_1 = 1, \qquad \xi_2 = \Big(\alpha(m-1)\Big)^{-\frac{1}{2}}, \qquad \text{if} \quad 0 < \alpha \le (m-1)^{-1}. \quad (4.14)$$

The following lower and upper estimations for f are proved in [9]:

$$C_4 t^{\alpha/(2-\alpha(m-1))} (\xi_3 - \xi)_+^{\frac{1}{m-1}} \le u \le C_5 t^{\alpha/(2-\alpha(m-1))} (\xi_4 - \xi)_+^{\frac{1}{m-1}}, \tag{4.15}$$

$$0 \le x < +\infty, \qquad 0 < t < +\infty,$$

where ξ_3 (respectively, ξ_4) is defined by the right-hand side of (4.13) and we replace ξ_*'' with $C^{\frac{m-1}{2-\alpha(m-1)}}\xi_1$ (respectively, with $C^{\frac{m-1}{2-\alpha(m-1)}}\xi_2$) and

$$C_4 = C^{2/(2-\alpha(m-1))} A_0 \xi_3^{1/(m-1)}, \qquad C_5 = C^{2/(2-\alpha(m-1))} A_0 \xi_4^{-1/(m-1)}.$$

In particular, if $\alpha = (m-1)^{-1}$, then the explicit solution of the problem (4.1), (4.4) with $b = 0, c = 0$ is

$$u(x,t) = C(\xi_* t - x)_+^{1/(m-1)}, \qquad \xi_* = C^{m-1} m (m-1)^{-1}. \tag{4.16}$$

and we have

$$\xi_1 = \xi_2, \quad \xi_*' = m(m-1)^{-1}, \quad f_0(x) = \big(\xi_*' - x\big)_+^{1/(m-1)} \tag{4.17}$$

The explicit formula (4.7) and (4.9) mean that the local behavior of the interface and solution along $x = \xi_\rho(t)$ coincide with those of the problem (4.1), (4.2) with $b = 0, c = 0$.

In the next theorem we identify the parameter range where reaction (or absorption) force dominates over both diffusion and convection forces, and accordingly the interface shrinks.

Theorem 4.2.2. Let $b > 0,\ c < 0,\ 0 < \beta < 1$ *and one of the following conditions be valid:*

(a) $\beta < p \leq (m+\beta)/2, \quad \alpha > 1/(p-\beta),$

(b) $p \geq (m+\beta)/2, \quad \alpha > 2/(m-\beta),$

If u_0 *satisfies* (4.3), *then interface shrinks and*

$$\eta(t) \sim -\ell_* t^{1/\alpha(1-\beta)} \ \text{as}\ t \to 0+ \tag{4.18}$$

where $\ell_* = C^{-1/\alpha}(b(1-\beta))^{1/\alpha(1-\beta)}$. *For arbitrary* $\ell > \ell_*$ *we have*

$$u(x,t) \sim \left[C^{1-\beta}(-x)_+^{\alpha(1-\beta)} - b(1-\beta)t\right]^{1/(1-\beta)} \ \text{as}\ t \to 0+ \tag{4.19}$$

along the curve $x = \eta_\ell(t) = -\ell t^{1/\alpha(1-\beta)}$.

Hence, the interface initially coincides with that of the solution

$$\bar{u}(x,t) = \left[C^{1-\beta}(-x)_+^{\alpha(1-\beta)} - b(1-\beta)t\right]_+^{1/(1-\beta)}$$

to the problem

$$\bar{u}_t + b\bar{u}^\beta = 0, \qquad \bar{u}(x,0) = C(-x)_+^\alpha.$$

Respective lower and upper estimations are given in section 4.6 (see (4.41a) and (4.41b) below).

4.3 Domination of Diffusion and Asymptotic Properties of Solutions

First we recall the well-known lemma (e.g. [9]) on the self-similar solution of the nonlinear diffusion equation:

Lemma 4.3.1. [9] *If* $b = 0, c = 0$ *and* $0 < \alpha < 2/(m-1)$, *then the solution* u *of the CP* (4.1), (4.4) *has a self-similar form* (4.10), *where the self-similarity function* f *satisfies* (4.12). *If* u_0 *satisfies* (4.3), *then the solution to CP* (4.1), (4.2) *satisfies* (4.7)-(4.9).

In the next lemma we identify the parameter ranges when diffusion dominates over both reaction and convection forces, and the asymptotic properties of the solution to the reaction-diffusion-convection equation coincides with the asymptotic properties of the diffusion equation.

Lemma 4.3.2. *Let* u *be a solution to the CP*(4.1), (4.2) *and* u_0 *satisfy* (4.3). *Let one of the following conditions be valid:*

(a) $0 < \beta < 1, \quad 0 < p < (m+1)/2, \quad 0 < \alpha < min\{1/(m-p), 2/(m-\beta)\}$.

(b) $0 < \beta < 1, \quad (m+1)/2 \leq p, \quad 0 < \alpha < 2/(m-\beta)$.

(c) $\beta \geq 1, \quad (m+\beta)/2 < p < (m+1)/2, \quad 0 < \alpha < 1/(m-p)$.

(d) $\beta \geq 1, \quad (m+1)/2 \leq p, \quad 0 < \alpha < 2/(m-1)$.

Then u *satisfies* (4.9).

Proof of Lemma 4.3.2. Suppose that u_0 satisfies (4.3). Then for arbitrary sufficiently small $\epsilon > 0$, there exists an $x_\epsilon < 0$ such that

$$(C - \epsilon/2)(-x)_+^\alpha \leq u_0(x) \leq (C + \epsilon/2)(-x)_+^\alpha, \quad x_\epsilon \leq x < +\infty \tag{4.20}$$

Let the conditions of one of the cases $(a), (c)$ and cases $(b), (d)$ with $p \leq m$ be valid.Then from results mentioned in Chapter 2, Theorem 2.1.1, 2.2.1 it follows that the existence and uniqueness of the (4.1), (4.2) with $u_0 = (C \pm \epsilon)(-x)_+^\alpha, T = +\infty$ hold. Let $u_\epsilon(x,t)$ (respectively, $u_{-\epsilon}(x,t)$) be a solution to the CP (4.1),(4.2) with initial data $(C + \epsilon)(-x)_+^\alpha$ (respectively , $(C - \epsilon)(-x)_+^\alpha$). Since the solution to the CP (4.1)-(4.2) is continuous there exists a number $\sigma = \sigma(\epsilon) > 0$ such that

$$u_{-\epsilon}(x_\epsilon, t) \leq u(x_\epsilon, t) \leq u_{+\epsilon}(x_\epsilon, t), \quad \text{for} \quad 0 \leq t \leq \sigma. \tag{4.21}$$

From (4.20),(4.21) and a comparison principle from Theorem 2.3.1 it follows that

$$u_{-\epsilon} \leq u \leq u_{+\epsilon}, \quad \text{for} \quad x_\epsilon \leq x < \infty \ , \quad 0 \leq t \leq \sigma. \tag{4.22}$$

If we consider a function

$$u_k^{\pm\epsilon}(x,t) = k u_{\pm\epsilon}(k^{-1/\alpha} x, k^{(\alpha(m-1)-2)/\alpha}), \qquad k > 0, \tag{4.23}$$

then $u_k^{\pm\epsilon}(x,t)$ satisfies the following problem:

$$u_t - (u^m)_{xx} + c k^{\frac{\alpha(m-p)-1}{\alpha}} (u^p)_x + b k^{\frac{\alpha(m-\beta)-2}{\alpha}} u^\beta = 0, x \in R, t > 0, \tag{4.24a}$$

$$u(x,0) = (C \pm \epsilon)(-x)_+^\alpha, \qquad x \in \mathbb{R}. \tag{4.24b}$$

There exists a unique solution to CP (4.24), which obeys a comparison principle also. Since $\alpha(m-p) - 1 < 0$ and $\alpha(m-\beta) - 2 < 0$, from [1] and the results of

Chapters 2 & 3 it follows that

$$\lim_{k\to+\infty} u_k^{\pm\epsilon}(x,t) = v_{\pm\epsilon}(x,t), \qquad x\in\mathbb{R}, \quad t\geq 0, \tag{4.25}$$

According to the lemma 4.3.1, $v_{\pm\epsilon}$ is a solution to the CP (4.1),(4.2) with $b = 0,\ c = 0,\ u_0 = (C\pm\epsilon)(-x)_+^{\alpha},\ T = +\infty$. By choosing $x = \xi_\rho(t) = \rho t^{1/(2-\alpha(m-1))}$ in (4.10), where ρ is an arbitrary fixed number satisfying $\rho < \xi_*$, we have

$$v_{\pm\epsilon}(\xi_\rho(t),t) = f(\rho; C\pm\epsilon)t^{\frac{\alpha}{2+\alpha(1-m)}}, \quad t\geq 0, \tag{4.26}$$

and from (4.25) it follows that

$$\lim_{k\to+\infty} ku_{\pm\epsilon}(k^{\frac{-1}{\alpha}}\xi_\rho(t), k^{\frac{\alpha(m-1)-2}{\alpha}}t) = f(\rho; C\pm\epsilon)t^{\frac{\alpha}{2-\alpha(m-1)}},\ t\geq 0. \tag{4.27}$$

If we takes $\tau = k^{(\alpha(m-1)-2)/\alpha}t$, then (4.27) implies

$$u_{\pm\epsilon}(\xi_\rho(\tau),\tau) \sim f(\rho; C\pm\epsilon)\tau^{\frac{\alpha}{2-\alpha(m-1)}},\ ,\ \text{as } \tau\to 0+. \tag{4.28}$$

Then (4.9) follows from (4.22), (4.28).

Now consider the cases (b) and (d) with $p > m$. Suppose that $u_{\pm\epsilon}$ is a solution of the Dirichlet problem

$$u_t - (u^m)_{xx} + a(u^p)_x + bu^\beta = 0, \quad |x| < |x_\epsilon|,\ 0 < t < \sigma, \tag{4.29a}$$

$$u(x,0) = (C\pm\epsilon)(-x)_+^{\alpha}, \quad |x| \leq |x_\epsilon|, \tag{4.29b}$$

$$u(x_\epsilon,t) = (C\pm\epsilon)(-x_\epsilon)^{\alpha}, \quad u(-x_\epsilon,t) = 0,\ \ 0\leq t\leq\sigma. \tag{4.29c}$$

The function $u_k^{\pm\epsilon}$, defined as in (4.23), satisfies the Dirichlet problem

$$u_t - (u^m)_{xx} + ck^{\frac{\alpha(m-p)-1}{\alpha}}(u^p)_x + bk^{\frac{\alpha(m-\beta)-2}{\alpha}}u^\beta = 0 \text{ in } D_\epsilon^k, \tag{4.30a}$$

$$u(k^{1/\alpha}x_\epsilon, t) = k(C \pm \epsilon)(-x)^\alpha, \quad u(-k^{1/\alpha}x_\epsilon, t) = 0, \; 0 \le t \le k^{\frac{2-\alpha(m-1)}{\alpha}}\sigma \tag{4.30b}$$

$$u(x, 0) = (C \pm \epsilon)(-x)_+^\alpha, \qquad |x| \le k^{1/\alpha}|x_\epsilon|, \tag{4.30c}$$

where

$$D_\epsilon^k = \{(x,t) : |x| < k^{1/\alpha}|x_\epsilon|, \quad 0 < t \le k^{\frac{2-\alpha(m-1)}{\alpha}}\sigma\}.$$

From Theorem 2.1.1, 2.2.1 it follows that there exists a number $\sigma > 0$ (which does not depend on k) such that both (4.29a)-(4.29c) and (4.30a)-(4.30c) have a unique solution. In view of finite speed of propagation a $\sigma = \sigma(\epsilon) > 0$ may be chosen such that

$$u(-x_\epsilon, t) = 0, \qquad 0 \le t \le \sigma. \tag{4.31}$$

By applying Comparison Theorem 2.3.1, from (4.20), (4.21) and (4.31),(4.22) follows for $|x| \le |x_\epsilon|, \; 0 \le t \le \sigma$. The next step consists in the proof of convergence of the sequences $\{u_k^{\pm\epsilon}\}$ as $k \to +\infty$. Consider a function

$$g(x,t) = (C+1)(1+x^2)^{\frac{\alpha}{2}}(1-\nu t)^{\frac{1}{1-m}}, \; x \in \mathbb{R}, \; 0 \le t \le t_0 = \frac{\nu^{-1}}{2}$$

where

$$\nu = h_* + 1, \; h_* = h_*(\alpha; m) = \max_{x\in\mathbb{R}} h(x),$$

$$h(x) = \alpha m(m-1)(C+1)^{m-1}(1+x^2)^{\frac{\alpha(m-1)-4}{2}}\Big((\alpha m - 1)x^2 + 1\Big)$$

Then we have

$$L_k g \equiv g_t - (g^m)_{xx} + ck^{\frac{\alpha(m-p)-1}{\alpha}}(g^p)_x + bk^{\frac{\alpha(m-\beta)-2}{\alpha}}(g^\beta)$$

$$= (C+1)(1+x^2)^{\alpha/2}(1-\nu t)^{\frac{m}{1-m}}(m-1)^{-1}S \ \text{ in } D_\epsilon^k,$$

$$S = \nu - h(x) + R_1 + R_2$$

$$R_1 = k^{\frac{\alpha(m-p)-1}{\alpha}} c\alpha p(m-1)(C+1)^{p-1}(1-\nu t)^{\frac{p-m}{1-m}} x(1+x^2)^{\frac{\alpha(p-1)}{2}-1}$$

$$R_2 = k^{\frac{\alpha(m-\beta)-2}{\alpha}} b(m-1)(C+1)^{\beta-1}(1-\nu t)^{\frac{\beta-m}{1-m}}(1+x^2)^{\frac{\alpha(\beta-1)}{2}}$$

and hence

$$S \geq 1 + R_1 + R_2 \text{ in } D_{0\epsilon}^k = D_\epsilon^k \cap \{0 < t \leq t_0\}, \tag{4.32}$$

where

$$R_1 = O(k^{(\alpha(m-p)-1)/\alpha}) \text{ uniformly for } (x,t) \in E_{o\epsilon}^k \text{ as } k \to \infty.$$

$$R_2 = O(k^{\theta}) \text{ uniformly for } (x,t) \in E_{o\epsilon}^k \text{ as } k \to \infty.$$

$$\theta = (\alpha(m-1)-2)/\alpha \ \text{ if } \ \beta \geq 1 \ \& \ \theta = (\alpha(m-\beta)-2)/\alpha \ \text{ if } \ 0 < \beta < 1$$

Moreover, we have for $0 < \epsilon \ll 1$

$$g(x,0) \geq u_k^{\pm\epsilon}(x,0) \text{ for } |x| \leq k^{1/\alpha}|x_\epsilon|, \tag{4.33a}$$

$$g(\pm k^{1/\alpha}x_\epsilon, t) \geq u_k^{\pm\epsilon}(\pm k^{1/\alpha}x_\epsilon, t) \text{ for } 0 \leq t \leq t_0. \tag{4.33b}$$

Hence, $\exists\, k_0 = k_0(\alpha; p)$ such that for $\forall k \geq k_0$ the comparison principle from Theorem 2.3.1 implies

$$0 \leq u_k^{\pm\epsilon}(x,t) \leq g(x,t) \text{ in } \bar{D}_{0\epsilon}^k. \tag{4.34}$$

Let G be an arbitrary fixed compact subset of

$$P = \big\{(x,t) : x \in \mathbb{R}, \quad 0 < t \leq t_0\big\}.$$

We take k_0 so large that $G \subset D^k_{0\epsilon}$ for $k \geq k_0$. From (4.34) it follows that the sequences $\{u_k^{\pm\epsilon}\}$, $k \geq k_0$, are uniformly bounded in G. As in Chapter 2, it may be proved that they are uniformly Höder continuous in G and that there exist function $v_{\pm\epsilon}$ such that for some subsequence k'

$$\lim_{k'\to+\infty} u_{k'}^{\pm\epsilon}(x,t) = v_{\pm\epsilon}(x,t), \qquad (x,t) \in P. \tag{4.35}$$

It may easily be checked that $v_{\pm\epsilon}$ is a solution to the CP (4.1), (4.2) with $b = 0$, $c = 0, T = t_0, u_0 = (C \pm \epsilon)(-x)_+^{\alpha}$. From (4.26), (4.27), (4.28) and (4.22), the required estimation (4.9) follows. The lemma is proved.

4.4 Domination by Reaction and Asymptotic Properties of Solutions

In the next lemma we identify the parameter ranges when absorption dominates over both diffusion and convection forces, and the asymptotic properties of the solution to the reaction-diffusion-convection equation coincides with the asymptotic properties of the reaction equation.

Lemma 4.4.1. *Let u be a solution to the CP* (4.1)-(4.3). *Let* $0 < \beta < 1$ *and one of the following conditions be valid:*

(a) $\beta < p \leq (m+\beta)/2, \quad \alpha > 1/(p-\beta)$,

(b) $p \geq (m+\beta)/2, \quad \alpha > 2/(m-\beta)$, *Then for arbitrary $\ell > \ell_*$ (see* (4.18)*) the asymptotic formula* (4.19) *is valid with* $x = \eta_\ell(t) = -\ell t^{1/\alpha(1-\beta)}$.

Proof of Lemma 4.4.1. As before, (4.20) and (4.21) follow from (4.3). Suppose that $u_{\pm\epsilon}$ solves the problem

$$v_t - (v^m)_{xx} + c(v^p)_x + bv^\beta = 0, \qquad |x| < |x_\epsilon|, \quad 0 < t \le \delta,$$

$$v(x,0) = (C \pm \epsilon)(-x)_+^\alpha, \qquad |x| \le |x_\epsilon|,$$

$$v(x_\epsilon, t) = (C \pm \epsilon)(-x_\epsilon)_+^\alpha, \quad v(-x_\epsilon, t) = u(-x_\epsilon, t), \qquad 0 \le t \le \delta.$$

Applying a comparison principle, from (4.20) and (4.21), (4.22) follows for $|x| \le |x_\epsilon|, \quad 0 \le t \le \delta$. If we rescale

$$u_k^{\pm\epsilon}(x,t) = k u_{\pm\epsilon}(k^{-1/\alpha}x, k^{\beta-1}t), \qquad k > 0,$$

then $u_k^{\pm\epsilon}$ satisfies the Dirichlet problem

$$v_t - k^{\frac{2-\alpha(m-\beta)}{\alpha}}(v^m)_{xx} + ak^{\frac{1-\alpha(p-\beta)}{\alpha}}(v^p)_x + bv^\beta = 0 \text{ in } E_\epsilon^k$$

$$v(x,0) = (C \pm \epsilon)(-x)_+^\alpha, \qquad |x| \le k^{1/\alpha}|x_\epsilon|$$

$$v(k^{1/\alpha}x_\epsilon, t) = k(C \pm \epsilon)(-x_\epsilon)_+^\alpha, \quad v(-k^{1/\alpha}x_\epsilon, t) = ku(-x_\epsilon, k^{\beta-1}t),$$

$$0 \le t \le k^{1-\beta}\delta$$

where

$$E_\epsilon^k = \{|x| < k^{1/\alpha}|x_\epsilon|, \ 0 < t \le k^{1-\beta}\delta\}.$$

The next step consists in proving the convergence of the sequences $\{u_k^{\pm\epsilon}\}$ as $k \to +\infty$. Considering the function $g(x,t) = (C+1)(1+x^2)^{\alpha/2} \exp t$, we have

$$\tilde{\mathrm{L}}_k g \equiv g_t - k^{\frac{2-\alpha(m-\beta)}{\alpha}} (g^m)_{xx} + ck^{\frac{1-\alpha(p-\beta)}{\alpha}} (g^p)_x + bg^\beta \geq$$

$$g\Big[1 - (C+1)^{m-1}\alpha m(1+x^2)^{\frac{\alpha(m-1)}{2}-2} \left(1 + (\alpha m - 1)x^2\right) k^{\frac{2-\alpha(m-\beta)}{\alpha}} e^{(m+1)t} +$$

$$ck^{\frac{1-\alpha(p-\beta)}{\alpha}} (\alpha\ p)(C+1)^{p-1} x(1+x^2)^{\frac{\alpha(p-1)}{2}-1} e^{(p-1)t}\Big] \text{ in } E_\epsilon^k. \tag{4.36}$$

Let $t_0 > 0$ be fixed and let $E_{0\epsilon}^k = E_\epsilon^k \cap \{(x,t) : \ 0 < t \leq t_0\}$. From (4.36) it follows that

$$\tilde{L}_k g \geq g(1 - h(x) + R), \qquad \text{in } \ E_0^k,$$

$$R = ck^{\frac{1-\alpha(p-\beta)}{\alpha}} (\alpha\ p)(C+1)^{p-1} e^{(p-1)t} x(1+x^2)^{\frac{\alpha(p-1)}{2}-1} + b(C+1)^{\beta-1} e^{(\beta-1)t} (1+x^2)^{\frac{\alpha(\beta-1)}{2}}$$

and, since

$$h(x) = k^{\frac{2-\alpha(m-\beta)}{\alpha}} \alpha m(C+1)^{m-1} e^{(m-1)t} (1+x^2)^{\frac{\alpha(m-1)}{2}-2} \Big[1 + (\alpha m - 1)x^2\Big],$$

$$h(x) = O(k^\theta) \text{ uniformly for } (x,t) \in E_{0\epsilon}^k \ \text{ as } \ k \to +\infty$$

$$\theta = \left(2 - \alpha(m-\beta)/\alpha\right) \quad \text{if } \ \alpha < 2/(m-1),$$

$$\theta = \beta - 1, \qquad\qquad \text{if } \alpha \geq 2/(m-1).$$

then

$$\tilde{L}_k g \geq g(1+R), \qquad \text{in } \ E_0^k,$$

where

$$R = O(k^\phi) \text{ uniformly for } (x,t) \in E_{o\epsilon}^k \text{ as } k \to \infty.$$

$$\phi = (1 - \alpha(p - \beta))/\alpha \quad \text{if} \ \ \alpha < 1/(p-1),$$

$$\phi = \beta - 1, \quad \text{if} \ \ \alpha \geq 1/(p-1).$$

Moreover, we have for $0 < \epsilon \ll 1$ that

$$g(x,0) = u_k^{\pm\epsilon}(x,0), \qquad \text{for} \quad |x| \leq k^{1/\alpha}|x_\epsilon|.$$

Since

$$u_k^{\pm\epsilon}(-k^{\frac{1}{\alpha}}x_\epsilon, t) = o(k), \ 0 \leq t \leq t_0 \text{ as } k \to \infty,$$

we also have

$$g(\pm k^{\frac{1}{\alpha}}x_\epsilon, t) \geq u_k^{\pm\epsilon}(\pm k^{\frac{1}{\alpha}}x_\epsilon, t), \ \ \text{for} \quad 0 \leq t \leq t_0$$

if k is chosen large enough. Hence, as in the proof of lemma 4.3.1, if k is large enough, a comparison principle (e.g. Theorem 3.2.3) implies (4.34) in $\bar{E}^k_{0\epsilon}$, where the respective functions $u_k^{\pm\epsilon}$ and g apply in the context of the this proof. As before, from the interior regularity results ([48]) it follows that the sequence of nonnegative and locally bounded solutions $\{u_k^{\pm\epsilon}\}$ is locally uniformly Hölder continuous on compact subsets of P. Thus there exist functions $v_{\pm\epsilon}$ such that for some subsequence k' , (4.35) is valid. It may be easily be demonstrated that the limit functions $v_{\pm\epsilon}$ are solutions to the problem

$$v_t + bv^\beta = 0, \ x \in \mathbb{R}, \ 0 < t \leq t_0; \ \ v(x,0) = (C \pm \epsilon)(-x)_+^\alpha, \ x \in \mathbb{R},$$

i.e.,

$$v_{\pm\epsilon}(x,t) = \Big[(C \pm \epsilon)^{1-\beta}(-x)_+^{\alpha(1-\beta)} - b(1-\beta)t\Big]_+^{\frac{1}{1-\beta}}.$$

Let $\ell > \ell_*$ be an arbitrary number and $\epsilon > 0$ be chosen such that

$$(C-\epsilon)^{1-\beta}\ell^{\alpha(1-\beta)} > b(1-\beta).$$

If we now take $x = \eta_\ell(t)$ and $\tau = k^{\beta-1}t$, it follows from (4.35) that

$$u_{\pm\epsilon}(\eta_\ell(\tau),\tau) \sim \Big[(C\pm\epsilon)^{1-\beta}\ell^{\alpha(1-\beta)} - b(1-\beta)\Big]^{\frac{1}{1-\beta}}\tau^{\frac{1}{1-\beta}} f \text{ as } \tau \to 0+. \qquad (4.37)$$

Since $\epsilon > 0$ is arbitrary, From (4.22) and (4.37), in view of arbitrariness of $\epsilon > 0$, the desired formula (4.19) follows. The lemma is proved.

4.5 Proof of Theorem 4.2.1

The formula (4.9) follows from Lemma 4.3.2. Since ρ is arbitrary, from Lemma 4.3.2 it follows that

$$u(\xi_\rho(t),t) \sim f(\rho)t^{\frac{\alpha}{2-\alpha(m-1)}}, \text{ as } t\downarrow 0, \ \forall \rho < \xi_*, \text{ where } \xi_\rho(t) = \rho t^{\frac{1}{2-\alpha(m-1)}}.$$

$$\lim_{t\downarrow 0}\frac{u(\xi_\rho(t),t)}{t^{\frac{\alpha}{2-\alpha(m-1)}}} = f(\rho) > 0$$

For $\epsilon > 0$, take $\rho = \xi_* - \epsilon$

$$\lim_{t\downarrow 0}\frac{u((\xi_*-\epsilon)t^{\frac{1}{2-\alpha(m-1)}},t)}{t^{\frac{\alpha}{2-\alpha(m-1)}}} = f(\xi_*-\epsilon) > 0$$

$$\exists \sigma > 0 \quad \forall t \in \big(0,\sigma\big], \quad u((\xi_*-\epsilon)t^{\frac{1}{2-\alpha(m-1)}},t) \geq \frac{f(\xi_*-\epsilon)}{2}t^{\frac{\alpha}{2-\alpha(m-1)}} > 0$$

$$\liminf_{t\downarrow 0}\, \eta(t)t^{\frac{1}{\alpha(m-1)-2}} \geq (\xi_*-\epsilon)$$

Let $\epsilon \downarrow 0$ then we get

$$\liminf_{t \to 0+} \eta(t) t^{\frac{1}{\alpha(m-1)-2}} \geq \xi_*. \qquad (4.38)$$

Take an arbitrary sufficiently small number $\epsilon > 0$. Let u_ϵ be a solution of the CP (4.1), (4.4) with $b = 0, c = 0$ and with C replaced by $C + \epsilon$. As before, the second inequality of (4.20) and the first inequality of (4.21) follow from (4.3). Suppose that $b > 0$. Since

$$Lu_\epsilon = c\left(u_\epsilon^p\right)_x + bu_\epsilon^\beta.$$

u_ϵ will be a supersolution if $c\left(u_\epsilon^p\right)_x + bu_\epsilon^\beta \geq 0$ and since $bu_\epsilon^\beta \geq 0$ and $c < 0$, so it is enough to prove $\left(u_\epsilon^p\right)_x \leq 0$. Construct a family of functions $u_{0,n}(x)$ such that

$$u_{0,n}(x) = u_0(x) \text{ if } x < x_n, \text{ and } \qquad u_{0,n}(x) = 1/n \text{ if } x \geq x_n,$$

where x_n is chosen such that $u_0(x_n) = 1/n$, and let $u_n(x,t)$ be a solution to (4.1) with $b = 0, c = 0$, and the initial function $u(x,0) = u_{0,n}(x)$. Since $u_{0,n}(x) \geq \frac{1}{n} > 0$, $u_n(x,t)$ is a classical solution and by the maximum principle we have

$$u_n(x,t) \geq u_{n+1}(x,t) \geq \cdots \geq 0,$$

and the following poiu wise limit exists:

$$u(x,t) = \lim_{t \to \infty} u_n(x,t).$$

Our aim is to prove that $(u_n)_x \leq 0$. In fact, $w = v_n = (u_n)_x$ is a solution of the second order linear parabolic PDE

$$w_t = mu^{m-1}w_{xx} + 3m(m-1)u^{m-2}u_x w_x + m(m-2)(m-1)u^{m-3}(u_x)^2 w.$$

in $x \in \mathbb{R}, t > 0$, and $w(x,0) \leq 0$. From the maximum principle it follows that $w \leq 0$ everywhere, and so $(u_n)_x \leq 0$, and accordingly we have $u_x \leq 0$. Therefore, we have

$$Lu_\epsilon \geq 0 \quad in \quad x \in \mathbb{R},\ t > 0.$$

From (4.20), (4.21), and a comparison principle, the second inequality of (4.22) follows. Thus $\forall \epsilon > 0\ \exists \sigma > 0$ such that

$$\eta(t) \leq (C+\epsilon)^{\frac{m-1}{2-\alpha(m-1)}} \xi_*' t^{1/(2-\alpha(m-1))}, \quad 0 \leq t \leq \sigma,$$

which implies

$$\limsup_{t \to 0^+} \eta(t) t^{\frac{1}{\alpha(m-1)-2}} \leq \xi_*. \tag{4.39}$$

From (4.38) and (4.39), (4.7) follows. Finally, (4.13),(4.14),(4.17) follow from (4.15). The theorem is proved. □

4.6 Proof of Theorem 4.2.2.

Take an arbitrary sufficiently small number $\epsilon > 0$. From (4.3), (4.20) follows. Then consider a function

$$g_\epsilon(x,t) = \left[(C+\epsilon)^{1-\beta}(-x)_+^{\alpha(1-\beta)} - (1-\beta)(1-\epsilon)t\right]_+^{1/(1-\beta)} \tag{4.40}$$

We estimate Lg in

$$M_1 = \left\{(x,t) : x_\epsilon < x < \eta_\ell(t), \quad 0 < t < \delta_1\right\},$$

$$\eta_\ell(t) = -\ell t^{1/\alpha(1-\beta)}, \ \ \ell(\epsilon) = (C+\epsilon)^{-1/\alpha}\left[b(1-\beta)(1-\epsilon)\right]^{1/\alpha(1-\beta)},$$

where $\delta_1 > 0$ is chosen such that $\eta_{\ell(\epsilon)}(\delta_1) = x_\epsilon$. We have

$$Lg_\epsilon = bg_\epsilon^\beta\{\epsilon + S\}$$

$$\begin{aligned}
S = &-b^{-1}\alpha m(\alpha(1-\beta)-1)(C+\epsilon)^{m-\beta}(-x)_+^{\alpha(m-\beta-2)}\Big\{g_\epsilon|x|^{-\alpha}\Big/(C+\epsilon)\Big\}^{m-1)} \\
&-b^{-1}\alpha^2 m(m+\beta)-1)(C+\epsilon)^{m-\beta}(-x)_+^{\alpha(m-\beta)-2}\Big\{g_\epsilon|x|^{-\alpha}\Big/(C+\epsilon)\Big\}^{m+\beta-2)} \\
&-b^{-1}cp\alpha(C+\epsilon)^{p-\beta}(-x)_+^{\alpha(p-\beta)-1}\Big\{g_\epsilon|x|^{-\alpha}\Big/(C+\epsilon)\Big\}^{p-1} \\
= &-b^{-1}\alpha m(C+\epsilon)^{m-\beta}(-x)_+^{\alpha(m-\beta)-2}\Big\{g_\epsilon|x|^{-\alpha}\Big/(C+\epsilon)\Big\}^{m+\beta-2}S_1 + S_2, \\
S_1 = &\Big\{\alpha(m+\beta-1) + (\alpha(1-\beta)-1)\Big[g_\epsilon|x|^{-\alpha}\Big/(C+\epsilon)\Big]^{1-\beta}\Big\}, \\
S_2 = &\Big\{-b^{-1}cp\alpha(C+\epsilon)^{p-\beta}(-x)_+^{\alpha(p-\beta)-1}\Big[g_\epsilon|x|^{-\alpha}\Big/(C+\epsilon)\Big]^{p-1}\Big\}.
\end{aligned}$$

If $m+\beta \geq 2$, then we can choose $x_\epsilon < 0$ such that (with sufficiently small $|x_\epsilon|$)

$$|S| < \epsilon/2 \quad \text{in} \quad M_3$$

Thus we have

$$\begin{aligned}
\mathrm{L}g_\epsilon &> b(\epsilon/2)g_\epsilon^\beta \quad (\text{respectively,} \quad \mathrm{L}g_{-\epsilon} < -b\big(\epsilon/2)g_{-\epsilon}^\beta\big) \quad \text{in} \quad M_3, \\
\mathrm{L}g_{\pm\epsilon} &= 0 \quad \text{for} \quad x > \eta_{\ell(\pm\epsilon)}(t), \quad 0 < t \leq \sigma_1, \\
g_\epsilon(x,0) &\geq u_0(x) \quad \big(\text{respectively,} \quad g_{-\epsilon}(x,0) \leq u_0(x)\big), \quad x \geq x_\epsilon.
\end{aligned}$$

Since u and g are continuous functions, $\sigma = \sigma(\epsilon) \in (0, \sigma_1]$ may be chosen such that

$$g_\epsilon(x_\epsilon, t) \geq u(x_\epsilon, t) \quad \big(\text{respectively,} \quad g_{-\epsilon}(x_\epsilon, t) \leq u(x_\epsilon, t)\big), \quad 0 \leq t \leq \sigma.$$

From Lemma 2.3.1 and Comparison Theorem 2.3.1 it follows that

$$g_{-\epsilon} \leq u \leq g_{\epsilon} \quad x \geq x_{\epsilon}, \;\; 0 \leq t \leq \sigma, \tag{4.41a}$$

$$\eta_{\ell(-\epsilon)}(t) \leq \eta(t) \leq \eta_{\ell(\epsilon)}, \;\; 0 \leq t \leq \sigma, \tag{4.41b}$$

which imply (4.18) and (4.19).

Let $m+\beta<2$. In this case the left-hand side of (4.41a), (4.41b) may be proved similarly. Moreover, we can replace $1+\epsilon$ with 1 in $g_{-\epsilon}$ and $\eta_\ell(-\epsilon)$.

To prove a relevant upper estimation, consider a function

$$g(x,t) = C_6\big(-\zeta_5 t^{\frac{1}{\alpha(1-\beta)}} - x\big)_+^{\alpha} \text{ in } G_{\ell,\sigma},$$

$$G_{\ell,\sigma} = \{(x,t) : \eta_\ell(t) < x < +\infty, \;\; 0 < t < \sigma\},$$

where $\ell \in (\ell_*, +\infty)$. From (4.19) it follows that for arbitrary $\ell > \ell_*$ and $\epsilon > 0$ there exists a $\sigma = \sigma(\epsilon,\ell) > 0$ such that

$$u(\eta_\ell(t),t) \leq [C^{1-\beta}\ell^{\alpha(1-\beta)} - b(1-\beta)(1-\epsilon)]^{\frac{1}{1-\beta}} t^{\frac{1}{1-\beta}}, \;\; 0 \leq t \leq \sigma. \tag{4.42}$$

Calculating $\mathrm{L}g$ in

$$G^+_{\ell,\sigma} = \{(x,t) : \eta_\ell(t) < x < -\zeta_5 t^{\frac{1}{\alpha(1-\beta)}}, \;\; 0 < t < \sigma\},$$

we have

$$\mathrm{L}g = bg^\beta S, \;\; S = 1 - (b(1-\beta))^{-1}\zeta_5 C_6^{\frac{1}{\alpha}}\{gt^{\frac{1}{\beta-1}}\}^{1-\beta-\frac{1}{\alpha}} - b^{-1}(\alpha m(\alpha m-1))C_6^{\frac{2}{\alpha}} g^{m-\beta-\frac{2}{\alpha}}$$

$$-b^{-1}c\alpha p C_6^{p-\beta} g^{p-\beta-\frac{1}{\alpha}}$$

Since

$$S_x \geq 0 \ \text{ in } \ G^+_{l,\sigma},$$

$$S \geq S|_{x=\eta_\ell(t)} = 1-(b(1-\beta))^{-1}\zeta_5 C_6^{1-\beta}(\ell-\zeta_5)^{\alpha(1-\beta)-1} - b^{-1}(\alpha m(\alpha m-1))C_6^{m-\beta}$$

$$\times\{(\ell-\zeta_5)t^{1/\alpha(1-\beta)}\}^{\alpha(m-\beta)-2} - b^{-1}a\alpha p C_6^{p-\beta}\{(\ell-\zeta_5)t^{1/(\alpha(1-\beta))}\}^{\alpha(p-\beta)-1}$$

Then we have

$$S \geq \epsilon - b^{-1}C_6^{m-\beta}(\alpha m(\alpha m-1))\{(\ell-\zeta_5)t^{1/\alpha(1-\beta)}\}^{\alpha(m-\beta)-2}$$

$$-b^{-1}c\alpha p C_6^{m-\beta}\{(\ell-\zeta_5)t^{1/\alpha(1-\beta)}\}^{\alpha(p-\beta)-1} \text{ in } G^+_{\ell,\sigma}.$$

where,

$$C_6 = \left[1-(\ell_*/\ell)^{\alpha(1-\beta)}(1-\epsilon)\right]^{-\alpha}\left[C^{1-\beta} - \ell^{-\alpha(1-\beta)}b(1-\beta)(1-\epsilon))\right]^{1/(1-\beta)},$$

$$\zeta_5 = (\ell_*/\ell)^{\alpha(1-\beta)}(1-\epsilon)\ell.$$

Hence, we can choose $\sigma = \sigma(\epsilon) > 0$ so small that

$$\text{L}g \ \geq \ b(\epsilon/2)g^\beta \ \text{ in } \ G^+_{\ell,\sigma}. \tag{4.43a}$$

Using (4.42), we can apply Comparison Theorem 2.3.1 in $G'_{\ell,\sigma} = G_{\ell,\sigma} \cap \{x < x_0\}$, for $\forall x_0 > 0$. We have

$$\text{L}g = 0 \ \text{ in } \ G'_{\ell,\sigma} \backslash \bar{G}^+_{\ell,\sigma}, \tag{4.43b}$$

$$u(\eta_\ell(t),t) \leq \left[C^{1-\beta}\ell^{\alpha(1-\beta)} - b(1-\beta)(1-\epsilon)\right]^{\frac{1}{1-\beta}} t^{\frac{1}{1-\beta}}$$

$$= C_6(\ell-\zeta_5)^\alpha t^{\frac{1}{(1-\beta)}} = g(\eta_\ell(t),t), \ 0 \leq t \leq \sigma. \tag{4.43c}$$

$$u(x_0,t) = g(x_0,t) = 0,\ 0 \le t \le \sigma,\quad u(x,0) \ = g(x,0) \ = 0,\ 0 \le x \le x_0. \quad (4.43d)$$

Since $x_0 > 0$ is arbitrary, from (4.43a)-(4.43d) and comparison principle it follows that for all $\ell > \ell_*$ and $\epsilon > 0$ there exists $\sigma = \sigma(\epsilon, \ell) > 0$ such that

$$u(x,t) \le C_6(-\zeta_5 t^{\frac{1}{\alpha(1-\beta)}} - x)_+^{\alpha} \ \text{ in } \ \bar{G}_{\ell,\sigma}. \quad (4.44)$$

Since (4.19) is valid along $x = \eta_\ell(t)$, σ may be chosen so small that

$$-\ell t^{1/\alpha(1-\beta)} \le \eta(t) \le -\zeta_5 t^{1/\alpha(1-\beta)},\ 0 \le t \le \sigma. \quad (4.45)$$

Since $\ell > \ell_*$ and $\epsilon > 0$ are arbitrary numbers, (4.18) follows from (4.45).

The theorem is proved. □

Chapter 5

Conclusions

- General theory of the one-dimensional reaction-diffusion-convection type nonlinear degenerate second order parabolic equations in general non-smooth and non-cylindrical time dependent domains is developed under the minimal regularity assumptions on the boundary curves. Cauchy-Dirichlet and Dirichlet problems with continuous initial-boundary functions are considered. The methods developed in *U.G. Abdulla, Journal of Differential Equations, 164, 2(2000), 321-354* for the reaction-diffusion equations are applied to prove the existence of weak solutions, boundary regularity, uniqueness and comparison theorems in the presence of convective flow.

- It is proved that the weak solutions are continuous up to the non-smooth boundary if at each interior point the left modulus of the lower (respectively upper) semicontinuity of the left (respectively right) boundary curve satisfies an upper (respectively lower) Hölder condition near zero with Hölder exponent $\nu > \frac{1}{2}$. The value $\frac{1}{2}$ is critical as in the classical theory of heat equation, and is independent of nonlinearity parameters, and from the degeneration or singularity of the PDE.

- Uniqueness and comparison theorems are proved under one-side Lipschitz condition on the boundary curves on the interior compact subsets of the time interval.

- General theory is applied to the problem on the initial development and asymptotics of the interfaces and local solutions near the interfaces for the reaction-diffusion-convection equation with compactly supported initial function. Depending on the relative strength of three competing forces such as diffusion, convection, and reaction, the interface may expand, shrink or remain stationary. The methods used are rescaling and blow-up techniques for the identification of the asymptotics of the solution along the class of interface type curves, construction of the barriers and application of the comparison theorem in non-cylindrical domains with characteristic boundary curves, as they are developed in papers *U.G. Abdulla & J.King, SIAM J. Math. Anal., 32, 2(2000), 235-260; U.G. Abdulla, Nonlinear Analysis, 50, 4(2002), 541-560.*

- The results of the dissertation are applicable to wide variety of physical, chemical and biological problems involving diffusion with a source or absorption, and accompanied with additional convective flow as for instance in modeling filtration in porous media, transport of thermal energy in a plasma, flow of a chemically reacting fluid from a flat surface, evolution of populations.

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