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Research Article

An Existence Principle for Nonlocal Difference Boundary Value Problems with φ-Laplacian and Its Application to Singular Problems

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The paper presents an existence principle for solving a large class of nonlocal regular discrete boundary value problems with the φ-Laplacian. Applications of the existence principle to singular discrete problems are given.

1. Introduction

Let $\mathbb{R}_+ = (0, \infty)$ and let $\mathbb{Z}$ denote the set of all integers. If $a, b \in \mathbb{Z}$, $a < b$, then $\mathbb{T}[a, b]$ denotes the discrete interval $\{a, a+1, \ldots, b\}$. Let $\Delta u(k) = u(k+1) - u(k)$ be the forward difference operator.

Let $T, N \in \mathbb{Z}$, $T < N$, and let X stand for the space of functions $u : \mathbb{T}[T-1, N+1] \to \mathbb{R}$ equipped with the norm $\|u\| = \max\{|u(k)| : k \in \mathbb{T}[T-1, N+1]\}$. Clearly, X is an $(N-T+3)$-dimensional Banach space.

Denote by $\mathcal{A}$ the set of continuous maps $\gamma : X \to \mathbb{R}$. We say that $\alpha, \beta \in \mathcal{A}$ *are compatible* if for each $\mu \in [0, 1]$ the problem

$$\Delta(\phi(\Delta u(k-1))) = 0, \quad k \in \mathbb{T}[T, N], \tag{1.1}$$

$$\alpha(u) - \mu\alpha(-u) = 0, \qquad \beta(u) - \mu\beta(-u) = 0 \tag{1.2}$$

has a solution; that is, there exists a function $u : \mathbb{T}[T-1, N+1] \to \mathbb{R}$ such that equality (1.1) holds for $k \in \mathbb{T}[T, N]$ and u satisfies (1.2). Here ϕ fulfils the following condition:

(H_1) $\phi \in C(\mathbb{R})$ is increasing such that $\phi(0) = 0$, $\phi(\mathbb{R}) = \mathbb{R}$.

Remark 1.1. It is easy to see that $u : \mathbb{T}[T-1, N+1] \to \mathbb{R}$ is a solution of (1.1) if and only if $\Delta u(k) = B$ for $k \in \mathbb{T}[T-1, N]$, where $B \in \mathbb{R}$. Hence u is a solution of (1.1) if and only if $u(k) = A + Bk$ for $k \in \mathbb{T}[T-1, N-1]$, where $A, B \in \mathbb{R}$. Consequently, problem (1.1)-(1.2) has a solution if and only if the system

$$\begin{aligned} \alpha(A+Bk) - \mu\alpha(-A-Bk) &= 0, \\ \beta(A+Bk) - \mu\beta(-A-Bk) &= 0 \end{aligned} \tag{1.3}$$

has a solution $(A, B) \in \mathbb{R}^2$. If $\alpha, \beta \in \mathcal{A}$ are linear, then system (1.3) has the form

$$\begin{aligned} A\alpha(1) + B\alpha(k) &= 0, \\ A\beta(1) + B\beta(k) &= 0 \end{aligned} \tag{1.4}$$

for each $\mu \in [0, 1]$.

Remark 1.2. Due to Remark 1.1, $\alpha, \beta \in \mathcal{A}$ are compatible if system (1.3) has a solution $(A, B) \in \mathbb{R}^2$ for each $\mu \in [0, 1]$. If α, β are linear, then they are compatible. Indeed, system (1.3) has the form of (1.4) for each $\mu \in \mathbb{R}$ and is always solvable in $\mathbb{R}^2$ because $(A, B) = (0, 0)$ is its solution.

Let ϕ satisfy (H_1) and let $h \in C(\mathbb{T}[T, N] \times \mathbb{R}^2)$. We discuss the nonlocal difference problem

$$\Delta(\phi(\Delta u(k-1))) = h(k, u(k), \Delta u(k)), \quad k \in \mathbb{T}[T, N], \tag{1.5}$$

$$\alpha(u) = 0, \quad \beta(u) = 0, \quad \alpha, \beta \in \mathcal{A}, \tag{1.6}$$

where α, β are compatible. We say that $u : \mathbb{T}[T-1, N+1] \to \mathbb{R}$ is *a solution of problem* (1.5)-(1.6) if u fulfils (1.6) and equality (1.5) holds for $k \in \mathbb{T}[T, N]$.

The first aim of this paper is to present an existence principle for solving problem (1.5)-(1.6) and the second aim is to give applications of this principle to singular problems with the ϕ-Laplacian, which include as special cases the Dirichlet problem and the mixed problem.

Singular discrete Dirichlet problems of the type

$$\begin{aligned} -\Delta\left(\phi_p(\Delta u(k-1))\right) = f(k, u(k)), \quad k \in \mathbb{T}[1, T], \\ u(0) = 0, \qquad u(T+1) = 0 \end{aligned} \tag{1.7}$$

were studied with $p = 2$ in [1] and [2–4], where $\phi_p(x) = |x|^{p-2}x$ $(p > 1)$ is the p-Laplacian, $f \in C(\mathbb{T}[1, T] \times (0, \infty))$, and $f(k, x)$ may be singular at $x = 0$. The existence of positive solutions is proved by variational methods [2] and by a combination of the lower and upper solutions method with a nonlinear alternative of Leray-Schauder type [1, 4] and an inequality theory [3]. In [1], the function f is nonnegative, while in [2–4] it may change sign. The paper [2] discusses also multiple positive solutions. The existence of multiple positive solutions is investigated also in [5, 6].

The paper [7] deals with the singular mixed problem

$$\begin{aligned} \Delta\left(\phi_p(\Delta u(k-1))\right) + f(k, u(k), \Delta u(k-1)) = 0, \quad k \in \mathbb{T}[1, T+1], \\ \Delta u(0) = 0, \qquad u(T+2) = 0, \end{aligned} \tag{1.8}$$

where $f \in C(\mathbb{T}[1, T+1] \times (0, \infty) \times \mathbb{R})$ and $f(k, x, y)$ may be singular at $x = 0$. The existence of a positive solution is proved by a combination of the lower and upper functions method with the Brouwer fixed-point theorem.

The rest of the paper is organized as follows. In Section 2, we present an existence principle for solving the discrete problem (1.5)-(1.6) (see Theorem 2.1). This principle is proved using the Brouwer degree and the Borsuk antipodal theorem (see, e.g., [8]). Notice that an analogous principle for continuous regular nonlocal problems with the ϕ-Laplacian was presented in [9, Theorem 2.1]. Section 3 is devoted to applications of the existence principle. We discuss the existence of positive solutions of the difference equation with the ϕ-Laplacian

$$\Delta(\phi(\Delta u(k-1))) = f(k, u(k), \Delta u(k)) \tag{1.9}$$

satisfying two types of nonlocal boundary conditions which include as special cases the Dirichlet problem and the mixed problem. Here f is continuous and $f(k, x, y)$ may be singular at $y = 0$. The existence of positive solutions is proved by a combination of regularization and sequential techniques with our existence principle. The results are demonstrated with examples.

2. Existence principle

The following theorem is an existence principle for problem (1.5)-(1.6).

Theorem 2.1. *Let (H_1) hold. Let $h \in C(\mathbb{T}[T, N] \times \mathbb{R}^2)$ and let $\alpha, \beta \in \mathcal{A}$ be compatible. Suppose that there exists a positive constant S independent of λ such that*

$$\|u\| < S \tag{2.1}$$

for any solution u of the problem

$$\begin{gathered}\Delta(\phi(\Delta u(k-1))) = \lambda h(k, u(k), \Delta u(k)), \quad \lambda \in [0,1], \\ \alpha(u) = 0, \qquad \beta(u) = 0.\end{gathered} \tag{2.2}$$

Also assume that there exists a positive constant Λ such that

$$\max\{|A|, |B|\} < \Lambda \tag{2.3}$$

for all solutions $(A, B) \in \mathbb{R}^2$ of system (1.3) for each $\mu \in [0,1]$.

Then problem (1.5)-(1.6) has a solution.

Proof. Put $L = (1 + \max\{|T-1|, |N+1|\}\Lambda$ and

$$\Omega = \{u \in X : \|u\| < \max\{S, L\}\}. \tag{2.4}$$

Then Ω is an open, bounded, and symmetric subset of the Banach space X with respect to $0 \in X$. Define an operator $\mathcal{P} : [0,1] \times \overline{\Omega} \to X$ by the formula

$$\mathcal{P}(\lambda, u)(k) = \sum_{j=T}^{k} \phi^{-1}\left(\phi(\Delta u(T-1) + \beta(u)) + \lambda \sum_{s=T}^{j-1} h(s, u(s), \Delta u(s))\right) + u(T-1) + \alpha(u) \tag{2.5}$$

for $k \in \mathbb{T}[T, N]$, where $\sum_{i=T}^{T-1} = 0$. It follows from the continuity of the functions ϕ, ϕ^{-1}, f and the maps α, β that $\mathcal{P}$ is a continuous operator. Suppose that u is a fixed point of $\mathcal{P}(\lambda, \cdot)$ for some $\lambda \in [0,1]$. Then

$$u(k) = \sum_{j=T}^{k} \phi^{-1}\left(\phi(\Delta u(T-1) + \beta(u)) + \lambda \sum_{s=T}^{j-1} h(s, u(s), \Delta u(s))\right) + u(T-1) + \alpha(u) \tag{2.6}$$

for $k \in \mathbb{T}[T,N]$. We set $k = T-1$ and $k = T$ in (2.6), and have $u(T-1) = u(T-1) + \alpha(u)$ and $u(T) = \Delta u(T-1) + \beta(u) + u(T-1) + \alpha(u)$. Hence $\alpha(u) = 0$ and $\beta(u) = 0$, which means that u satisfies the boundary conditions (1.6). In addition,

$$\Delta u(k) = u(k+1) - u(k) = \phi^{-1}\left(\phi(\Delta u(T-1) + \beta(u)) + \lambda \sum_{s=T}^{k} h(s, u(s), \Delta u(s))\right), \tag{2.7}$$

and consequently

$$\Delta(\phi(\Delta u(k-1))) = \phi(\Delta u(k)) - \phi(\Delta u(k-1)) = \lambda h(k, u(k), \Delta u(k)) \tag{2.8}$$

for $k \in \mathbb{T}[T,N]$. Hence u is a solution of the equation in (2.2). We have proved that for each $\lambda \in [0,1]$ any fixed point of the operator $\mathcal{P}(\lambda,\cdot)$ is a solution of problem (2.2). In particular, any fixed point of $\mathcal{P}(1,\cdot)$ is a solution of problem (1.5)-(1.6). In order to prove the solvability of problem (1.5)-(1.6), it suffices to show, by the Brouwer degree theory, that

$$d(\mathcal{I} - \mathcal{P}(1,\cdot), \Omega, 0) \neq 0, \tag{2.9}$$

where "d" stands for the Brouwer degree and $\mathcal{I}$ is the identical operator on X. We know that $\mathcal{P}$ is a continuous operator and, by the assumptions of our theorem, for each $\lambda \in [0,1]$ and any fixed point u of $\mathcal{P}(\lambda,\cdot)$ the estimate (2.1) is true with a positive constant S. Hence for each $\lambda \in [0,1]$, the operator $\mathcal{P}(\lambda,\cdot)$ is fixed point free on the boundary $\partial\Omega$ of Ω. Consequently, by the homotopy property,

$$d(\mathcal{I} - \mathcal{P}(1,\cdot), \Omega, 0) = d(\mathcal{I} - \mathcal{P}(0,\cdot), \Omega, 0). \tag{2.10}$$

We now define an operator $\mathcal{L} : [0,1] \times \overline{\Omega} \to X$ by the formula

$$\mathcal{L}(\mu,u)(k) = \begin{cases} u(T-1) + \alpha(u) - \mu\alpha(-u) \\ \quad +(k+1-T)[\Delta u(T-1) + \beta(u) - \mu(\beta(-u))] \\ \qquad\qquad\qquad \text{for } k \in \mathbb{T}[T-1, N+1]. \end{cases} \tag{2.11}$$

The operator $\mathcal{L}$ is continuous because of the continuity of α, β. In addition, $\mathcal{L}(0,\cdot) = \mathcal{P}(0,\cdot)$ and $\mathcal{L}(1,\cdot)$ is an odd operator, that is, $\mathcal{L}(1,-u) = -\mathcal{L}(1,u)$ for $u \in \overline{\Omega}$. Suppose that u_0 is a fixed point of $\mathcal{L}(\mu,\cdot)$ for some $\mu \in [0,1]$. Then

$$u_0(k) = \begin{cases} u_0(T-1) + \alpha(u_0) - \mu\alpha(-u_0) \\ \quad +(k+1-T)[\Delta u_0(T-1) + \beta(u_0) - \mu(\beta(-u_0))] \\ \qquad\qquad\qquad \text{for } k \in \mathbb{T}[T-1, N+1]. \end{cases} \tag{2.12}$$

Therefore

$$u_0(T-1) = u_0(T-1) + \alpha(u_0) - \mu\alpha(-u_0), \tag{2.13}$$

$$u_0(T) = u_0(T-1) + \alpha(u_0) - \mu\alpha(-u_0) + \Delta u_0(T-1) + \beta(u_0) - \mu\beta(-u_0), \tag{2.14}$$

$$u_0(k+1) - u_0(k) = \Delta u_0(T-1) + \beta(u_0) - \mu\beta(-u_0), \quad k \in \mathbb{T}[T,N]. \tag{2.15}$$

Then, by (2.13) and (2.14),

$$\alpha(u_0) - \mu\alpha(-u_0) = 0, \qquad \beta(u_0) - \mu\beta(-u_0) = 0, \tag{2.16}$$

which combined with (2.15) yield $\Delta u_0(k) = \Delta u_0(T-1)$ for $k \in \mathbb{T}[T,N]$. Hence

$$u_0(k) = A + kB \quad \text{for } k \in \mathbb{T}[T-1, N+1], \tag{2.17}$$

where $A = u_0(T-1) + (1-T)\Delta u_0(T-1)$ and $B = \Delta u_0(T-1)$. It follows from (2.16) and (2.17) that (A,B) is a solution of system (1.3) and therefore $\max\{|A|,|B|\} < \Lambda$ by the assumptions of our theorem. From this we conclude that $\|u_0\| < (1+\max\{|T-1|,|N+1|\}\Lambda$. As a result for each $\mu \in [0,1]$ and any fixed point u of $\mathcal{L}(\mu,\cdot)$, we have $u \notin \partial\Omega$. Hence, by the Borsuk antipodal theorem and the homotopy property,

$$d(\mathcal{I} - \mathcal{L}(1,\cdot), \Omega, 0) \neq 0, \qquad d(\mathcal{I} - \mathcal{L}(0,\cdot), \Omega, 0) = d(\mathcal{I} - \mathcal{L}(1,\cdot), \Omega, 0). \tag{2.18}$$

Relation (2.9) follows from $\mathcal{L}(0,\cdot) = \mathcal{P}(0,\cdot)$ and from (2.10) and (2.18). □

3. Applications of the existence principle

Theorem 2.1 presents an existence principle which can be used for a large class of nonlocal boundary value problems. In this section, we apply Theorem 2.1 to prove the existence of positive solutions of a generalized singular Dirichlet problem and a generalized singular mixed problem. Both of these problems are called "generalized" since by the special choice of their boundary conditions we obtain the Dirichlet conditions $u(-N-1) = C$, $u(N+1) = C$ and the mixed conditions $\Delta u(0) = 0$, $u(N+1) = C$.

3.1. Generalized singular Dirichlet problem

Denote by $\mathcal{C}_1$ the set of functions $q \in C(\mathbb{R}^2)$ such that

(i) $q(x,y)$ is increasing in x and nondecreasing in y,

(ii) $q(x,y) = -q(-x,-y)$ for $(x,y) \in \mathbb{R}^2$,

(iii) $\lim_{x\to\infty} q(x,0) = \infty$.

It is obvious that for each $q \in \mathcal{C}_1$ we have $q(0,0) = 0$ and $q(x,y) > 0$ for $(x,y) \in \mathbb{R}^2_+$.

Let $N \geq 1$ be a positive integer. We discuss the singular boundary value problem

$$\Delta(\phi(\Delta u(k-1))) = f(k, u(k), \Delta u(k)), \quad k \in \mathbb{T}[-N,N], \tag{3.1}$$

$$\begin{gathered} q(u(-N-1), -\Delta u(-N-1)) = C, \\ q(u(N+1), \Delta u(N)) = C, \quad q \in \mathcal{C}_1,\ C > 0, \end{gathered} \tag{3.2}$$

where ϕ satisfies (H_1) and f satisfies the condition

(H_2) $f \in C(\mathbb{T}[-N,N] \times \mathfrak{D})$, $\mathfrak{D} = [0,\infty) \times (\mathbb{R} \setminus \{0\})$, $f(k,x,y) > 0$ for $k \in \mathbb{T}[-N,N]$, $(x,y) \in \mathbb{R}_+ \times (\mathbb{R} \setminus \{0\})$, $f(k,0,y) = 0$ for $k \in \mathbb{T}[-N,N]$, $y \in \mathbb{R} \setminus \{0\}$, and for each $k \in \mathbb{T}[-N,N]$, $\lim_{y\to 0} f(k,x,y) = \infty$ locally uniformly on $\mathbb{R}_+$.

We say that $u \in \mathbb{T}[-N-1, N+1] \to \mathbb{R}$ is *a solution of problem* (3.1)-(3.2) if u satisfies the boundary conditions (3.2) and fulfils equality (3.1) for $k \in \mathbb{T}[-N, N]$.

Notice that a special case of the boundary conditions (3.2) is the Dirichlet conditions $u(-N-1) = C$, $u(N+1) = C$ which we get by setting $q(x, y) = x$.

We apply sequential and regularization methods to show the existence of a solution of problem (3.1)-(3.2). To this end, for each $n \in \mathbb{N}$ define $f_n \in C(\mathbb{T}[-N, N] \times \mathbb{R}^2)$ by the formula

$$f_n(k, x, y) = \begin{cases} f_*(k, x, y) \quad \text{for } k \in \mathbb{T}[-N, N],\ (x, y) \in \mathbb{R} \times \left(\mathbb{R} \setminus \left[-\dfrac{1}{n}, \dfrac{1}{n}\right]\right), \\ \dfrac{n}{2}\left[f_*\left(k, x, \dfrac{1}{n}\right)\left(y + \dfrac{1}{n}\right) - f_*\left(k, x, -\dfrac{1}{n}\right)\left(y - \dfrac{1}{n}\right)\right] \\ \qquad \text{for } k \in \mathbb{T}[-N, N],\ (x, y) \in \mathbb{R} \times \left[-\dfrac{1}{n}, \dfrac{1}{n}\right], \end{cases} \tag{3.3}$$

where

$$f_*(k, x, y) = \begin{cases} f(k, x, y) & \text{for } k \in \mathbb{T}[-N, N],\ (x, y) \in \mathfrak{D}, \\ 0 & \text{for } k \in \mathbb{T}[-N, N],\ (x, y) \in (-\infty, 0) \times (\mathbb{R} \setminus \{0\}). \end{cases} \tag{3.4}$$

If condition (H_2) holds, then

$$f_n(k, x, y) > 0 \quad \text{for } k \in \mathbb{T}[-N, N],\ (x, y) \in \mathbb{R}_+ \times \mathbb{R}, \tag{3.5}$$

$$f_n(k, x, y) = 0 \quad \text{for } k \in \mathbb{T}[-N, N],\ (x, y) \in (-\infty, 0] \times \mathbb{R}, \tag{3.6}$$

$$\lim_{n \to \infty} f_n(k, x, y) = f(k, x, y) \quad \text{for } k \in \mathbb{T}[-N, N],\ (x, y) \in [0, \infty) \times (\mathbb{R} \setminus \{0\}). \tag{3.7}$$

Throughout this section, X denotes the Banach space of functions $u : \mathbb{T}[-N-1, N+1] \to \mathbb{R}$ with the norm $\|u\| = \max\{|u(k)| : k \in \mathbb{T}[-N-1, N+1]\}$.

Keeping in mind the boundary conditions (3.2), put

$$\begin{aligned} \alpha(u) &= q(u(-N-1), -\Delta u(-N-1)) - C, \\ \beta(u) &= q(u(N+1), \Delta u(N)) - C, \quad q \in \mathcal{C}_1,\ C > 0, \end{aligned} \tag{3.8}$$

for $u \in X$. Then $\alpha, \beta \in \mathcal{A}$ and we can write the boundary conditions (3.2) in the form of (1.6).

Lemma 3.1. *Let $\alpha, \beta \in \mathcal{A}$ be defined in* (3.8). *Then for each $\mu \in [0, 1]$ system* (1.3) *has a unique solution $(A, B) \in \mathbb{R}^2$ and there exists a positive constant Λ independent of μ such that*

$$\max\{|A|, |B|\} < \Lambda. \tag{3.9}$$

Proof. Using property (ii) of $q \in \mathcal{C}_1$ we can write system (1.3) in the form

$$\begin{aligned} q(A - (N+1)B, -B) &= \frac{(1-\mu)C}{1+\mu}, \\ q(A + (N+1)B, B) &= \frac{(1-\mu)C}{1+\mu}. \end{aligned} \tag{3.10}$$

Suppose that some $(A,B) \in \mathbb{R}^2$ is a solution of (3.10). If $B \neq 0$, then $q(A-(N+1)B,-B) \neq q(A+(N+1)B,B)$ due to property (i) of functions belonging to the set $\mathcal{C}_1$, which is impossible. Hence $B=0$ and $q(A,0)=(1-\mu)C/(1+\mu)$. Put

$$p(x) = q(x,0) \quad \text{for } x \in \mathbb{R}. \tag{3.11}$$

Then $p \in C(\mathbb{R})$ is increasing and odd on $\mathbb{R}$ and $\lim_{x\to\infty} p(x) = \infty$. Therefore $A = p^{-1}((1-\mu)C/(1+\mu))$ is the unique solution of the equation $q(A,0) = (1-\mu)C/(1+\mu)$. It is easy to check that $(A,B) = (p^{-1}((1-\mu)C/(1+\mu)),0)$ is a solution of system (1.3) for each $\mu \in [0,1]$. This proves that system (1.3) has the unique solution $(A,B) = (p^{-1}((1-\mu)C/(1+\mu)),0)$ for each $\mu \in [0,1]$. It follows from the inequality $0 \le p^{-1}((1-\mu)C/(1+\mu)) \le P^{-1}(C)$ that (A,B) fulfils the estimate (3.9) with $\Lambda = p^{-1}(C)+1$. □

Remark 3.2. Due to Lemma 3.1 and Remark 1.2 the boundary conditions (3.2) are compatible.

The following result gives the properties of solutions to a regular problem depending on a parameter λ.

Lemma 3.3. *Let* (H_1) *and* (H_2) *hold. Let* u *be a solution of the equation*

$$\Delta(\phi(\Delta u(k-1))) = \lambda f_n(k,u(k),\Delta u(k)), \quad k \in \mathbb{T}[-N,N],\ \lambda \in (0,1], \tag{3.12}$$

fulfilling the boundary conditions (3.2). *Then there exists a positive constant* S *independent of* n *and* λ *such that*

$$0 < u(k) < S \quad \text{for } k \in \mathbb{T}[-N-1,N+1], \tag{3.13}$$

$$\Delta u(k-1) < \Delta u(k) \quad \text{for } k \in \mathbb{T}[-N,N], \tag{3.14}$$

$$\Delta u(-N-1) < 0, \qquad \Delta u(N) > 0. \tag{3.15}$$

Proof. Suppose that $u(N+1) \le 0$. If $\Delta u(N) \le 0$, then $q(u(N+1),\Delta u(N)) \le q(0,0)=0$, contrary to $q(u(N+1),\Delta u(N)) = C > 0$. Hence $\Delta u(N) > 0$ and therefore $u(N) < u(N+1) \le 0$, which gives $\Delta(\phi(\Delta u(N-1))) = 0$ because $f_n(N,u(N),\Delta u(N)) = 0$ by (3.6). It follows from $\Delta(\phi(\Delta u(N-1))) = 0$, $\Delta u(N) > 0$, and from condition (H_1) that $\Delta u(N-1) = \Delta u(N) > 0$, and consequently $u(N-1) < u(N) < 0$. Applying the above arguments repeatedly, we get $\Delta u(j) = \Delta u(N)$ for $j \in \mathbb{T}[-N-1,N]$. Then $\Delta u(-N-1) > 0$ and $u(-N-1) < u(N) < 0$, which yields $q(u(-N-1),-\Delta u(-N-1)) < 0$, contrary to $q(u(-N-1),-\Delta u(-N-1)) = C > 0$ by (3.2). Hence $u(N+1) > 0$. Suppose that there exists $j \in \mathbb{T}[-N-1,N]$ such that $u(j) \le 0$ and $u(j+1) > 0$. If $j > -N-1$, then $\Delta(\phi(\Delta u(j-1))) = \lambda f_n(j,u(j),\Delta u(j)) = 0$ and therefore $\Delta u(j-1) = \Delta u(j)$, which gives $u(j-1) < u(j)$ because $\Delta u(j) > 0$. Essentially, the same reasoning as in the above part of the proof yields $\Delta u(k) = \Delta u(j) > 0$ for $k \in \mathbb{T}[-N-1,j]$. In particular, $u(-N-1) < u(j) \le 0$ and $\Delta u(-N-1) > 0$. Consequently, $q(u(-N-1),-\Delta u(-N-1)) < 0$, which is impossible by (3.2). If $j = -N-1$, then $u(-N-1) \le 0$ and $\Delta(-N-1) > 0$, which gives $q(u(-N-1),-\Delta u(-N-1)) \le 0$, contrary to (3.2). We have

$$u(k) > 0 \quad \text{for } k \in \mathbb{T}[-N-1,N+1]. \tag{3.16}$$

Then $f_n(k,u(k),\Delta u(k)) > 0$ for $k \in \mathbb{T}[-N,N]$ by (3.5) and so $\Delta(\phi(\Delta u(k-1))) > 0$ for these k, which means that inequality (3.14) is true.

We now prove that inequality (3.15) holds. Suppose that $\Delta u(-N-1) \geq 0$. Then $\Delta u(k) > \Delta u(-N-1) \geq 0$ for $k \in \mathbb{T}[-N, N]$ by (3.14) and $u(N+1) - u(-N-1) = \sum_{k=-N}^{N} \Delta u(k) > 0$. In particular, $\Delta u(N) > 0$ and

$$u(N+1) > u(-N-1). \tag{3.17}$$

Hence $C = q(u(-N-1), -\Delta u(-N-1)) \leq q(u(-N-1), 0)$, $C = q(u(N+1), \Delta u(N)) \geq q(u(N+1), 0)$. Therefore $q(u(-N-1), 0) \geq q(u(N+1), 0)$, which contradicts (3.17), because the function $q(\cdot, 0)$ is increasing on $\mathbb{R}$. We have shown that the first inequality in (3.15) holds. In order to prove that the second inequality in (3.15) is true we assume, on the contrary, that $\Delta u(N) \leq 0$. By (3.14), $\Delta u(k) < \Delta u(N) \leq 0$ for $k \in \mathbb{T}[-N-1, N-1]$ and so $u(N+1) - u(-N-1) = \sum_{k=-N}^{N} \Delta u(k) < 0$. It follows from $C = q(u(-N-1), -\Delta u(-N-1)) \geq q(u(-N-1), 0)$ and $C = q(u(N+1), \Delta u(N)) \leq q(u(N+1), 0)$ that $q(u(-N-1), 0) \leq q(u(N+1), 0)$, which contradicts $u(N+1) < u(-N-1)$, because $q(\cdot, 0)$ is increasing on $\mathbb{R}$.

It remains to prove that $u(k) < S$ for $k \in \mathbb{T}[-N-1, N+1]$, where S is a positive constant independent of n and λ. We see from (3.14) and (3.15) that there exists $j \in \mathbb{T}[-N, N-1]$ such that

$$\Delta u(k) < 0 \quad \text{for } k \in \mathbb{T}[-N-1, j-1], \qquad \Delta u(k) > 0 \quad \text{for } k \in \mathbb{T}[j+1, N]. \tag{3.18}$$

Hence $u(k) \leq \max\{u(-N-1), u(N+1)\}$ for $k \in \mathbb{T}[-N-1, N+1]$. We conclude from $C = q(u(-N-1), -\Delta u(-N-1)) \geq q(u(-N-1), 0)$, $C = q(u(N+1), \Delta u(N)) \geq q(u(N+1), 0)$ that $q(u(-N-1), 0) \leq C$, $q(u(N+1), 0) \leq C$, and consequently $\max\{u(-N-1), u(N+1)\} \leq p^{-1}(C)$, where p^{-1} is the inverse function to p given in (3.11). Therefore estimate (3.13) holds with $S = p^{-1}(C) + 1$. □

Remark 3.4. Problem (3.12)–(3.2) with $\lambda = 0$ has the unique solution u, $u(k) = p^{-1}(C)$, for $k \in \mathbb{T}[-N-1, N+1]$, where p is given in (3.11). This fact follows from Remark 1.1 and from the proof of Lemma 3.1 with $\mu = 0$.

The next lemma gives an existence result for problem (3.19)–(3.2), where

$$\Delta(\phi(\Delta u(k-1))) = f_n(k, u(k), \Delta u(k)), \quad k \in \mathbb{T}[-N, N]. \tag{3.19}$$

Lemma 3.5. *Let (H_1) and (H_2) hold. Then for each $n \in \mathbb{N}$ there exists a solution of problem* (3.19)–(3.2) *and any of its solutions u_n fulfils the inequalities*

$$0 < u_n(k) < S \quad \text{for } k \in \mathbb{T}[-N-1,\ N+1], \tag{3.20}$$

where S is a positive constant independent of n, and

$$\Delta u_n(k-1) < \Delta u_n(k) \quad \text{for } k \in \mathbb{T}[-N, N], \tag{3.21}$$

$$\Delta u_n(-N-1) < 0, \qquad \Delta u_n(N) > 0. \tag{3.22}$$

Proof. Let us choose $n \in \mathbb{N}$. Put $h(k, x, y) = f_n(k, x, y)$ for $k \in \mathbb{T}[-N, N]$, $(x, y) \in \mathbb{R}^2$ and let $\alpha, \beta \in \mathcal{A}$ be given in (3.8). By Remark 3.2, the boundary conditions (3.2) are compatible. Due to Lemma 3.3 and Remark 3.4 there exists a positive constant S such that $\|u\| < S$ for all solutions u of problem (2.2). By Lemma 3.1, there exists a positive constant Λ such that estimate (3.9) is true for any solutions $(A, B) \in \mathbb{R}^2$ of problem (1.3) for each $\mu \in [0, 1]$. Hence the conditions of Theorem 2.1 are satisfied and therefore problem (3.19)–(3.2) has a solution. In addition, any of its solutions u_n fulfils inequalities (3.20)–(3.22) by Lemma 3.3. □

The main existence result for problem (3.1)-(3.2) is given in the following theorem.

Theorem 3.6. *Let* (H_1) *and* (H_2) *hold. The problem* (3.1)-(3.2) *has a solution* u *and* $u(k) > 0$ *for* $k \in \mathbb{T}[-N-1, N+1]$.

Proof. By Lemma 3.5, for each $n \in \mathbb{N}$ there exists a solution u_n of problem (3.19)–(3.2) satisfying inequalities (3.20)–(3.22). As a result, the sequence $\{u_n(k)\}$ is bounded for $k \in \mathbb{T}[-N-1, N+1]$, and therefore by the Bolzano-Weierstrass compactness theorem, there exist a subsequence $\{\ell_n\}$ of $\{n\}$ and some $u \in X$ such that $\lim_{n\to\infty} u_{\ell_n} = u$. Letting $n \to \infty$ in (3.20)–(3.22) (with ℓ_n instead of n) and in the boundary conditions $q(u_{\ell_n}(-N-1), -\Delta u_{\ell_n}(-N-1)) = C$, $q(u_{\ell_n}(N+1), -\Delta u_{\ell_n}(N1)) = C$, we obtain

$$0 \le u(k) \le S \quad \text{for } k \in \mathbb{T}[-N-1, N+1], \tag{3.23}$$

$$\Delta u(k-1) \le \Delta u(k) \quad \text{for } k \in \mathbb{T}[-N, N], \tag{3.24}$$

$$\Delta u(-N-1) \le 0, \qquad \Delta u(N) \ge 0, \tag{3.25}$$

and u satisfies the boundary conditions (3.2).

If $u(N+1) = 0$, then $u(N) = -\Delta u(N)$, and since $u(N) \ge 0$ by (3.23) and $\Delta u(N) \ge 0$ by (3.25), we have $\Delta u(N) = 0$. Hence $q(u(N+1), \Delta u(N)) = q(0,0) = 0$, contrary to (3.2). We have $u(N+1) > 0$. In order to prove that $u(k) > 0$ for $k \in \mathbb{T}[-N-1, N]$ we first assume that there exists $j \in \mathbb{T}[-N, N]$ such that $u(j) = 0$ and $u(k) > 0$ for $k \in \mathbb{T}[j+1, N+1]$. Then $\Delta u(j) > 0$ and therefore

$$\lim_{n\to\infty} \Delta\big(\phi\big(\Delta u_{\ell_n}(j-1)\big)\big) = \lim_{n\to\infty} f_{\ell_n}\big(j, u_{\ell_n}(j), \Delta u_{\ell_n}(j)\big) = f(j, 0, \Delta u(j)) = 0, \tag{3.26}$$

by (3.7) and (H_2). Since $\lim_{n\to\infty}\Delta(\phi(\Delta u_{\ell_n}(j-1))) = \Delta(\phi(\Delta u(j-1)))$, we have $\Delta(\phi(\Delta u(j-1))) = 0$. Consequently, $\Delta u(j-1) = \Delta u(j) > 0$, which contradicts $u(j-1) = -\Delta u(j-1) < 0$ and (3.23). We have proved that $u(k) > 0$ for $k \in \mathbb{T}[-N, N+1]$. If $u(-N-1) = 0$, then it follows from $u(-N) \ge 0$, and $\Delta u(-N-1) \le 0$ by (3.23) and (3.25) that $u(-N) = 0$, $\Delta u(-N-1) = 0$, and consequently $q(u(-N-1), \Delta u(-N-1)) = q(0,0) = 0$, contrary to (3.2). Hence $u(-N-1) > 0$. To summarize, we have

$$u(k) > 0 \quad \text{for } k \in [-N-1,\ N+1]. \tag{3.27}$$

We now prove that

$$\Delta u(k) \ne 0 \quad \text{for } k \in [-N, N]. \tag{3.28}$$

On the contrary, suppose that $\Delta u(j) = 0$ for some $j \in \mathbb{T}[-N, N]$. Then $\lim_{n\to\infty} f_{\ell_n}(j, u_{\ell_n}(j), \Delta u_{\ell_n}(j)) = \infty$ by (H_2) since $\lim_{n\to\infty} u_{\ell_n}(j) = u(j) > 0$ and $(\ell_n/2)\max\{\Delta u_{\ell_n}(j) + 1/\ell_n, -\Delta u_{\ell_n}(j) + 1/\ell_n\} \ge 1/2$ for each n such that $|\Delta u_{\ell_n}(j)| \le 1/\ell_n$. Therefore $\lim_{n\to\infty}\Delta(\phi(\Delta u_{\ell_n}(j-1))) = \lim_{n\to\infty} f_{\ell_n}(j, u_{\ell_n}(j), \Delta u_{\ell_n}(j)) = \infty$, which contradicts $\lim_{n\to\infty}\Delta(\phi(\Delta u_{\ell_n}(j-1))) = \Delta(\phi(\Delta u(j-1))) \in \mathbb{R}$.

Keeping in mind (3.27) and (3.28), we have

$$\begin{aligned} \Delta(\phi(\Delta u(k-1))) &= \lim_{n\to\infty} \Delta\big(\phi\big(\Delta u_{\ell_n}(k-1)\big)\big) \\ &= \lim_{n\to\infty} f_{\ell_n}\big(k, u_{\ell_n}(k), \Delta u_{\ell_n}(k)\big) \\ &= f(k, u(k), \Delta u(k)) \end{aligned} \tag{3.29}$$

for $k \in \mathbb{T}[-N, N]$, which means that u is a solution of (3.1). Hence u is a positive solution of problem (3.1)-(3.2). □

Example 3.7. Let $a, b, c \in \mathbb{R}_+$, $\mu \geq 0$, and $n \in \mathbb{N}$. Then $f(k,x,y) = e^k \arctan x + x^a + x^b/|y|^c$, $k \in \mathbb{T}[-N,N]$, $(x,y) \in [0,\infty)\times(\mathbb{R}\setminus\{0\})$, satisfies condition (H$_2$) and $q(x,y) = x^{2n-1}+\mu(e^y-e^{-y})$, $(x,y) \in \mathbb{R}^2$, belongs to the set $\mathcal{C}_1$. If ϕ fulfils (H$_1$) then, by Theorem 3.6, the singular equation

$$\Delta(\phi(\Delta u(k-1))) = e^k \arctan(u(k)) + (u(k))^a + \frac{(u(k))^b}{|\Delta u(k)|^c}, \quad k \in \mathbb{T}[-N,N], \tag{3.30}$$

has a positive solution fulfilling the boundary conditions

$$\begin{gathered}(u(-N-1))^{2n-1} + \mu\left(e^{-\Delta u(-N-1)} - e^{\Delta u(-N-1)}\right) = C,\\ (u(N+1))^{2n-1} + \mu\left(e^{\Delta u(N)} - e^{-\Delta u(N)}\right) = C, \quad C > 0.\end{gathered} \tag{3.31}$$

3.2. Generalized singular mixed problem

In this section, $N \in \mathbb{N}$, $N > 1$. Denote by $\mathcal{C}_2$ the set of functions $Q \in C(\mathbb{R}^{N+1})$ such that

(i) $Q(x_1,\dots,x_{N+1})$ is nondecreasing in its arguments $x_1,\dots,x_N$ and increasing in x_{N+1},

(ii) $Q(x_1,\dots,x_{N+1}) = -Q(-x_1,\dots,-x_{N+1})$ for $(x_1,\dots,x_{N+1}) \in \mathbb{R}^{N+1}$,

(iii) $\lim_{x_{N+1}\to\infty} Q(0,\dots,0,x_{N+1}) = \infty$.

It is clear that for each $Q \in \mathcal{C}_2$ we have $Q(0,\dots,0) = 0$ and $Q(x_1,\dots,x_{N+1}) > 0$ for $(x_1,\dots,x_{N+1}) \in \mathbb{R}_+^{N+1}$.

Consider the nonlocal singular boundary value problem

$$\Delta(\phi(\Delta(u(k-1))) = f(k,u(k),\Delta u(k)), \quad k \in \mathbb{T}[1,N], \tag{3.32}$$

$$\Delta u(0) = 0, \qquad Q(u(1),\dots,u(N+1)) = C, \quad Q \in \mathcal{C}_2,\ C > 0, \tag{3.33}$$

where ϕ satisfies (H$_1$) and f fulfils the condition

(H$_3$) $f \in C(\mathbb{T}[1,N] \times \mathfrak{D})$, $\mathfrak{D} = [0,\infty) \times \mathbb{R}_+$, $f(k,x,y) > 0$ for $k \in \mathbb{T}[1,N]$, $(x,y) \in \mathbb{R}_+^2$, $f(k,0,y) = 0$ for $k \in \mathbb{T}[1,N]$, $y \in \mathbb{R}_+$, and $\lim_{y\to 0^+} f(1,x,y) = \infty$ locally uniformly on $\mathbb{R}_+$.

We say that $u \in \mathbb{T}[0,N+1] \to \mathbb{R}$ is a solution of problem (3.32)-(3.33) if u satisfies (3.33) and fulfils equality (3.32) for $k \in \mathbb{T}[1,N]$.

Notice that a special case of the boundary conditions (3.33) is the mixed conditions $\Delta u(0) = 0$, $u(N+1) = C$ which we get by setting $Q(x_1,\dots,x_{N+1}) = x_{N+1}$.

The existence of a solution to problem (3.32)-(3.33) is proved by regularization and sequential techniques. To this end, for each $n \in \mathbb{N}$ define $f_n \in C(\mathbb{T}[1,N] \times \mathbb{R}^2)$ by the formula

$$f_n(k,x,y) = f^*\left(k,x,\max\left\{\frac{1}{n},y\right\}\right), \quad k \in \mathbb{T}[1,N],\ (x,y) \in \mathbb{R}^2, \tag{3.34}$$

where

$$f^*(k,x,y) = \begin{cases} f(k,x,y) & \text{for } k \in \mathbb{T}[1,N],\ (x,y) \in [0,\infty) \times \mathbb{R}_+, \\ 0 & \text{for } k \in \mathbb{T}[1,N],\ (x,y) \in (-\infty,0) \times \mathbb{R}_+. \end{cases} \tag{3.35}$$

Under condition (H_3), we have

$$f_n(k,x,y) > 0 \quad \text{for } k \in \mathbb{T}[1,N],\ (x,y) \in (0,\infty) \times \mathbb{R}, \tag{3.36}$$

$$f_n(k,x,y) = 0 \quad \text{for } k \in \mathbb{T}[1,N],\ (x,y) \in (-\infty,0] \times \mathbb{R}, \tag{3.37}$$

$$\lim_{n\to\infty} f_n(k,x,y) = f(k,x,y) \quad \text{for } k \in \mathbb{T}[1,N],\ (x,y) \in [0,\infty) \times \mathbb{R}_+. \tag{3.38}$$

Throughout this section, X denotes the Banach space of functions $u : \mathbb{T}[0,N+1] \to \mathbb{R}$ equipped with the norm $\|u\| = \max\{|u(k)| : k \in \mathbb{T}[0,N+1]\}$.

Finally, let $\alpha, \beta \in \mathcal{A}$ be defined on X by

$$\alpha(u) = \Delta u(0), \qquad \beta(u) = Q(u(1),\dots,u(N+1)) - C, \quad Q \in \mathcal{C}_2,\ C > 0. \tag{3.39}$$

Then we can write the boundary conditions (3.33) in the form of (1.6).

Lemma 3.8. *Let $\alpha, \beta \in \mathcal{A}$ be defined in* (3.39). *Then for each $\mu \in [0,1]$ system* (1.3) *has a unique solution $(A,B) \in \mathbb{R}^2$ and there exists a positive constant Λ independent of μ such that*

$$\max\{|A|,|B|\} < \Lambda. \tag{3.40}$$

Proof. Since α is a linear map and Q is an odd function, we can write system (1.3) in the form

$$\begin{gathered}(1+\mu)B = 0,\\ (1+\mu)Q(A+B,\dots,A+(N+1)B) = (1-\mu)C.\end{gathered} \tag{3.41}$$

In particular, $B = 0$ and A is a solution of the equation

$$Q(A,\dots,A) = \frac{(1-\mu)C}{1+\mu}. \tag{3.42}$$

Put $p(x) = Q(x,\dots,x)$ for $x \in \mathbb{R}$. Then $p \in C(\mathbb{R})$ is increasing on $\mathbb{R}$, $p(0) = 0$ and $\lim_{x\to\infty} p(x) = \infty$. Hence $A = p^{-1}((1-\mu)C/(1+\mu))$ is the unique solution of (3.42), and for each $\mu \in [0,1]$ we have $0 < A \le p^{-1}(C)$. To summarize, for each $\mu \in [0,1]$ system (1.3) has a unique solution $(A,B) = (p^{-1}((1-\mu)C/(1+\mu)),0)$ and the estimate (3.40) is true with $\Lambda = p^{-1}(C) + 1$. □

Remark 3.9. By Lemma 3.8 and Remark 1.2, the boundary conditions (3.33) are compatible.

Lemma 3.10. *Let (H_1) and (H_3) hold. Let $u : \mathbb{T}[1,N] \to \mathbb{R}$ be a solution of the equation*

$$\Delta(\phi(\Delta u(k-1))) = \lambda f_n(k,u(k),\Delta u(k)), \quad k \in \mathbb{T}[1,N],\ \lambda \in (0,1], \tag{3.43}$$

fulfilling the boundary conditions (3.33). *Then there exists a positive constant S independent of n and λ such that*

$$0 < u(k) < S \quad \textit{for } k \in \mathbb{T}[0,N+1], \tag{3.44}$$

$$\Delta u(k-1) < \Delta u(k) \quad \textit{for } k \in \mathbb{T}[1,N]. \tag{3.45}$$

Proof. Suppose that $u(0) \leq 0$. Then $u(1) = u(0) \leq 0$ and, by equality (3.37), $\Delta(\phi(\Delta u(0))) = \lambda f_n(1, u(1), \Delta u(1)) = 0$. Hence $\Delta u(1) = \Delta u(0) = 0$ and so $u(2) = u(0) \leq 0$. Applying the above arguments repeatedly, we have $\Delta u(j-1) = \Delta u(0) = 0$ and $u(j) = u(0) \leq 0$ for $j \in \mathbb{T}[2, N+1]$. Therefore $Q(u(1), \ldots, u(N+1)) \leq Q(0, \ldots, 0) = 0$, which contradicts the fact that $Q(u(1), \ldots, u(N+1)) = C > 0$ by (3.33). Consequently, $u(0) = u(1) > 0$. By (3.36) and (3.37), $f_n(k, u(k), \Delta u(k)) \geq 0$ for $k \in \mathbb{T}[1, N]$, which gives $\Delta(\phi(\Delta u(k-1))) \geq 0$ for these k. Therefore $\Delta u(k) \geq \Delta u(k-1)$ for $k \in \mathbb{T}[1, N]$. This and $\Delta u(0) = 0$ and $u(1) > 0$ yield

$$u(k) > 0 \quad \text{for } k \in [0, N+1]. \tag{3.46}$$

Then $\Delta(\phi(\Delta u(k-1))) = \lambda f_n(k, u(k), \Delta u(k)) > 0$ by (3.36), and consequently inequality (3.45) is true, which means that the sequence $\{u(k)\}_{k=1}^{N+1}$ is increasing and $\max\{u(k) : k \in \mathbb{T}[0, N+1]\} = u(N+1)$. It remains to prove that $u(N+1) < S$, where S is a positive constant independent of n and λ. To this end, put $r(x) = Q(0, \ldots, 0, x)$ for $x \in \mathbb{R}$. Then $C = Q(u(1), \ldots, u(N), u(N+1)) \geq Q(0, \ldots, 0, u(N+1)) = r(u(N+1))$. Since $r \in C(\mathbb{R})$ is increasing on $\mathbb{R}$ and $\lim_{x\to\infty} r(x) = \infty$, it follows from the inequality $C \geq r(u(N+1))$ that $u(N+1) \leq r^{-1}(C)$. Hence $u(N+1) < S$, where $S = r^{-1}(C) + 1$. Clearly, S is independent of n and λ. □

Remark 3.11. Let $\lambda = 0$ in (3.43). Then problem (3.43)–(3.33) has a unique solution u, $u(k) = p^{-1}(C)$, for $k \in \mathbb{T}[0, N+1]$, where p^{-1} is the inverse function to p defined by $p(x) = Q(x, \ldots, x)$ for $x \in \mathbb{R}$. This fact follows from Remark 1.1 and the proof of Lemma 3.8 with $\mu = 0$. Since $p(x) \geq r(x)$ for $x \in \mathbb{R}_+$, we have $p^{-1}(C) \leq r^{-1}(C)$. Here $r(x) = Q(0, \ldots, 0, x)$ for $x \in \mathbb{R}$. Hence $0 < u(k) < S$ for $k \in \mathbb{T}[0, N+1]$, where $S = r^{-1}(C) + 1$.

The next lemma gives an existence result for problem (3.47)–(3.33), where

$$\Delta(\phi(\Delta u(k-1))) = f_n(k, u(k), \Delta u(k)), \quad k \in \mathbb{T}[1, N]. \tag{3.47}$$

Lemma 3.12. *Let (H_1) and (H_3) hold. Then for each $n \in \mathbb{N}$ there exists a solution of problem* (3.47)–(3.33) *and any of its solutions u_n satisfies the estimate*

$$0 < u_n(k) < S \quad \textit{for } k \in \mathbb{T}[0, N+1], \tag{3.48}$$

where S is a positive constant independent of n and

$$\Delta u_n(k-1) < \Delta u_n(k) \quad \textit{for } k \in \mathbb{T}[1, N]. \tag{3.49}$$

Proof. Let us choose $n \in \mathbb{N}$. Put $h(k, x, y) = f_n(k, x, y)$ for $k \in \mathbb{T}[1, N]$, $(x, y) \in \mathbb{R}^2$, and let $\alpha, \beta \in \mathcal{A}$ be given in (3.39). By Remark 3.9, the boundary conditions (3.33) are compatible, and it follows from Lemma 3.10 and Remark 3.11 that there exists a positive constant S independent of n such that $\|u\| < S$ for any solution u of problem (3.43)–(3.33), where $\lambda \in [0, 1]$. Besides, by Lemma 3.8, there exists a positive constant Λ such that estimate (3.40) holds for all solutions $(A, B) \in \mathbb{R}^2$ of problem (1.3) for each $\mu \in [0, 1]$. Therefore the conditions of Theorem 2.1 are fulfilled, and consequently problem (3.47)–(3.33) has a solution. In addition, any of its solutions u_n satisfies inequalities (3.48) and (3.49) by Lemma 3.10. □

We are now in a position to give our result for the solvability of problem (3.32)-(3.33).

Theorem 3.13. *Let (H_1) and (H_3) hold. Then problem* (3.32)-(3.33) *has a positive solution.*

Proof. Due to Lemma 3.12, for each $n \in \mathbb{N}$ there exists a solution u_n of problem (3.47)–(3.33) satisfying inequalities (3.48) and (3.49). Hence the sequence $\{u_n(k)\}$ is bounded for each $k \in \mathbb{T}[0, N+1]$, and consequently by the Bolzano-Weierstrass compactness theorem, there exists a subsequence $\{\ell_n\}$ of $\{n\}$ and $u \in X$ such that $\lim_{n\to\infty} u_{\ell_n} = u$. Letting $n \to \infty$ in (3.48) and (3.49) (with ℓ_n instead of n) and in the boundary conditions $\Delta u_{\ell_n}(0) = 0$, $Q(u_{\ell_n}(1), \dots, u_{\ell_n}(N+1)) = C$, we have

$$0 \le u(k) \le S \quad \text{for } k \in \mathbb{T}[0, N+1], \tag{3.50}$$

$$\Delta u(k-1) \le \Delta u(k) \quad \text{for } k \in \mathbb{T}[1, N], \tag{3.51}$$

and u satisfies the boundary conditions (3.33). It follows from $\Delta u(0) = 0$ and inequalities (3.50)-(3.51) that

$$0 \le u(0) = u(1) \le u(2) \le \cdots \le u(N+1) \le S. \tag{3.52}$$

If $u(N+1) = 0$, then $u(k) = 0$ for $k \in \mathbb{T}[0, N+1]$. Therefore $Q(u(1), \dots, u(N+1)) = Q(0, \dots, 0) = 0$, contrary to (3.33). We have $u(N+1) > 0$. Suppose now that $u(N) = 0$. Then $\Delta u(N) = u(N+1) > 0$ and

$$\begin{aligned}
\lim_{n\to\infty} \Delta\big(\phi\big(\Delta u_{\ell_n}(N-1)\big)\big) &= \lim_{n\to\infty} f_{\ell_n}\big(N, u_{\ell_n}(N), \Delta u_{\ell_n}(N)\big) \\
&= \lim_{n\to\infty} f\left(N, u_{\ell_n}(N), \max\left\{\frac{1}{\ell_n}, \Delta u_{\ell_n}(N)\right\}\right) \\
&= f(N, 0, \Delta u(N)) \\
&= 0.
\end{aligned} \tag{3.53}$$

Since $\lim_{n\to\infty} \Delta(\phi(\Delta u_{\ell_n}(N-1))) = \Delta(\phi(\Delta u(N-1)))$, we have $\Delta(\phi(\Delta u(N-1))) = 0$. This gives $\Delta u(N-1) = \Delta u(N) > 0$ and therefore $u(N-1) = -\Delta u(N-1) < 0$, which is impossible. Hence $u(N) > 0$. Repeated application of the above arguments yields $u(k) > 0$ for $k \in \mathbb{T}[0, N-1]$. Hence

$$u(k) > 0 \quad \text{for } k \in \mathbb{T}[0, N+1]. \tag{3.54}$$

We proceed to show that

$$\Delta u(k) > 0 \quad \text{for } k \in \mathbb{T}[1, N]. \tag{3.55}$$

Suppose that $0 = \Delta u(0) = \Delta u(1)$. Then

$$\lim_{n\to\infty} \Delta\big(\phi\big(\Delta u_{\ell_n}(0)\big)\big) = \Delta(\phi(\Delta u(0))) = 0. \tag{3.56}$$

Since $\lim_{n\to\infty} u_{\ell_n}(1) = u(1) > 0$, we have

$$\begin{aligned}
\lim_{n\to\infty} \Delta\big(\phi\big(\Delta u_{\ell_n}(0)\big)\big) &= \lim_{n\to\infty} f_{\ell_n}\big(1, u_{\ell_n}(1), \Delta u_{\ell_n}(1)\big) \\
&= \lim_{n\to\infty} f\left(1, u_{\ell_n}(1), \max\left\{\frac{1}{\ell_n}, \Delta u_{\ell_n}(1)\right\}\right) \\
&= \infty
\end{aligned} \tag{3.57}$$

by (H_3), contrary to (3.56). Hence $\Delta u(1) > 0$. From this and from (3.51), it follows that inequality (3.55) is true. Having in mind (3.55), we get

$$\begin{aligned}\Delta(\phi(\Delta u(k-1))) &= \lim_{n\to\infty}\Delta\left(\phi\left(\Delta u_{\ell_n}(k-1)\right)\right)\\ &= \lim_{n\to\infty} f\left(k, u_{\ell_n}(k), \max\left\{\frac{1}{\ell_n}, \Delta u_{\ell_n}(k)\right\}\right)\\ &= f(k, u(k), \Delta u(k))\end{aligned} \tag{3.58}$$

for $k \in \mathbb{T}[1,N]$. In particular, u is a solution of (3.32). Since u satisfies (3.33) and (3.41), it follows that u is a positive solution of problem (3.32)-(3.33). □

Example 3.14. Let $a, b, a_{N+1} \in \mathbb{R}_+$ and $a_j \in [0,\infty)$ for $j \in \mathbb{T}[1,N]$. Then $f(k,x,y) = (e^x - 1)(\ln k + x^a + 1/y^b)$, $k \in \mathbb{T}[1,N]$, $(x,y) \in [0,\infty)\times\mathbb{R}_+$, satisfies condition ($H_3$), and the function $Q(x_1,\dots,x_{N+1}) = \sum_{j=1}^{N+1} a_j x_j^{2j-1}$ belongs to the set $\mathcal{C}_2$. If ϕ fulfils (H_1) then, by Theorem 3.13, the singular problem

$$\begin{gathered}\Delta(\phi(\Delta u(k-1))) = \left(e^{u(k)}-1\right)\left(\ln k + (u(k))^a + \frac{1}{(\Delta u(k))^b}\right), \quad k \in \mathbb{T}[1,N],\\ \Delta u(0) = 0, \qquad \sum_{j=1}^{N+1} a_j (u(j))^{2j-1} = C, \quad C > 0,\end{gathered} \tag{3.59}$$

has a positive solution.

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References

[1] R. P. Agarwal and D. O'Regan, "Singular discrete boundary value problems," *Applied Mathematics Letters*, vol. 12, no. 4, pp. 127–131, 1999.
[2] R. P. Agarwal, K. Perera, and D. O'Regan, "Multiple positive solutions of singular discrete p-Laplacian problems via variational methods," *Advances in Difference Equations*, vol. 2005, no. 2, pp. 93–99, 2005.
[3] H. Lü and D. O'Regan, "Construction of upper and lower solutions for singular discrete initial and boundary value problems via inequality theory," *Advances in Difference Equations*, vol. 2005, no. 2, pp. 205–214, 2005.
[4] H. Lü, D. O'Regan, and R. P. Agarwal, "An eigenvalue interval of solutions for a singular discrete boundary value problem with sign changing nonlinearities," *Journal of Difference Equations and Applications*, vol. 12, no. 7, pp. 717–730, 2006.
[5] Y. Liu and W. Ge, "Twin positive solutions of boundary value problems for finite difference equations with p-Laplacian operator," *Journal of Mathematical Analysis and Applications*, vol. 278, no. 2, pp. 551–561, 2003.
[6] Y. Tian, D. Ma, and W. Ge, "Multiple positive solutions of four point boundary value problems for finite difference equations," *Journal of Difference Equations and Applications*, vol. 12, no. 1, pp. 57–68, 2006.
[7] I. Rachůnková and L. Rachůnek, "Singular discrete second order BVPs with p-Laplacian," *Journal of Difference Equations and Applications*, vol. 12, no. 8, pp. 811–819, 2006.
[8] K. Deimling, *Nonlinear Functional Analysis*, Springer, Berlin, Germany, 1985.
[9] R. P. Agarwal, D. O'Regan, and S. Staněk, "General existence principles for nonlocal boundary value problems with ϕ-Laplacian and their applications," *Abstract and Applied Analysis*, vol. 2006, Article ID 96826, 30 pages, 2006.