FORMULATION OF THE ENERGY EQUATION IN FLUID DYNAMICS

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FORMULATIONS for the energy equation exist in variety in fundamental and applied texts and monographs on fluid dynamics. It is often difficult to establish the set of conditions leading to a given formulation. More deleterious is the fact that in some cases an energy equation as point of departure does not have the generality implied by the accompanying text.

In two instances (Chandrasekhar, 1961; Newman and Pierson, 1966) the term $(\alpha T/\kappa_T)\,\bar{\nabla}\cdot\bar{v}$ in formula (7) is replaced by $p\bar{\nabla}\cdot\bar{v}$ (for the symbols used see the list at the end of this note). For an incompressible fluid this term is zero, admittedly, while for an ideal gas one has $\alpha T/\kappa_T = p$, and the difference disappears. It would appear, though, to be a didactic drawback to start from an inexact formula, even though the final approximation is unaffected. Moreover, it is quite possible to imagine situations where neither the incompressible fluid nor the ideal gas would be a satisfactory approximation to reality. In this note a hierarchy of formulations is presented for reference, from the general to some restricted cases. A few directions are given to guide in the derivations.

The point of departure is the energy equation for the flow of a simple fluid system. (For a derivation from first principles see e.g. Aris, 1962, p. 121)

$$\rho\,(du/dt) = -\bar{\nabla}\cdot\bar{q} + \bar{\bar{T}};(\bar{\nabla}\bar{v}) \qquad (1)$$

in dyadic notation, or

$$\rho\,(du/dt) = -(\delta q_i/\delta x_i) + T_{ij}\,(\delta v_i/\delta x_j)$$

in tensor notation.

For a Stokesian fluid this reduces to

$$\rho\,(du/dt) = -\bar{\nabla}\cdot\bar{q} - p\bar{\nabla}\cdot\bar{v} + Y \qquad (2)$$

where Y is the viscous dissipation function.

For a linear Stokesian (Newtonian) fluid, one finds

$$Y = (\lambda + 2\mu)(\bar{\nabla}\cdot\bar{v})^2 - 4\mu\,\Phi \qquad (3)$$

Inserting the relationship between entropy and energy, and enthalpy and energy, into (2) one finds the expressions for the time rate of change of these functions as

$$\rho\ T\ (\ ds/dt\ )\ =\ -\ \bar{\nabla}\cdot\bar{q}\ +\ Y, \tag{4}$$

and

$$\rho\ (\ dh/dt\ )\ =\ -\ \bar{\nabla}\cdot\bar{q}\ +\ dp/dt\ +\ Y \tag{5}$$

Often one utilizes the corresponding equations in terms of the temperature as the main variable ("heat conduction equation"). The transition takes place through standard manipulation (see e.g. Callen, 1960, Chapter 7). It is sketched here, since this appears to be the source of the discrepancies in references (Chandrasekhar, 1961; Newman and Pierson, 1966).

For enthalpy,

$$d\,h = c_p\, dT\ +\ (\ \delta h/\delta p\ )_T\, dp.$$

Since $(\ \delta h/\delta p\ )_T = (\ 1/\rho\ )(\ 1 - \alpha T\ )$,

we find by inserting into (5), if one also assumes Fourier's law for heat flow, $q = -k\bar{\nabla}T$,

$$\rho\ c_p\ (\ dT/dt\ )\ =\ \bar{\nabla}\cdot(\ k\bar{\nabla}T\ ) + \alpha T\ (\ dp/dt\ ) + Y \tag{6}$$

A similar expression follows from (2) by noticing that

$$(\ \delta u/\delta\, v\ )_T = -p\ +\ \alpha T/\kappa_T,$$

namely

$$\rho\ c_v\ (\ dT/dt\ ) = \bar{\nabla}\cdot(\ k\bar{\nabla}T\ ) - (\ \alpha T/\kappa_T\ )\bar{\nabla}\cdot\bar{v}\ +\ Y \tag{7}$$

Since for an ideal gas $\alpha T\ =\ 1$, and $\kappa_T\ =\ 1/p$, we have the corresponding expressions for that special case.

Ideal gas

$$\rho\ c_p\, dT/dt = \bar{\nabla}\cdot(\ k\bar{\nabla}T\ ) + dp/dt\ +\ Y \tag{8}$$

and

$$\rho\ c_v\, dT/dt = \bar{\nabla}\cdot(k\bar{\nabla}T) - p\bar{\nabla}\cdot\bar{v}\ +\ Y. \tag{9}$$

Notice that in the above derivations it was not assumed that c_v, c_p, k are independent of position.

List of Symbols

α	=	$(1/v)(\delta v/\delta T)_p$, coefficient of thermal expansion
c_p	=	$(\delta h/\delta T)_p$, specific heat at constant pressure (per unit mass)
c_v	=	$(\delta u/\delta T)_v$, specific heat at constant volume (per unit mass)
(d/dt)	=	$(\delta/\delta t + v_x \delta/\delta x + v_y \delta/\delta y + v_z \delta/\delta z)$, total time derivative
e_{ij}	=	$(1/2(\delta v_i/\delta x_j + \delta v_j/\delta x_i)$, rate of strain tensor
Φ	,	sum of cofactors of diagonal terms in e_{ij}
h	,	specific enthalpy (unit mass)
k	,	thermal conductivity
κ_T	=	$-(1/v)(\delta v/\delta p)_T$, isothermal compressibility
λ	,	viscosity coefficient in $T_{ij} = (-p + \lambda\bar{\nabla}\cdot\bar{v})\delta_{ij} + 2\mu e_{ij}$
μ	,	dynamic viscosity
p	,	thermodynamic pressure
$\bar{q}$	,	heat flux vector
ρ	,	density
s	,	specific entropy (unit mass)
T	,	temperature
$\bar{\bar{T}}$, T_{ij}	,	stress tensor
t	,	time
Y	,	viscous dissipation function
u	,	specific internal energy (unit mass)
$\bar{v}$	,	velocity vector
v	,	specific volume

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ERRATUM: *1971 Photoelectric Observations of Beta Lyrae* [Florida Scientist 37:100 (1974)] The authors and the institutions with which they are associated were incorrectly listed in this paper. Correct addresses are given here: *R. M. Williamon, T. H. Morgen and D. H. Martins,* Rosemary Hill Observatory, Department of Physics and Astronomy, University of Florida, Gainesville, Florida 32611; *T. F. Collins,* Department of Physics, Clemson University, Clemson, South Carolina 29631; *H. R. Miller,* Department of Physics, Georgia State University, Atlanta, Georgia 30303.