ON FINITE ELEMENT METHODS FOR
THE EULER-POISSON-DARBOUX EQUATION*

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To the fond memory of my beloved mother, Sonya, Z"L, I humbly dedicate this work.

Abstract. In this work we deal primarily with the derivation of various convergence estimates for some semidiscrete and fully discrete procedures which might be used in the approximation of exact solutions of initial-boundary value problems with homogeneous Dirichlet boundary conditions for the Euler–Poisson–Darboux equation. These procedures include the ordinary Galerkin method based on conforming finite element subspaces as well as certain methods which do not require such restrictions. Although the equation is of hyperbolic type, the results are somewhat analogous to those known for parabolic equations. This is due to the presence of a limited “smoothing” property. This paper contains $L_2$ estimates, maximum norm estimates, interior estimates of difference quotients and superconvergence estimates of the error. Most of the proofs are based on different modifications of an energy method, some of them intrinsically depend on the Weinstein recursion formulae and on known results for the associated elliptic problem. Several of these estimates are obtained for positive time under weak assumptions on the initial data.

1. Introduction. Let $\Omega$ be a bounded domain in $\mathbb{R}^N, N \geq 1$, with sufficiently smooth boundary $\partial \Omega$. In this work we shall consider the following initial boundary value problem for the Euler–Poisson–Darboux equation:

$$D_t u^p + \frac{2p+1}{t} D_t u^p + \mathcal{L} u^p = 0 \quad \text{in } \Omega \times (0, t^*],$$

(1.1)

$$u^p = 0 \quad \text{on } \partial \Omega \times (0, t^*],$$

$$u^p|_{t=0} = g \quad \text{and } \quad D_t u^p|_{t=0} = 0 \quad \text{in } \Omega.$$

Here we assume that $p > -\frac{1}{2}$, $g|_{\partial \Omega} = 0$ and $\mathcal{L}$ denotes a second-order positive definite uniformly elliptic operator

$$\mathcal{L} w = - \sum_{i,j=1}^{N} \frac{\partial}{\partial x_i} \left( a_{ij}(x) \frac{\partial w}{\partial x_j} \right) + a_0(x) w,$$

where $a_{ij} = a_{ji} \in C^\infty(\bar{\Omega}), 1 \leq i, j \leq N$, $a_0 \in C^\infty(\bar{\Omega})$ and $a_0 \geq 0$ on $\bar{\Omega}$.

First of all, note that throughout this work $u^p$ will always denote a solution of (1.1), which corresponds to the parameter $p$ appearing in the equation. Also we would like to remark that the choice of the initial condition $D_t u^p|_{t=0} = 0$ was dictated by the presence of a singular coefficient in the equation, i.e. any other choice would have adversely affected the question of well-posedness of the problem in the functional classes we shall be working with. Next, following Carroll and Showalter [12], we ruled out the values of $p < -\frac{1}{2}$, which choice would have also been inadmissible for its negative, in general, effect upon existence and uniqueness of a solution in the above mentioned setting. Since the case of $p = -\frac{1}{2}$ corresponds to the wave equation, whose numerical solution has been extensively studied by many authors (Baker [2], Baker and Bramble [3], Dupont [13]), it became very natural to restrict our analysis of the problem (1.1) to the case when $p > -\frac{1}{2}$.

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Using the Fourier method of separation of variables, one can easily show that the formal solution of (1.1) has the following spectral representation:

\[ u^P(x, t) = \Gamma(p + 1) \sum_{n=1}^{\infty} g_n \phi_n(x) \left( \frac{\sqrt{\lambda_n t}}{2} \right)^{-p} J_p(\sqrt{\lambda_n t}), \]

where \( g_n = (g, \phi_n) \), \((\cdot, \cdot)\) denotes the scalar product in \( L_2 = L_2(\Omega) \), \( \Gamma(\cdot) \) is the gamma function, \( J_p(\cdot) \) denotes the Bessel function of the first kind and of order \( p \), and \( \{\lambda_n\} \) with \( \{\phi_n\}, 1 \leq n < \infty \), are respectively the eigenvalues in nondecreasing order and \( L_2 \) orthonormalized eigenfunctions of the associated Sturm–Liouville problem \( \mathcal{L} w = \lambda w \) in \( \Omega \), \( w = 0 \) on \( \partial\Omega \). It is well known [14] that under given conditions all \( \lambda_n \)'s are positive and tend to infinity with \( n \).

The following pieces of notation will be used throughout. As usual, \( H_s = W^s_2(\Omega) \), \( s \geq 0 \), denotes the Sobolev space of order \( s \) of real-valued functions on \( \Omega \). In addition we introduce certain subspaces of \( H_s \) which, as it turns out, serve as a very natural setting for our analysis of the problem (1.1). So let \( \dot{H}_s = H_s(\Omega) \) be the space of \( w \in L_2 \), for which functions

\[ \| w \|_s = \left( \sum_{n=1}^{\infty} \lambda_n^s w_n^2 \right)^{1/2} < \infty, \quad w_n = (w, \phi_n). \]

It can be shown [14] that for \( 0 \leq s \in \mathbb{Z} \) the space \( \dot{H}_s \) consists of \( w \in H_s \) which satisfy the boundary condition \( \mathcal{L}^m w = 0 \) on \( \partial\Omega \) for \( 0 \leq m \leq [(s-1)/2] \), and that the norm \( \| \cdot \|_s \) is equivalent to the standard norm in \( H_s \). In particular, \( \dot{H}_s = \bigcap_{s \geq 0} H_s \) contains those and only those \( w \in C^\infty(\bar{\Omega}) \), for which \( \mathcal{L}^m w = 0 \) on \( \partial\Omega \) for all \( m \geq 0 \). Obviously, \( \dot{H}_0 = L_2 \) and then we always write \( \| \cdot \| = \| \cdot \|_0 \). For \( s > 0 \) we also define \( \dot{H}_{-s} \) to be the dual of \( \dot{H}_s \) with respect to the inner product in \( L_2 \). Hence

\[ \| w \|_{-s} = \left( \sum_{n=1}^{\infty} \lambda_n^{-s} w_n^2 \right)^{1/2}, \quad w_n = (w, \phi_n), \]

for \( w \in L_2 \). It is worth noting [14] that \( \phi_n \in H_s \) for all \( 1 \leq n < \infty \), whenever \( \partial\Omega \in C^s \). Thus the spectral representation (1.2) remains meaningful also for \( g \in \dot{H}_{-s}, s \geq 0 \).

Following the ideas of Bramble, Schatz, Thomée and Wahlbin [8], we introduce the solution operator of the elliptic problem \( \mathcal{L} w = f \) in \( \Omega \), \( w = 0 \) on \( \partial\Omega \), defined as a map \( T : L_2 \rightarrow L_2 \) such that \( w = T f \). According to [14] this operator possesses the following eigenfunction expansion:

\[ Tf = \sum_{n=1}^{\infty} \mu_n f_n \phi_n, \]

where \( f_n = (f, \phi_n) \) and \( \mu_n = \lambda_n^{-1} \), with \( \phi_n \) and \( \lambda_n \) as above. Then it follows at once that \( T \) is a bounded operator from \( \dot{H}_s \) into \( \dot{H}_{s+2} \) for any \( s \in \mathbb{R} \). In terms of \( T \) we may reformulate the problem (1.1) in the following equivalent form,

\[ D_t^2 T u^p + \frac{2p+1}{t} D_t T u^p + u^p = 0, \quad 0 < t \leq t^*, \quad p > -\frac{1}{2}, \]

\[ u^p|_{t=0} = g, \quad D_t u^p|_{t=0} = 0. \]

For the purpose of approximating the solution of this problem we shall employ a family \( \{S_h\}, 0 < h < 1 \), of finite element subspaces of \( H_1 \), such that

\[ \inf_{\chi \in S_h} \{ \| w - \chi \| + h \| w - \chi \|_1 \} \leq C h^2 \| w \|_s, \]

\( \text{METHODS FOR THE EULER-POISSON-DARBOUX EQUATION 1081} \)
for $1 \leq s \leq r$ and for all $w \in H_s$. We shall also assume that we are given a corresponding family of operators $T_h : L_2 \rightarrow S_h$ such that:

(1.5) $T_h$ is self-adjoint, positive semidefinite on $L_2$ and positive definite on $S_h$,

(1.6) there exist $2 \leq r \in \mathbb{Z}$ and a constant $C > 0$ such that

$$\| (T_h - T) w \| \leq Ch^{s+2} \| w \|_s, \quad 0 \leq s \leq r - 2,$$

and/or

(1.7) there are $3 \leq r \in \mathbb{Z}$ and a constant $C > 0$ such that

$$\| (T_h - T) w \|_{-q} \leq Ch^{q+s+2} \| w \|_s, \quad 0 \leq s, q \leq r - 2.$$

The semidiscrete solution is defined as a mapping $u_h^p : [0, t^*) \rightarrow S_h$ such that

$$2p + 1 \int_0^{t^*} D_t^2 T_h u_{h,p} + D_t T_h u_{h,p} + u_{h,p} = 0, \quad 0 < t < t^*, \quad p > -\frac{1}{2},$$

$$u_{h,p}|_{t=0} = g_h^h, \quad D_t u_{h,p}|_{t=0} = 0,$$

where $g_h^h \in S_h$ is a suitable approximation to $g$. Next, we consider $\mathcal{L}_h : S_h \rightarrow S_h$ defined as $\mathcal{L}_h = T_h^{-1}$. It is reasonable to assume that there exists a constant $C > 0$ such that

(1.9) $\| w \|_1^2 \leq C(\mathcal{L}_h w, w),$

for all $w \in S_h$, $0 < h < 1$. Observe that with the help of just introduced $\mathcal{L}_h$ we can reformulate (1.9) as

$$2p + 1 \int_0^{t^*} D_t^2 u_{h,p} + D_t \mathcal{L}_h u_{h,p} + \mathcal{L}_h u_{h,p} = 0, \quad 0 < t \leq t^*, \quad p > -\frac{1}{2},$$

$$u_{h,p}|_{t=0} = g_h^h, \quad D_t u_{h,p}|_{t=0} = 0.$$

In addition to the above defined norms $\| \cdot \|_1$, we shall use different norms in several other Banach spaces of functions on $\Omega$ and subsets $\Omega \subset \Omega$. The function space will then in general appear as a subscript, with the convention that whenever the basic domain of the space is all of $\Omega$, then it will be omitted. For the norm in $W^k_\infty(\Omega)$ we always write $\| \cdot \|_{W^k_\infty(\Omega)}$, and the norms in $L_2(\Omega)$ and $L_\infty(\Omega)$ are denoted by $\| \cdot \|_{L_2(\Omega)}$ and $\| \cdot \|_{L_\infty(\Omega)}$, respectively, again with the above convention whenever $\Omega = \Omega$. Finally, for $B$ being some Banach space we shall use the following norms,

$$\| \cdot \|_{L_2(B)} = \left( \int_0^{t^*} \| \cdot \|^2_B dt \right)^{1/2} \quad \text{and} \quad \| \cdot \|_{L_\infty(B)} = \sup_{0 \leq t \leq t^*} \| \cdot \|_B.$$

Error estimates in various norms for semidiscrete and fully discrete Galerkin problems for the Euler–Poisson–Darboux equation are derived in this work. The results are valid for a large class of conforming and nonconforming finite element spaces defined on quasi-uniform meshes which are widely used in practice. One of the most interesting features of the Euler–Poisson–Darboux equation is the presence of hyperbolicity of the equation and a “smoothing” property at the same time. The framework of the present investigation, which is similar to the one for parabolic equations [8], is essentially motivated by the above-mentioned feature. However, in our case the solution operators do not form a semigroup, so we cannot utilize many of the techniques employed in [8]. Thus most of spectral methods exploited quite successfully by Bramble, Schatz, Thomée and Wahlbin in [8] were unavailable to us. In this situation an energy method and its various combinations with the Weinstein recursion formulae proved to be a principal tool of the present work.
The outline of our paper is as follows. In §2 we obtain an eigenfunction expansion of the semidiscrete solution \( u^{h,p} \). The "smoothing" property result is contained in Lemma 2.1, which states that

\[ \| u^p(t) \|_s \leq Ct^{-s} \| g \|, \]

for \( t > 0 \) and \( 0 \leq s \leq p + \frac{1}{2} \). This tells us that for positive values of time \( u^p(t) \in \dot{H}_s \), \( 0 \leq s \leq p + \frac{1}{2} \), even for example with \( g \in L_2 \). In the same lemma we establish the \( L_2 \) stability result for the semidiscrete case. More precisely, the following holds:

\[ \| u^{h,p} \|_{L_2(L_2)} \leq C \| g^h \|, \]

for \( p > -\frac{1}{2} \). In Lemma 2.2 we prove certain auxiliary inequalities in time-weighted norms, which are exploited later in §3. In the conclusion of §2 we derive various recursion formulae using the above-mentioned Weinstein formulae as a starting point. These formulae will play a crucial role in subsequent sections, especially in §4 and §5.

In §3 we deal primarily with the \( L_2 \) estimates of the error in the semidiscrete case. There, under essentially as weak regularity assumptions as possible, we derive \( L_2 \) error estimates of optimal rate of convergence. More precisely, Theorem 3.1 yields

\[ \| u^{h,p} - u^p \|_{L_2(L_2)} \leq C \| u^h(t) \|_{L_2} \]

and according to Theorem 3.2 we have

\[ \| \partial_t (u^{h,p} - u^p) \|_{L_2(L_2)} \leq C \| u^h(t) \|_{L_2}, \]

where in both cases \( p \geq \frac{1}{2} \) and \( 0 < \varepsilon < 1 \).

Next, §4 is entirely devoted to an original and quite tedious procedure in which we successfully combine the results of §2 and §3. It allows us to obtain optimal \( L_2 \) error estimates for nonsmooth data under the assumption that \( g^h = P_0 g \), where \( P_0 \) is the standard \( L_2 \) projection onto \( S_h \). For instance, Theorem 4.1 contains the following inequality,

\[ \| u^{h,p}(t) - u^p(t) \| \leq C \| u^h(t) \|_{L_2}, \quad 0 < t \leq t^*, \]

for \( p \geq r + \frac{1}{2} \) and \( \varepsilon \) as above. The result

\[ \| \mathcal{L}_h^m u^{h,p}(t) - \mathcal{L}_h^m u^p(t) \| \leq C \| u^h(t) \|_{L_2}, \]

with \( p \geq r + 2m + \frac{1}{2} \) and \( t, \varepsilon \) as above, is proven in Theorem 4.2. It should be mentioned that the latter estimate will play an important role in §6.

The next section, §5, contains global \( \dot{H}_{r-2} \) error estimates deduced via a certain manipulation of an energy method, whose principal ideas are somewhat similar to those of §3 and §4. The results are of optimal order \( O(h^{2r-2}) \) in \( \dot{H}_{r-2} \) and are derived under the assumption that the initial data \( g \in \dot{H}_{r-2,0} \) and \( g^h = P_0 g \). The main result of the section is proved in Theorem 5.1 and used later on in §8.

The results of §6 reflect our efforts to obtain error estimates in the maximum norm, provided some \( L_2 \) and \( L_\infty \) inequalities are known for the elliptic problem. This task is accomplished by modifying an analogous procedure in [8] and combining it with certain results of §4. In Theorem 6.1 we show that choosing \( g^h = P_0 g \), where \( P_0 = T_0 P_0 L \) is the "elliptic projection" onto \( S_h \), one can derive optimal \( O(h^r) \) error estimates in the \( L_\infty \) norm, although for \( 1 \leq N \leq 3 \). In Theorem 6.2 we generalize the above result to arbitrary \( N \geq 1 \) under the assumption that \( g^h \) is chosen in such a way that \( \| \mathcal{L}_h^m g^h - \mathcal{L}_h^m g \| \leq C h^r \), with \( M = M(N) \). Finally, Theorem 6.3 contains the following interior in time result,

\[ |u^{h,p}(t) - u^p(t)| \leq C \| \gamma(h) + h^r \| \| g \|_{L_\infty}. \]
for $0 < t_0 \leq t \leq t^*$, provided $g^h = P_0 g$ and $p$ is large enough, with $\gamma(h)$ being “in practice” either $h^r$ or $h^r \ln (1/h)$ (see [8] for details).

In § 7 we define the fully discrete problem and obtain $L_2$ error estimates of optimal order $O(h^r + (\Delta t)^2)$. However, it should be mentioned that the time discretization is a bit spoiled by the presence of a singular time-dependent coefficient in the equation. This results in slightly more stringent regularity assumptions on the involved data than those of § 3 for the semidiscrete case. Although these assumptions can be weakened at the expense of a constant $C$, which would have been replaced in such a case by $C \ln (1/\Delta t)$, in the present framework we preferred to choose the first option. Thus the discretization used is of Crank–Nicolson type and similar to that employed by Dupont [13]. Theorem 7.1 contains $L_2$ error estimates of optimal order for any $N \geq 1$.

Next, Corollary 10.1 consists of global error estimates in the $L_\infty$ norm of nearly optimal order $O(h \ln (1/h) + (\Delta t)^2)$, for $N = 2, 3$.

Finally, in § 8 we shall demonstrate very briefly that an averaging process described in [7] and [8], after some modification, can be combined with the results of § 5 to yield certain superconvergent $O(h^{2r-2})$ approximations of the exact solution, provided $g \in H_{r-2+\epsilon}$, $g^h = P_0 g$ and $p$ is sufficiently large. Then the principal result can be found in Theorem 8.1. Since the ideas and arguments of that section follow quite closely those of [8], we shall omit the proofs except for the cases when some originality is present.

In conclusion of this section we wish to remark that instead of Dirichlet boundary conditions in (1.1) we could consider, for instance, homogeneous Neumann type boundary conditions. Assuming in such a case that $a_0 > 0$ on $\partial \Omega$, the operator $L_v$ is again positive definite, so the spaces $H_v$ may be defined in an analogous fashion. According to [14] it turns out that for such a problem $H_v$ will consist of those $w \in H_v$, for which functions the following boundary conditions are fulfilled,

$$\frac{\partial}{\partial \nu} L^m w|_{\partial \Omega} = 0,$$

with $0 \leq m \leq [s/2] - 1$ and $\partial/\partial \nu$ denoting the derivative with respect to the outward normal to $\partial \Omega$. The “smoothing” property still holds here, so we may again introduce $T$ and $T_h$ and then consider both semidiscrete and fully discrete problems. The analysis below also covers this case of boundary conditions.

Throughout this work $C$ denotes a positive generic constant, not necessarily the same in any two places.

2. Preliminary results. Having outlined in § 1 the principal results of this work, we would like to demonstrate that the solution $u^{h_p}$ of (1.8) or, equivalently, of (1.10), has a spectral representation similar to (1.2). Let us observe that in view of (1.5)

$$\varphi_h = T_h^{-1}$$

is positive definite on $S_h$. Then, considering (1.10) and applying the Fourier method of separation of variables in a similar fashion as in the continuous case, we shall discover that

$$u^{h_p}(x, t) = \Gamma(p + 1) \sum_{n=1}^{N_h} g_n^h \phi_n^h(x) \left(\frac{\sqrt{\lambda_n^h} t}{2}\right)^{-p} J_p\left(\sqrt{\lambda_n^h} t\right),$$

where $N_h = \text{dim } S_h$ and $\{\lambda_n^h\}$ with $\{\phi_n^h\}$, $1 \leq n \leq N_h$, are eigenvalues and eigenfunctions of $\varphi_h$, respectively. Also it is worth remembering that $\lambda_n^h = 1/\mu_n^h$, where $\{\mu_n^h\}$, $1 \leq n \leq N_h$, is the set of eigenvalues of $T_h$. 
Next, we shall introduce some additional norms and their equivalent representations. For any $s \in \mathbb{R}$ let
\[
\| w \|_{s,h} = \left( \sum_{n=1}^{N_h} \left( \lambda_n^h \right)^s (w_n^h)^2 \right)^{1/2}, \quad \text{with } w_n^h = (w, \phi_n^h),
\]
be a norm on $S_h$. Note that if $s = 0$ then $\| \cdot \|_{0,h}$ becomes the usual $L_2$ norm, independent of $h$. For $s \in \mathbb{Z}$ it is obvious that this norm can be written in the equivalent form as
\[
\| w \|_{s,h} = (T^{-s} w, w)^{1/2}. \quad \text{Furthermore, an analogous representation exists for } \| \cdot \|_s, \text{ so }
\]
\[
\| w \|_s = (T^{-s} w, w)^{1/2} \quad \text{whenever } s \in \mathbb{Z}.
\]

The following lemma contains the smoothing property among its several assertions.

**Lemma 2.1.** Assume that $0 \leq s \leq p + \frac{1}{2}$. Then there exists a constant $C > 0$ such that for $t > 0$

\begin{align}
(2.2) \quad &\| u^p(t) \|_s \leq C t^{-s} \| g \|, \\
(2.3) \quad &\| u^{p}(t) \| \leq C t^{-s} \| g \|_{-s}, \\
(2.4) \quad &\| u^{h,p}(t) \|_{s,h} \leq C t^{-s} \| g^h \|, \\
(2.5) \quad &\| u^{h,p}(t) \| \leq C t^{-s} \| g^h \|_{-s,h}.
\end{align}

**Proof.** Set $0 \leq s \leq p + \frac{1}{2}$ and consider the function
\[
F_{s,p}(z) = \left( \frac{z}{2} \right)^{s-p} J_p(z) = \sum_{k=0}^{\infty} \frac{(-1)^k (z/2)^{2k+s}}{k! \Gamma(p + k + 1)}, \quad 0 \leq z \in \mathbb{R}.
\]

Using both continuity and asymptotic behavior of this function, one can easily show that there exists a constant $C_{s,p} > 0$ such that $|F_{s,p}(z)| \leq C_{s,p}$ uniformly for all $z \geq 0$, provided $0 \leq s \leq p + \frac{1}{2}$. Employing this just derived bound, we may conclude that
\[
\| u^p(t) \|_s^2 \leq C_p \sum_{n=1}^{\infty} \lambda_n^s g_n^2 \left( \frac{\sqrt{\lambda_n t}}{2} \right)^{-2p} J_p^2(\sqrt{\lambda_n t}) \leq C_p t^{-2s} \sum_{n=1}^{\infty} g_n^2 \left( \frac{\sqrt{\lambda_n t}}{2} \right)^{2(s-p)} J_p^2(\sqrt{\lambda_n t}) \leq C t^{-2s} \sum_{n=1}^{\infty} g_n^2 = C t^{-2s} \| g \|_s^2, \quad t > 0,
\]

where $C$ depends exclusively on $p$ and $s$. This confirms the proposition (2.2). Following more or less along the lines of the above argument, we shall obtain (2.3)–(2.5). Q.E.D.

Observe that if we set $s = 0$ in (2.4) the resulting inequality will express the presence of $L_2$ stability that holds for the semidiscrete solution (2.1). Furthermore, it is valid for any starting function $g^h \in S_h$.

In the next step we shall prove the following technical result which will be used on certain occasions in subsequent sections.

**Lemma 2.2.** Suppose that $p \geq \frac{1}{2}$, $0 < \varepsilon < 1$ and $s \in \mathbb{R}$. Then there is a constant $C > 0$ such that

\begin{align}
(2.6) &\int_0^t \tau^{1-s} \| D_s u^p(\tau) \|_{s+\varepsilon}^2 \, d\tau \equiv C \| g \|_{s+\varepsilon}^2, \\
(2.7) &\int_0^t \tau^{-1} \| D_s u^p(\tau) \|_{s+1+\varepsilon}^2 \, d\tau \equiv C \| g \|_{s+1+\varepsilon}^2, \\
(2.8) &\int_0^t \tau \| D_s^2 u^p(\tau) \|_{s+1+\varepsilon}^2 \, d\tau \equiv C \| g \|_{s+1+\varepsilon}^2.
\end{align}
Proof. It is well known [4] that
\[ \frac{d}{dz}(z^{-p}J_p(z)) = -z^{-p}J_{p+1}(z), \]
which together with the spectral expansion (1.2) yields
\[ (2.9) \quad D_t u^p = -2^{-1}\Gamma(p+1)t \sum_{n=1}^{\infty} g_n \varphi_n \lambda_n \left( \frac{\sqrt{\lambda_n} t}{2} \right)^{-p} J_{p+1}(\sqrt{\lambda_n} t). \]

Next note that
\[ \left( \frac{z}{2} \right)^{-(p+1)} J_{p+1}(z) \leq K_{p, \epsilon} < \infty \]
uniformly with respect to \( z \geq 0 \) by the same type of argument as one employed in the proof of Lemma 2.1. Thus by virtue of (2.9) and the above bound we may conclude that
\[ \int_0^{t^*} \tau^{-1-\epsilon} \| D_t u^p(\tau) \|^2 d\tau \leq C_p \int_0^{t^*} \sum_{n=1}^{\infty} g_n^2 \lambda_n^{2(p+1)} \tau^{3-\epsilon} \left( \frac{\sqrt{\lambda_n} \tau}{2} \right)^{-(p+1)} J_{p+1}(\sqrt{\lambda_n} \tau) \, d\tau \leq C_p K_{p, \epsilon} \sum_{n=1}^{\infty} g_n^2 \lambda_n^{2(p+1)} \int_0^{t^*} \tau^{-1+\epsilon} \, d\tau \leq C \| g \|_{s+\epsilon}^2, \]
with \( C = C(p, \epsilon, t^*) \). This establishes (2.6) and the remaining two assertions (2.7)–(2.8) can be deduced via an obvious adaptation of the above proof. Q.E.D.

The following recursion formulae, first established by Weinstein [22]–[23],
\[ (2.10) \quad D_t u^p = -t \mathcal{L} u^{p+1} \]
and
\[ (2.11) \quad D_t u^p = \frac{2p}{t} (u^{p-1} - u^p), \]
can easily be proven by using the spectral expansion (1.2). Formulae (2.10) and (2.11) were systematically exploited by Carroll (see [12] for references) in a general existence-uniqueness theory of the Euler–Poisson–Darboux equations and it turns out that they have a group-theoretic significance. In this part of the section we shall derive several additional recursion formulae and their counterparts for the semidiscrete case. Then they will be extensively used later on in the subsequent sections, in particular in §§ 4 and 5.

First, combining the equation in (1.1) with (2.10), we deduce that
\[ (2.12) \quad D_t^2 u^p = \frac{2p+1}{2(p+1)} \mathcal{L} u^{p+1} - \mathcal{L} u^p. \]

Second, equating (2.10) and (2.11), we obtain
\[ (2.13) \quad \mathcal{L} u^p = \frac{4p(p-1)}{t^2} (u^{p-1} - u^{p-2}). \]
Next, differentiating (2.13) and applying (2.10) with (2.11) to the result, we immediately arrive at

\[ \mathcal{L}D_t u^p = \frac{4p^2}{t^2} D_t u^{p-1} - \frac{4p(p-1)}{t^2} D_t u^{p-2}. \]  

(2.14)

Proceeding further, we have, in view of (1.1),

\[ \frac{2p+1}{t} D_t u^p = -\mathcal{L}u^p - D_t^2 u^p. \]  

(2.15)

Using the spectral representation (2.1) and following the lines of the proof of (2.10) and (2.11), we deduce at once

\[ D_t u^{h,p} = -\frac{t \mathcal{L}_h u^{h,p+1}}{2(p+1)} \]  

(2.16)

and

\[ D_t u^{h,p} = \frac{2p}{t} (u^{h,p-1} - u^{h,p}). \]  

(2.17)

Having the basic formulae (2.16) and (2.17) at our disposal, we can similarly derive the rest of the semidiscrete analogues, that is

\[ D_t^2 u^{h,p} = \frac{2p+1}{2(p+1)} \mathcal{L}_h u^{h,p+1} - \mathcal{L}_h u^{h,p}, \]  

(2.18)

\[ \mathcal{L}_h u^{h,p} = \frac{4p(p-1)}{t^2} (u^{h,p-1} - u^{h,p-2}), \]  

(2.19)

\[ \mathcal{L}_h D_t u^{h,p} = \frac{4p^2}{t^2} D_t u^{h,p-1} - \frac{4p(p-1)}{t^2} D_t u^{h,p-2}, \]  

(2.20)

\[ \frac{2p+1}{t} D_t u^{h,p} = -\mathcal{L}_h u^{h,p} - D_t^2 u^{h,p}. \]  

(2.21)

3. \textit{L}_2 \textit{ estimates of the error for smooth data.} The following manipulation of an energy method allows us to derive optimal in \( L_2 \) results similar to those in [8] obtained for the parabolic case. This technique is different from the argument in [3] in the more careful use of a priori regularity assumptions (see Lemma 2.2).

Let us suppose that the starting function is chosen in such a way that

\[ \|g^h - g\| \leq Ch\|g\|. \]  

(3.1)

\textbf{Theorem 3.1.} Assume that (1.5), (1.6) and (3.1) hold. If \( p \geq \frac{1}{2} \) and \( 0 < \epsilon < 1 \), then there exists a constant \( C > 0 \) such that

\[ \|u^{h,p} - u^p\|_{L_2(L_2)} \leq Ch \epsilon \|g\|_{L_2(L_2)}. \]

Proof. Introduce the error function \( e^p = u^{h,p} - u^p \), then, in view of (1.3) and (1.8), it will satisfy the following problem,

\[ D_t^2 T_h e^p + \frac{2p+1}{t} D_t T_h e^p + \epsilon^p = \rho^p, \quad 0 < t \leq t^*, \quad p > -\frac{1}{2}, \]  

(3.2)

\[ e^p|_{t=0} = g^h - g, \quad D_t e^p|_{t=0} = 0, \]
where \( \rho^p = [T - T_h](D_2^2 u^p + ((2p + 1)/t) D_1 u^p) \) which, in light of (1.1), can be rewritten as \( \rho^p = (T_h - T) \mathcal{L} u^p \). Multiplying (3.2) by \( D_1 \rho^p \) and taking into account (1.5), we arrive at

\[
\frac{d}{dt} \| \rho^p \|^2 \leq 2 \frac{d}{dt} \left( \rho^p, e^p \right) - 2(D_1 \rho^p, e^p),
\]

where \( \| \cdot \|^2 = \| \cdot \|^2 + (D_1 T_h, D_1 \cdot) \) is introduced as an auxiliary norm on \( L_2 \). Next, integrate (3.3) from 0 to \( t \) to obtain

\[
\| e^p(t) \|^2 \leq 2(\rho^p(t), e^p(t)) + 2\| (T_h - T) \mathcal{L} g \|^2 \\
+ 3\| g^h - g \|^2 + 2 \int_0^t \| (D_1 \rho^p(\tau), e^p(\tau)) \| d\tau
\]

\[
\leq \frac{1}{2} \| e^p(t) \|^2 + C\left\{ \| \rho^p(t) \|^2 + \| (T_h - T) \mathcal{L} g \|^2 + 2\| g^h - g \|^2 \\
+ \int_0^t \tau^{-1+\varepsilon} \| D_1 \rho^p(\tau) \|^2 d\tau + \int_0^t \tau^{-1+\varepsilon} \| e^p(\tau) \|^2 d\tau \right\},
\]

where \( 0 < \varepsilon < 1 \). In the following step observe that \( \| \cdot \| \leq \| \cdot \| \) which in combination with a kickback argument and the Gronwall inequality leads us to

\[
\| e^p(t) \|^2 \leq C\left\{ \exp \int_0^t C\tau^{-1+\varepsilon} d\tau \right\}
\]

\[
\cdot \left\{ \sup_{0 \leq \tau \leq t} \| \rho^p(t) \|^2 + \| g^h - g \|^2 + \int_0^t \tau^{-1+\varepsilon} \| D_1 \rho^p(\tau) \|^2 d\tau \right\},
\]

for all \( 0 \leq t \leq t^* \). Denote \( C_{\varepsilon, t^*} = \int_0^{t^*} C\tau^{-1+\varepsilon} d\tau \); then (3.5) together with (1.6), (2.6) and (3.1) yields the sought for result. Also note that proceeding along the lines of the above proof, one can easily obtain a corresponding result for \( \| e^p \| \). Q.E.D.

**Remark 3.1.** It can be easily shown that the optimal rate of convergence \( O(h^r) \) in \( L_2 \) is obtained by using \( g^h = P_0 g \), for then \( \| P_0 g - g \| \leq C h^r \| g \| \). Another good choice of starting values is \( g^h = P_0 g \). Then we have \( \| P_0 g - g \| \leq C h^r \| g \| \), as well.

For our future needs we shall derive \( L_2 \) estimates of the first time derivative of the error. Only this time we shall employ a certain modification of the ideas used by Dupont in [13].

**Theorem 3.2.** Suppose that (1.4), (1.5) and (1.9) hold. Then for \( 0 < \varepsilon < 1 \) there is a constant \( C > 0 \) such that

\[
\| D_1 (u_d - u^p) \|_{L_\infty(L_2)} \leq C h^r \| g \|_{r+1+\varepsilon},
\]

provided \( g^h = P_0 g \) and \( p \geq \frac{1}{2} \).

**Proof.** Define \( w^p : [0, t^*] \to S_h \) by \( w^p = P_0 u^p \). Denote \( \eta^p = w^p - u^p \), then it is easy to observe that \( w^p \) satisfies the following equation,

\[
D_1^2 w^p + \frac{2p+1}{t} D_1 w^p + \mathcal{L}_h w^p = P_0 \left( D_1^2 \eta^p + \frac{2p+1}{t} D_1 \eta^p \right).
\]

Introduce \( \xi^p = u_d - u^p \); clearly \( \xi^p \in S_h \) and it is a solution of the problem

\[
D_1^2 \xi^p + \frac{2p+1}{t} D_1 \xi^p + \mathcal{L}_h \xi^p = -P_0 \left( D_1^2 \eta^p + \frac{2p+1}{t} D_1 \eta^p \right),
\]

\[
\xi^p|_{t=0} = 0, \quad D_1 \xi^p|_{t=0} = 0.
\]
Multiplying (3.7) scalarly by $D_x^p$ and using the assumptions (1.5) and (1.9) together with a kickback argument, we obtain

$$\frac{1}{2} \frac{d}{dt} \left\| \zeta^p \right\|^2 + \frac{2p+1}{t} \left\| D_x^p \right\|^2 \leq C_p \{ t \left\| D_x^2 \eta^p \right\|^2 + t^{-1} \left\| D_t \eta^p \right\|^2 \} + \frac{2p+1}{t} \left\| D_x^p \right\|^2,$$

(3.8)

where $\left\| \cdot \right\|_2 = \left\| D_x \cdot \right\|^2 + \left\| \cdot \right\|_2$ is introduced as an auxiliary norm on $S_n$. Then accounting for the initial data in (3.7) and integrating (3.8) from 0 to $t$, we easily obtain

$$\left\| \zeta^p(t) \right\|_2^2 \leq C_p \int_0^t \left( \tau \left\| D_x^2 \eta^p(\tau) \right\|^2 + \tau^{-1} \left\| D_t \eta^p(\tau) \right\|^2 \right) d\tau,$$

(3.9)

for all $0 \leq t \leq t^*$. Next observe that

$$\left\| D_x^i \eta^p(t) \right\| = \left\| (P_e - I) D_x^i u^p(t) \right\| \leq Ch^r \left\| D_x^i u^p(t) \right\|,$$

(3.10)

with $0 \leq i \leq 2$. Combining (3.9), (3.10) and Lemma 2.2, we are ready to conclude that

$$\left\| \zeta^p(t) \right\|_1 \leq Ch^r \left\| g \right\|_{r+1+\varepsilon}, \quad 0 \leq t \leq t^*.$$

(3.11)

Finally, note that $u^{h,p} - u^p = \xi^p + \eta^p$, implying that

$$\left\| D_t (u^{h,p} - u^p) \right\| \leq \left\| D_x^p \right\| + \left\| D_t \eta^p \right\|.$$

(3.12)

Now digress for a moment to find, in view of (2.9), that

$$\left\| D_x u^p(t) \right\|_2 \leq C \left\| g \right\|_{s+1}, \quad 0 \leq t \leq t^*.$$

(3.13)

Combining (3.12) with (3.10), (3.11), (3.13) and the fact that $\left\| D_t \cdot \right\| \leq \left\| \cdot \right\|_1$, we prove the assertion of this theorem. Q.E.D.

**Remark 3.2.** Note that according to Theorem 3.2 the optimal rate of convergence $0(h^r)$ in $L_2$ is obtained only if $\zeta^p(0) = D_x^p(0) = 0$ or, equivalently, if $g^h = P_x g$.

### 4. Error estimates for nonsmooth data.

Our main objective in this section consists in obtaining interior in time $L_2$ error estimates for the case when $g \in H^s_e$ with arbitrary $0 < e < 1$. Since the technique used is intrinsically linked to the original idea of employing Weinstein recursion formulae, all proofs in this section will be presented in greater detail than before.

At the first step we shall observe that for any $1 \leq m \in \mathbb{Z}$ the following holds,

$$T^m_h - T^m = (T^{m-1}_h - T^{m-1})(T_h - T) + T^{m-1}(T_h - T) + (T^{m-1}_h - T^{m-1}) T.$$

(4.1)

Applying (4.1) repeatedly $m-2$ times to the last term of its own right-hand side, we easily arrive at the following operator decomposition:

$$T^m_h - T^m = \sum \left\{ (T^{m-k-1}_h - T^{m-k-1})(T_h - T) T^k + T^{m-k-1}(T_h - T) T^k \right\}.$$

(4.2)

As an easy consequence of (4.2) we have the following set of estimates.

**Lemma 4.1.** Suppose that $1 \leq m \in \mathbb{Z}$ and (1.6) with (1.7) hold. Then there is a constant $C > 0$ such that

$$\left\| (T^m_h - T^m) w \right\| \leq C h^m \| w \| \quad \text{for} \quad m \leq \frac{r}{2}, \quad r = 2, 4, \ldots,$$

(4.3)

$$\left\| (T^m_h - T^m) w \right\|_{-1} \leq C h^{m+1} \| w \| \quad \text{for} \quad m \leq \frac{r-1}{2}, \quad r = 3, 5, \ldots.$$
Proof. Let $m = 1$; then (1.6) and (1.7) imply
\[ \| (T_h - T) w \| \leq Ch^2 \| w \| \quad \text{and} \quad \| (T_h - T) w \|_{-1} \leq Ch^3 \| w \|, \]
respectively. Now let us make an induction assumption that for $1 \leq l \leq m - 1$ the following inequality holds,
\[ \| (T^l_h - T^l) w \|_{-l} \leq Ch^{2l+i} \| w \|, \]
with $i = 0, 1$. Then for $m \geq 2$ we obtain, via (4.5) combined with (1.6), the following result,
\[ \| (T^{m-k-1}_h - T^{m-k-1}) (T_h - T) T^k w \|_{-i} \leq Ch^{2m+i} \| w \|. \]
Next, it follows, by virtue of (1.7), that
\[ \| (T^{m-k-1}_h - T^{m-k-1}) (T_h - T) T^k w \|_{-i} \leq Ch^{2m+i} \| w \|. \]
Note that (4.2) yields
\[ \| (T^m_h - T^m) w \|_{-i} \leq \sum \{ \| (T^{m-k-1}_h - T^{m-k-1}) (T_h - T) T^k w \|_{-i} + \| T^{m-k-1}_h - T^{m-k-1} (T_h - T) T^k w \|_{-i} \}. \]
Substituting (4.6)–(4.7) into (4.8) and setting $i = 0, 1$ proves the assertions (4.3) and (4.4), respectively. Q.E.D.

Employing the above “tools”, we are ready to derive the sought for estimates. Because of different techniques used in the cases of even and odd values of $r$ we shall first consider the case when $r$ is assumed to be even. So recalling that $\mathcal{L} = T^{-1}$ and $\mathcal{L}_h = T_h^{-1}$, then inverting (2.13) and (2.19), we have
\[ u^p = \frac{4p(p-1)}{t^2} (T^{p-1} - T^{p-2}) \]
and
\[ u^{h,p} = \frac{4p(p-1)}{t^2} (T_h^{h,p-1} - T_h^{h,p-2}). \]
In view of (4.9) and (4.10) the error can be represented as
\[ u^{h,p} - u^p = \frac{4p(p-1)}{t^2} [(T_h^{h,p-1} - T^{p-1}) - (T_h^{h,p-2} - T^{p-2})]. \]
Applying (4.9) with (4.10) repeatedly $m - 1$ times to the right-hand side of (4.11), we eventually arrive at
\[ u^{h,p} - u^p = t^{-2m} \sum_{k=0}^{m} C_{p,m,k} (T_h^{h,p-m-k} - T^m u^{h,p-m-k}). \]
Throughout this section we shall always assume that $g^h = P_0 g$. Introduce an auxiliary function $\bar{u}^p$, defined as a solution of (1.3) with the initial data $g^h$. Next, consider $\tilde{\bar{u}}^p = T^m \bar{u}^p$ and $v^{h,p} = T_h^{h,p} \bar{u}^p$ which, apparently, satisfy
\[ D_t^2 \tilde{\bar{u}}^p + \frac{2p+1}{t} D_t \tilde{\bar{u}}^p + \tilde{\bar{u}}^p = 0, \quad 0 < t \leq t^*, \]
\[ \tilde{\bar{u}}^p|_{t=0} = T^m g^h, \quad D_t \tilde{\bar{u}}^p|_{t=0} = 0, \]
and

\[ D_t^2 T_h v^{h,p} + \frac{2p+1}{t} D_t T_h v^{h,p} + v^{h,p} = 0, \quad 0 < t \leq t^*, \]

(4.14)

\[ v^{h,p} \big|_{t=0} = T_h^n g^h, \quad D_t v^{h,p} \big|_{t=0} = 0. \]

Setting \( 2m = r \) and \( w = g^h \) in (4.3) leads us to

\[ \| (T_h^{r/2} - T_t^{r/2}) g^h \| \leq C h^r \| g^h \|, \]

which, in view of the fact that \( T_t^{r/2} g^h \in \dot{H}_r \), permits us to apply Theorem 3.1 to the problems (4.13) and (4.14), with \( 2m = r \). Then we obtain, for \( p \geq \frac{1}{2}, 0 < \epsilon < 1, r = 2, 4, \ldots, \) that

\[ \| T_h^{r/2} u^{h,p}(t) - T_t^{r/2} u^p(t) \| \leq C h^r \| g^h \|, \quad 0 \leq t \leq t^*. \]

(4.15)

Here we took into account that \( \| P_0 w \|_\infty \leq C \| w \|_\infty \), which fact can be proven by interpolation. As a natural consequence of (4.15) we have the following result for positive \( t \).

**Lemma 4.2.** Assume that \( p \geq r + \frac{1}{2} \) with \( r = 2, 4, \ldots \); then there exists a constant \( C > 0 \) such that

\[ \| u^{h,p}(t) - u^p(t) \| \leq \frac{1}{t} \| g \|, \quad 0 < t \leq t^*. \]

Proof. Indeed, choose \( 2m = r \) and consider (4.12) with \( \tilde{u}^p \) instead of \( u^p \). Then it easily follows, if we set \( C_{p,r} = \max_{0 \leq k \leq r/2} \| C_{p,r,k} \| \), that

\[ \| u^{h,p} - \tilde{u}^p \| \leq C_{p,r} t^{-r} \sum_{k=0}^{r/2} \| T_h^{r/2} u^{h,p-r/2-k} - T_t^{r/2} \tilde{u}^{p-r/2-k} \|. \]

(4.16)

The desired result follows by applying (4.15) to every term on the right-hand side of (4.16). Q.E.D.

**Lemma 4.3.** Let \( p \geq r + \frac{1}{2} \) with any \( 2 \leq r \in \mathbb{Z} \); then there is a constant \( C > 0 \) such that

\[ \| u^p(t) - \tilde{u}^p(t) \| \leq \frac{1}{t} \| g \|, \quad 0 < t \leq t^*. \]

Proof. Introduce \( w^p = u^p - \tilde{u}^p \), which apparently satisfies the following problem,

\[ D_t^2 T w^p + \frac{2p+1}{t} D_t T w^p + w^p = 0, \quad 0 < t \leq t^*, \]

(4.17)

\[ w^p \big|_{t=0} = g - P_0 g, \quad D_t w^p \big|_{t=0} = 0. \]

Observe that \( \| P_0 g - g \|_{-r} \leq C h^r \| g \| \); then, applying (2.3) to \( w^p \), we obtain the sought for estimate. Q.E.D.

Finally, we shall complete the case of even values of \( r \) by combining the triangle inequality with Lemmas 4.2 and 4.3.

**Lemma 4.4.** Suppose that \( g^h = P_0 g, 0 < \epsilon < 1 \) and \( p \geq r + \frac{1}{2} \) with \( r = 2, 4, \ldots \); then there exists a constant \( C > 0 \) such that

\[ \| u^{h,p}(t) - u^p(t) \| \leq \frac{1}{t} \| g \|, \quad 0 < t \leq t^*. \]

In the following part of this section we shall concern ourselves with the extension of the above result to the case of odd values of \( r \). This time, in order to sustain the optimal rate of convergence \( O(h^r) \) in \( L_2 \), we shall temporarily employ an auxiliary starting function \( \tilde{g} = \mathcal{S}_n^m P_\epsilon T^m g \), with suitable \( m \in \mathbb{Z} \).
So, let us invert (2.14) and (2.20) to obtain

\[
(4.18) \quad D_t u^p = \frac{4p^2}{t^2} D_t T u^{p-1} - \frac{4p(p-1)}{t^2} D_t T u^{p-2},
\]

\[
(4.19) \quad D_t u^{h,p} = \frac{4p^2}{t^2} D_t T_h u^{h,p-1} - \frac{4p(p-1)}{t^2} D_t T_h u^{h,p-2}.
\]

Next, by inverting (2.10) and (2.16), we deduce

\[
(4.20) \quad u^p = -\frac{2p}{t} D_t T u^{p-1},
\]

\[
(4.21) \quad u^{h,p} = -\frac{2p}{t} D_t T_h u^{h,p-1},
\]

which allow us to express the error as

\[
(4.22) \quad u^{h,p} - u^p = -\frac{2p}{t} D_t (T_h u^{h,p-1} - T u^{p-1}).
\]

Applying (4.18) and (4.19) repeatedly \( m - 1 \) times to (4.22), we eventually arrive at

\[
(4.23) \quad u^{h,p} - u^p = t^{-2m+1} \sum_{k=1}^{m} C_{p,m,k} D_t (T_h u^{h,p-m-k+1} - T u^{p-m-k+1}).
\]

In the next step we shall introduce an auxiliary function \( \tilde{u}^{h,p} \), defined as a solution of (1.8) with \( g^h = \tilde{g}^h \). We then again consider \( v^p = T^m u^p \) and \( \tilde{v}^{h,p} = T_h \tilde{u}^{h,p} \), which functions satisfy

\[
(4.24) \quad D_t^2 v^p + \frac{2p+1}{t} D_t v^p + \mathcal{L} v^p = 0, \quad 0 < t \equiv t^*,
\]

\[
(4.25) \quad v^p|_{t=0} = T^m g, \quad D_t v^p|_{t=0} = 0
\]

and

\[
(4.26) \quad D_t^2 \tilde{v}^{h,p} + \frac{2p+1}{t} D_t \tilde{v}^{h,p} + \mathcal{L}_h \tilde{v}^{h,p} = 0, \quad 0 < t \equiv t^*,
\]

\[
(4.27) \quad \tilde{v}^{h,p}|_{t=0} = T^m \tilde{g}^h, \quad D_t \tilde{v}^{h,p}|_{t=0} = 0,
\]

respectively. According to Remark 3.2 we see that Theorem 3.2 gives us the optimal rate of convergence \( O(h^r) \) in \( L^2 \) only if \( \tilde{g}^h \) is chosen in such a fashion that \( T^m \tilde{g}^h = P_t T^m g \). Then we obtain

\[
(4.26) \quad \|D_t (T_h \tilde{u}^{h,p} - T^m u^p)\| \leq C h^r \|g\|_{r+1/2-2m},
\]

for \( 0 \leq t \equiv t^* \) and \( p \geq \frac{1}{2} \), provided

\[
(4.27) \quad \tilde{g}^h = \mathcal{L}_h^m P_t T^m g.
\]

Let \( 2m - 1 = r \) in (4.23), (4.26) and (4.27), then

\[
(4.28) \quad \|\tilde{u}^{h,p} - u^p\| \leq C_{p,m} t^{-r} \sum_{k=1}^{(r+1)/2} \|D_t (T_h^{r+1/2} u^{h,p-(r+1)/2-k+1} - T^{(r+1)/2} u^{p-(r+1)/2-k+1})\|,
\]
where once more
\[ C_{p,r} = \max_{1 \leq k \leq (r+1)/2} |C_{p,r,k}|. \]

Furthermore, we have
\[
\| D_t (T_h^{(r+1)/2} \tilde{u}^{h,p}(t) - T^{(r+1)/2} u^p(t)) \| \leq Ch^r \| g \|, \]

for \( p \geq \frac{1}{2}, 0 < \varepsilon < 1 \) and \( 0 \leq t \leq t^* \) with \( r = 3, 5, \cdots \), provided
\[ \tilde{g}^h = T_h^{(r+1)/2} p_0 T^{(r+1)/2} g. \]

Applying (4.29) to every term on the right-hand side of (4.28), we deduce the following result.

**Lemma 4.5.** Assume that (4.27) holds with \( m = (r+1)/2 \). Let \( 0 < \varepsilon < 1 \) and \( p \geq r + \frac{1}{2} \), where \( r = 3, 5, \cdots \); then there exists a constant \( C > 0 \) such that
\[ \| \tilde{u}^{h,p}(t) - u^p(t) \| \leq Ch^r \varepsilon \| g \|, \quad 0 < t \leq t^*. \]

Next, one can easily see that
\[
\| w \|_{-1,h} \leq C (h \| w \| + \| w \|_{-1}).
\]

The following two observations are trivial,
\[
(4.31) \quad \mathcal{L}_h P_e = P_0 \mathcal{L} \quad \text{or, equivalently,} \quad P_e T = T_h P_0,
\]
and
\[
(4.32) \quad P_0 g = \mathcal{L}_h P_0^s T_h^s g \quad \text{for any} \quad 0 \leq s \in \mathbb{Z}.
\]

This enables us to compare the above chosen \( \tilde{g}^h \) with \( g^h = P_0 g \), for then (4.31) implies
\[ \tilde{g}^h = T_h^{(r-1)/2} P_0 T^{(r-1)/2} g. \]

Hence, setting \( s = (r-1)/2 \) in (4.32) and subtracting it from \( \tilde{g}^h \), we arrive at
\[
(4.33) \quad \tilde{g}^h - g^h = T_h^{(r-1)/2} P_0 (T^{(r-1)/2} - T_h^{(r-1)/2}) g.
\]

By virtue of (4.30), (4.33) combined with
\[
\| Pw \|_{-1} \leq C (h \| w \| + \| w \|_{-1})
\]
we find ourselves in a position to proceed as follows,
\[
\| \tilde{g}^h - g^h \|_{-r,h} = \| \mathcal{L}_h^{(r-1)/2} P_0 (T_h^{(r-1)/2} - T^{(r-1)/2}) g \|_{-r,h}
\]
\[
\leq \| P_0 (T_h^{(r-1)/2} - T^{(r-1)/2}) g \|_{-1,h}
\]
\[
\leq C \{ \| (T_h^{(r-1)/2} - T^{(r-1)/2}) g \|_{-1} + h \| (T_h^{(r-1)/2} - T^{(r-1)/2}) g \| \}.
\]

Now set \( m = (r-1)/2 \) in (4.3) and (4.4), then apply the result to the right-hand side
of (4.34). It follows that
\[
(4.35) \quad \| \tilde{g}^h - g^h \|_{-r,h} \leq Ch^r \| g \|.
\]

Next, introduce \( w^{h,p} = u^{h,p} - \tilde{u}^{h,p} \) and note that it is a solution of
\[
D_t^2 w^{h,p} + \frac{2p+1}{t} D_t w^{h,p} + \mathcal{L}_h w^{h,p} = 0, \quad 0 < t \leq t^*,
\]
\[
w^{h,p} \big|_{t=0} = g^h - \tilde{g}^h, \quad D_t w^{h,p} \big|_{t=0} = 0.
\]
This fact allows us to apply (2.5) to $w^{h,p}$, then

\[(4.37) \quad \|u^{h,p}(t) - \tilde{u}^{h,p}(t)\| \leq Ct^{-r}\|g - \tilde{g}\|_{r, h},\]

for $p \geq r + \frac{1}{2}$ and $0 < t \leq t^*$. Combining (4.35) with (4.37), we immediately deduce the following technical result.

**Lemma 4.6.** Assume that $g^h = P_0 g$ and $p \geq r + \frac{1}{2}$ with $3 \leq r \in \mathbb{Z}$. Then there is a constant $C > 0$ such that

\[\|u^{h,p}(t) - \tilde{u}^{h,p}(t)\| \leq Ct^{-r}\|g\|, \quad 0 < t \leq t^*.\]

Finally, via the triangle inequality and Lemmas 4.5 and 4.6 we are ready to complete the case of odd values of $r$ with the following analogue of Lemma 4.4.

**Lemma 4.7.** Suppose that $g^h = P_0 g$, $0 < \varepsilon < 1$ and $p \geq r + 1/2$ with $3 \leq r \in \mathbb{Z}$, then there exists a constant $C > 0$ such that

\[\|u^{h,p}(t) - u^p(t)\| \leq Ct^{-r}\|g\|_\varepsilon, \quad 0 < t \leq t^*.\]

Combining the assertions of Lemmas 4.4 and 4.7, we eventually deduce the following general theorem, which holds for any $2 \leq r \in \mathbb{Z}$.

**Theorem 4.1.** Assume that $g^h = P_0 g$, $0 < \varepsilon < 1$ and $p \geq r + \frac{1}{2}$ with $2 \leq r \in \mathbb{Z}$. Then there is a constant $C > 0$ such that

\[\|u^{h,p}(t) - u^p(t)\| \leq Ct^{-r}\|g\|_\varepsilon, \quad 0 < t \leq t^*.\]

In conclusion of this section we shall derive certain $L_2$ estimates of derivatives of the error. Due to specific needs of the following sections, in which these estimates will be systematically exploited, we shall limit our attention only to some spatial derivatives of “even” orders.

**Theorem 4.2.** Suppose that $g^h = P_0 g$, $0 < \varepsilon < 1$ and $p \geq r + 2l + \frac{1}{2}$ with $2 \leq r, l + 2 \in \mathbb{Z}$. Then there exists a constant $C > 0$ such that for all $0 < t \leq t^*$

\[\|\mathcal{L}_l u^{h,p}(t) - \mathcal{L}_l u^p(t)\| \leq Ct^{-(r+2l)}\|g\|_\varepsilon.\]

**Proof.** Observe that the case of $l = 0$ coincides with the assertion of Theorem 4.1. Subtract (2.13) from (2.19) to obtain

\[(4.38) \quad \mathcal{L}_h u^{h,p} - \mathcal{L} u^p = \frac{4p(p-1)}{l^2}[(u^{h,p-1} - u^{p-1}) - (u^{h,p-2} - u^{p-2})],\]

from which it easily follows that for $l \geq 1$

\[(4.39) \quad \mathcal{L}_h^{l+1} u^{h,p} - \mathcal{L}_h^{l} u^p = \frac{4p(p-1)}{l^2}[(\mathcal{L}_h^{l-1} u^{h,p-1} - \mathcal{L}_h^{l-1} u^{p-1}) - (\mathcal{L}_h^{l-1} u^{h,p-2} - \mathcal{L}_h^{l-1} u^{p-2})].\]

Applying (4.39) repeatedly $l - 1$ times to its own right-hand side, we arrive, provided $p + \frac{1}{2} > l$, at

\[(4.40) \quad \mathcal{L}_h^{l} u^{h,p} - \mathcal{L}_h^{l} u^p = t^{-2l} \sum_{k=0}^{l} C_{p,l,k} (u^{h,p-l-k} - u^{p-l-k}).\]

As usual we set $C_{p,l} = \max_{0 \leq k \leq l} |C_{p,l,k}|$. Then in view of (4.40) combined with the result of Theorem 4.1 we obtain the sought for estimate for $l \geq 1$. Q.E.D.
5. Negative norm estimates. In this section we shall derive error estimates in the $\mathcal{H}_{-(r-2)}$ norm that subsequently will enable us to demonstrate the superconvergence phenomenon later on, in § 8. Throughout this entire section it will be assumed that $g \in \mathcal{H}_{-(r-2+\varepsilon)}$, $0 < \varepsilon < 1$, and $g^h = P_0 g$. The following lemma constitutes a certain negative norm analogue of Theorem 3.1, and it is derived via essentially the same technique.

**Lemma 5.1.** Let $0 < \varepsilon < 1$, $p \geq 1/2$ and $2 \leq r \in \mathbb{Z}$. Also assume that (1.5)-(1.7) hold and $g^h = P_0 g$. Then there exists a constant $C > 0$ such that

$$\| T_h u^{h,p} - T u^p \|_{L^\infty(\mathcal{H}_{-(r-2)})} \leq C h^{2r-2} \| g \|_{r-2+\varepsilon}.$$  

**Proof.** Observe that $v^p = T u^p$ and $v^{h,p} = T_h u^{h,p}$ are the solutions of two corresponding problems,

$$D_t^2 T v^p + \frac{2p+1}{t} D_t T v^p + v^p = 0, \quad 0 < t \leq t^*,$$  

$$v^p|_{t=0} = T g, \quad D_t v^p|_{t=0} = 0,$$

and

$$D_t^2 T_h v^{h,p} + \frac{2p+1}{t} D_t T_h v^{h,p} + v^{h,p} = 0, \quad 0 < t \leq t^*,$$

$$v^{h,p}|_{t=0} = T_h g^h, \quad D_t v^{h,p}|_{t=0} = 0.$$  

Set, as usual, $e^p = v^{h,p} - v^p$ and note that $T_h g^h = T_h P_0 g = P_e T g$. Then it becomes apparent that $e^p$ satisfies the following problem,

$$D_t^2 T_h e^p + \frac{2p+1}{t} D_t T_h e^p + e^p = \rho^p, \quad 0 < t \leq t^*,$$

$$e^p|_{t=0} = (P_e - I) g, \quad D_t e^p|_{t=0} = 0,$$

where $\rho^p = (T_h - T) \mathbb{L} v^p$.

Note that by virtue of (1.5) we have that for any $w \in L^2$ and $2 \leq r \in \mathbb{Z}$

$$(T_h w, w)_{-(r-2),h} \geq 0.$$  

Next, multiplying the equation in (5.3) by $D_t e^p$ in the $(\cdot, \cdot)_{-(r-2),h}$ sense, we easily obtain

$$\frac{1}{2} \frac{d}{dt} \| e^p \|_{-(r-2),h}^2 + \frac{2p+1}{t} (D_t T_h e^p, D_t e^p)$$

$$= \frac{d}{dt} (\rho^p, e^p)_{-(r-2),h} - (D_t \rho^p, e^p)_{-(r-2),h},$$

where in similarity to the previous cases

$$\| \cdot \|_{-(r-2),h} \leq \| \cdot \|_{-(r-2),h} + (T_h D_t \cdot, D_t \cdot)_{-(r-2),h}$$

is defined as an auxiliary norm on $L^2$. Taking into account that $\| \cdot \|_{-(r-2),h} \leq \| \cdot \|_{-(r-2),h}$, we shall integrate (5.4) from 0 to $t$. Consequently we obtain

$$\| e^p(t) \|_{-(r-2),h} \leq C_{p,e,t^*} \left\{ \| e^p(0) \|_{-(r-2),h}^2 + \sup_{0 \leq t \leq t^*} \| \rho^p(t) \|_{-(r-2),h}^2 + \int_0^{t^*} \tau^{1-\varepsilon} \| D_t \rho^p(\tau) \|_{-(r-2),h}^2 d\tau \right\},$$

$$\| e^p(t) \|_{-(r-2),h} \leq C_{p,e,t^*} \left\{ \| e^p(0) \|_{-(r-2),h}^2 + \sup_{0 \leq t \leq t^*} \| \rho^p(t) \|_{-(r-2),h}^2 + \int_0^{t^*} \tau^{1-\varepsilon} \| D_t \rho^p(\tau) \|_{-(r-2),h}^2 d\tau \right\},$$

$$\| e^p(t) \|_{-(r-2),h} \leq C_{p,e,t^*} \left\{ \| e^p(0) \|_{-(r-2),h}^2 + \sup_{0 \leq t \leq t^*} \| \rho^p(t) \|_{-(r-2),h}^2 + \int_0^{t^*} \tau^{1-\varepsilon} \| D_t \rho^p(\tau) \|_{-(r-2),h}^2 d\tau \right\}.$$
for all \(0 \leq t \leq t^*\). Digress for a moment and observe that in view of Lemma 4.1
\[
\|w\|_{-(r-2),h} \leq C(h^{2r-2}\|w\| + \|w\|_{-(r-2)}),
\]
for any \(w \in L_2\). Next, it is well known that
\[
\|(P_0 - I)w\|_{-r} \leq Ch^{2r-2}\|w\|_{r-2},
\]
implying that
\[
\|(P_0 - I)w\|_{-(r-2)} \leq \|(T_0P_0 - T)\mathcal{L}w\|_{-(r-2)} + \|(T_0 - T)\mathcal{L}w\|_{-(r-2)} + \|T(P_0 - I)\mathcal{L}w\|_{-(r-2)}
\]
\[
\leq Ch\|(P_0 - I)\mathcal{L}w\|_{-r} + Ch^{2r-2}\|w\|_{r-2}.
\]
Thus, applying (5.6) and (5.8) to the right-hand side of (5.5), we arrive at
\[
\|e_0(t)\|_{-(r-2),h} \leq Ch^{2r-2}\|g\|_{r-2},
\]
\[
\|\rho_0(t)\|_{-(r-2),h} \leq Ch^{2r-2}\|g\|_{r-2}
\]
and
\[
\|\rho^p(t)\|_{-(r-2),h} \leq Ch^{2r-2}\|\rho^p(t)\|_{r-2},
\]
for all \(0 \leq t \leq t^*\). Hence, using Lemma 2.2 and (5.11), we deduce that for \(p \geq \frac{1}{2}\)
\[
\int_0^{t^*} \tau^{r-2}\|\rho^p(\tau)\|^2_{-(r-2),h} d\tau \leq C_p, \epsilon h^{4r-4}\|g\|_{r-2}^2,
\]
Combining (5.5) with (5.9), (5.10) and (5.12), we are allowed to conclude that
\[
\|e_0(t)\|_{-(r-2),h} \leq Ch^{2r-2}\|g\|_{r-2+\epsilon}, \quad 0 \leq t \leq t^*.
\]
Now we shall digress for a moment to observe that some analogue to (5.6) is true. More precisely, for \(w \in L_2\)
\[
\|w\|_{-(r-2)} \leq C(h^{2r-2}\|w\| + \|w\|_{-(r-2),h}).
\]
Next, recall that by Theorem 3.1
\[
\|e_0(t)\| \leq Ch^r\|v^0(0)\|_{r+\epsilon},
\]
for \(0 \leq t \leq t^*\). Consequently, altogether (5.13)–(5.15) imply the sought for estimate. Q.E.D.

As an almost trivial consequence of Lemma 5.1 we have the following “non-smooth” data result.

**Corollary 5.1.** Suppose that \(0 < \epsilon < 1\), \(p \geq \frac{5}{2}\), \(2 \leq r \in \mathbb{Z}\) and \(g^h = P_0 g\). Then there is a constant \(C > 0\) such that
\[
\|u^{(p)}(t) - u^p(t)\|_{-(r-2)} \leq Ch^{2r-2}t^{-2}\|g\|_{r-2+\epsilon}, \quad 0 < t \leq t^*.
\]
**Proof.** The desired result follows by a straightforward application of the triangle inequality and Lemma 5.11 to the expression (4.11). Q.E.D.

Finally we shall conclude this section with the following result, which will play an important role in § 8.
Theorem 5.1. If \( p \geq \frac{3}{2} + 2l \) and \( 0 \leq r - 2, l \in \mathbb{Z} \), then there exists a constant \( C > 0 \) such that

\[
\| \mathcal{L}_{\phi}^h u_{h}^{p}(t) - \mathcal{L}_{\phi}^l u_{l}^{p}(t) \|_{-r-2} \leq C h^{2r-2} t^{-(2l+2)} \| g \|_{r-2+e},
\]

for \( 0 \leq t \leq t^* \), provided \( g^h = P_0 g \).

Proof. The sought for inequality easily follows by the triangle inequality and Corollary 5.1 applied to (4.40). Q.E.D.

Remark 5.1. Slightly modifying the proof of Lemma 5.1 and using essentially the same assumptions, one can easily arrive at the following negative norm analogue to Theorem 3.1,

\[
u_{h}^{p} - u_{l}^{p} \|_{\mathcal{H}(-r-2)} \leq C h^{r-2} \| g \|_{r+e},
\]

where \( p \geq \frac{1}{2} \) and \( g \in \mathcal{H}_{r+e} \), 0 < \( \varepsilon < 1 \).

Remark 5.2. Following along the lines of the proof of Lemma 5.1, one can easily convert it to obtain the following \( \mathcal{H}(-1) \) estimate of the error,

\[
u_{h}^{p} - u_{l}^{p} \|_{\mathcal{H}(-1)} \leq C \| g \|_{r-1+e},
\]

for \( p \geq \frac{1}{2} \), \( g \in \mathcal{H}_{r-1+e} \) and 0 < \( \varepsilon < 1 \).

6. Error estimates in the maximum norm. Motivated by the results obtained by Bramble, Schatz, Thomee and Wahlbin in [8], we shall derive similar \( L_{\infty} \) estimates of the error for the case of the Euler–Poisson–Darboux equations.

Throughout this section (with the only exception for Theorem 6.1) it will be assumed that the family \( \{T_h\}, 0 < h < 1 \), possesses the property that there exists a function \( \gamma(h) \) and a constant \( C > 0 \) such that for sufficiently small values of \( h \)

\[
(6.1) \quad |T_h w| \leq C |w|_1 \quad \text{and} \quad \| T_h w \| \leq C \| w \|_1,
\]

\[
(6.2) \quad |(T_h - T) w| \leq \gamma(h) |T w|_1.
\]

Before we begin dealing with the main objective of this section, namely, derivation of global and interior in time \( L_{\infty} \) estimates of the error in the general \( N \)-dimensional case, we would like to show that in the particular case of \( 1 \leq N \leq 3 \) it is possible to deduce quite sharp inequalities by using just (6.2) and replacing (6.1) with certain stronger estimates.

So, for the sake of a practically important theorem, which will be given below, let us state the following assumptions on \( \{T_h\} \),

\[
(6.3) \quad \| T_h w \|_1 \leq C \| T w \|_1,
\]

\[
(6.4) \quad |T_h w| \leq C \ln \frac{1}{h} \| T_h w \|_1 \quad \text{for} \ N = 2,
\]

\[
(6.5) \quad |T_h w| \leq C \ln \frac{1}{h} |T w| \quad \text{for} \ N = 3;
\]

the first one is trivial and the other two are proven in [19] and [20].

Theorem 6.1. Let \( g \in \mathcal{H}_{r+2+e} \) with \( 2 \leq r \in \mathbb{Z} \) and \( 0 < \varepsilon < 1 \). Also assume that \( g^h = P_g, p \geq \frac{1}{2} \) and (6.2) holds. Then there exists a constant \( C > 0 \) such that

\[
(6.6) \quad \| u_{h}^{p} - u_{l}^{p} \|_{L_{\infty}(I_{\infty})} \leq C \{ \gamma(h) + h' \} \| g \|_{r+1+e},
\]

for \( N = 1 \), provided (6.3) holds;

\[
(6.7) \quad \| u_{h}^{p} - u_{l}^{p} \|_{L_{\infty}(I_{\infty})} \leq C \left\{ \gamma(h) + h' \ln \frac{1}{h} \right\} \| g \|_{r+1+e},
\]

for \( N = 2 \) and

\[
(6.8) \quad \| u_{h}^{p} - u_{l}^{p} \|_{L_{\infty}(I_{\infty})} \leq C \left\{ \gamma(h) + h' \ln \frac{1}{h} \right\} \| g \|_{r+1+e},
\]

for \( N = 3 \).
whenever $N = 2$ and (6.4) is fulfilled;

(6.8) \[ \| u^{h,p} - u^p \|_{L^\infty(t_\infty)} \leq C \left( \gamma(h) + h' \ln \frac{1}{h} \right) \| g \|_{r+2+\varepsilon}, \]

for $N = 3$ under the assumption (6.5).

Proof. Observe that since $g \in H_{r+1+\varepsilon}$, $r \geq 2$, the "elliptic" projection $g_h = P_0 g$ is well defined. Consider two auxiliary functions $v^p = \mathcal{L}u^p$ and $v^{h,p} = \mathcal{L}_h u^{h,p}$, which obviously solve the following problems,

(6.9) \[ D_t^2 T v^p + \frac{2p+1}{t} D_t T v^p + v^p = 0, \quad 0 < t \leq t^*, \]

(6.10) \[ D_t^2 T_h v^{h,p} + \frac{2p+1}{t} D_t T_h v^{h,p} + v^{h,p} = 0, \quad 0 < t \leq t^*, \]

respectively.

Next we note that $\mathcal{L}_h P_e g = P_0 \mathcal{L} g$, implying that

(6.11) \[ \| \mathcal{L}_h g^h - \mathcal{L} g \|_{r-i} = \| (P_0 - I) \mathcal{L} g \|_{r+2-i} \leq C h' \| g \|_{r+2-i+\varepsilon}, \]

where $i = 0, 1$. By virtue of (6.11) we are allowed to use Theorem 3.2 and Remark 5.2, corresponding to $i = 0$ and $i = 1$. Then we obtain

(6.12) \[ \| \mathcal{L}_h u^{h,p} - \mathcal{L} u^p \|_{L^\infty(H_{r-i})} \leq C h' \| g \|_{r+2-i+\varepsilon}, \]

for $p \geq \frac{1}{2}$. Now we rewrite the equation in (3.2) as

(6.13) \[ u^{h,p} - u^p = \rho^p + T_h (\mathcal{L}_h u^{h,p} - \mathcal{L} u^p), \]

where $\rho^p = (T_h - T) \mathcal{L} u^p$. In what follows we treat the cases of $1 \leq N \leq 3$ separately.

First, by (6.3) we have

(6.14) \[ |T_h w| \leq C \| T_h w \|_1 \leq C \| T w \|_1 \leq C \| w \|_{r-1}, \]

for $N = 1$. Setting $w = \mathcal{L}_h u^{h,p} - \mathcal{L} u^p$ in (6.14) and applying the result to (6.13) yields

(6.15) \[ |u^{h,p} - u^p| \leq \| (T_h - T) \mathcal{L} u^p \| + C \| \mathcal{L}_h u^{h,p} - \mathcal{L} u^p \|. \]

Then using (6.2) combined with Sobolev's inequality and (6.12) with $i = 1$, we estimate the right-hand side of (6.15). Then assertion (6.6) follows at once.

In the case of $N = 2$ the assumption (6.4) implies

(6.16) \[ |T_h w| \leq C \ln \frac{1}{h} \| w \|_{r-1}, \]

which result allows us to mimic the entire proof of the previous case and obtain (6.7).

Finally, by virtue of (6.5) we deduce

(6.17) \[ |T_h w| \leq C \ln \frac{1}{h} \| w \|, \]

for $N = 3$. And the final inequality (6.8) follows as above, provided we employ (6.12) with $i = 0$. Q.E.D.
To treat the general $N$-dimensional case we shall need the following technical result, whose proof is based on certain ideas employed in [8].

**Lemma 6.1.** There exist $0 \leq M \in \mathbb{Z}$ and a constant $C > 0$, both dependent on $N$, such that

$$
|u^{h,p}(t) - u^p(t)| \leq C \left\{ \gamma(h) \sum_{m=1}^{M} \| \mathcal{L}^{m-1} u^p(t) \|_{r} + \| \mathcal{L}^{M} u^{h,p}(t) - \mathcal{L}^{M} u^p(t) \| \right\},
$$

for $p \geq \frac{1}{2}$ and $0 \leq t \leq t^*$, provided (6.1) and (6.2) hold.

**Proof.** By virtue of (1.3) and (1.8) we have

$$
\mathcal{L}^{m} u^{h,p} - \mathcal{L}^{m} u^p = \rho^{m,p} - \frac{2p+1}{t} T_h D_t (\mathcal{L}^{m} u^{h,p} - \mathcal{L}^{m} u^p),
$$

(6.18)

with $\rho^{m,p} = (T_h - T) \mathcal{L}^{m+1} u^p$ and $0 \leq m \in \mathbb{Z}$. Combining (2.15) and (2.21) with (6.18) we arrive at

$$
\mathcal{L}^{m} u^{h,p} - \mathcal{L}^{m} u^p = \rho^{m,p} + T_h (\mathcal{L}^{m+1} u^{h,p} - \mathcal{L}^{m+1} u^p).
$$

(6.19)

The rest of the argument is a simple modification of the proof of [8, Thm. 4.1]. Q.E.D.

Now under the assumption that $g^h$ is chosen in such a way that

$$
\| \mathcal{L}^{m} g^h - \mathcal{L}^{m} g \| \leq Ch^{r} \| g \|_{r+2M},
$$

(6.20)

which holds for instance with $g^h = T_h^M P_0 \mathcal{L}^{M} g$, $g^h = T_h^M P_e \mathcal{L}^{M} g$ or $g^h = T_h^M \mathcal{L}^{M} g$, we shall obtain the following $N$-dimensional analogue to Theorem 6.1.

**Theorem 6.2.** Let $g \in \dot{H}_s$ with $s = \max \{ r + 2M - 2 + N_0, r + 2M + \epsilon \}$, $2 \leq r \in \mathbb{Z}$, $0 < \epsilon < 1$ and $N_0 = [N/2] + 1$. In addition assume that (6.20) holds. Then there is a constant $C > 0$ such that

$$
\| u^{h,p} - u^p \|_{L^{\infty}(L^{\infty})} \leq C \{ h^{r} + \gamma(h) \} \| g \|_{s},
$$

provided $p \geq \frac{1}{2}$.

**Proof.** It can easily be seen that $v^p = \mathcal{L}^{M} u^p$ and $v^{h,p} = \mathcal{L}^{h} u^{h,p}$ are the respective solutions of the following auxiliary problems,

$$
D_t^2 T v^p + \frac{2p+1}{t} D_t T v^p + v^p = 0, \quad 0 < t \leq t^*,
$$

(6.21)

$$
v^p|_{t=0} = \mathcal{L}^{M} g, \quad D_t v^p|_{t=0} = 0,
$$

and

$$
D_t^2 T_h v^{h,p} + \frac{2p+1}{t} D_t T_h v^{h,p} + v^{h,p} = 0, \quad 0 < t \leq t^*,
$$

(6.22)

$$
v^{h,p}|_{t=0} = \mathcal{L}^{h} g^h, \quad D_t v^{h,p}|_{t=0} = 0.
$$

Since (6.20) is fulfilled, we are permitted to use Theorem 3.1, which implies that

$$
\| \mathcal{L}^{M} u^{h,p} - \mathcal{L}^{M} u^p \|_{L^{\infty}(L^{2})} \leq C h^{r} \| g \|_{r+2M+\epsilon},
$$

(6.23)

for $p \geq \frac{1}{2}$. By virtue of Sobolev's inequality we obtain in the next step that

$$
\sup_{0 \leq m \leq M} \| \mathcal{L}^{m-1} u^p(t) \|_{r} \leq C_{M,p} \| g \|_{r+2M+N_0-2},
$$

(6.24*)
where \( p > -\frac{1}{2} \). The final result follows by a direct application of (6.23) and (6.24*) to Lemma 6.1. Q.E.D.

We conclude this section by combining Theorem 4.2 with Lemma 6.1. As a result we obtain the following interior in time estimate for the case when \( g \in H_r \).

**Theorem 6.3.** Let \( g \in H_r \), \( 0 < s < 1 \), \( g^h = P_0 g \) and suppose that (6.1)-(6.2) hold. Then for any \( t_0 > 0 \) there exists a constant \( C > 0 \) such that

\[
|u^h - u(t)| \leq C \| \gamma (h) + h' \| \| g \|, \quad t_0 \leq t \leq t^*,
\]

provided \( p \geq \max \{ r + 2M + \frac{1}{2}, r + 2M - 5/2 + N_0 \} \).

**Proof.** Using Sobolev's inequality and Lemma 2.1, one can easily deduce

\[
(6.24) \quad \sum_{m=1}^{M} |\mathcal{L}^{m-1} u^h(t)| \leq C t^{-r+2M+2+1/2} \| g \|
\]

for \( p \geq r + 2M - 5/2 + N_0 \) and \( 0 < t \leq t^* \). In addition Theorem 4.2 implies

\[
(6.25) \quad \| \mathcal{L}_h u^h - \mathcal{L}_h u^h(t) \| \leq C t^{-r+2M+2+N_0} \| g \|
\]

where \( p \geq r + 2M + \frac{1}{2} \) and \( 0 < t \leq t^* \). Finally, fix any \( t_0 > 0 \), then the desired result follows by virtue of (6.24) and (6.25) combined with Lemma 6.1. Q.E.D.

### 7. Time-discrete procedure and error estimates.

In the present section we shall consider the Crank–Nicolson discrete procedure analogous to the one employed by Dupont for the wave equation [13]. Let \( \Delta t = t^*/K \), where \( 1 \leq K \in \mathbb{Z} \). For any function \( w \) defined at times \( t_k = k \Delta t \), \( 0 \leq k \leq K \), denote \( w_k = w(t_k) \). Furthermore, we shall use the following pieces of notation.

\[
\begin{align*}
\omega_{k+1/2} & = (w_{k+1} + w_k)/2, \\
\omega_{k, \theta} & = \theta w_{k+1} + (1 - 2\theta) w_k + \theta w_{k-1}, \quad 0 \leq \theta \leq 1, \\
\omega_{k+1/2} & = (w_{k+1} - w_k)/\Delta t, \\
\omega_{k+1/2} & = (w_{k+1} + w_k - w_k - w_{k-1})/2 \Delta t, \\
\omega_{k+1/2} & = (w_{k+1} - 2w_k + w_{k-1})/(\Delta t)^2.
\end{align*}
\]

In the current step we observe that the “continuous” solution \( u^p \) satisfies the following fully discrete equation,

\[
(7.1) \quad (\partial_t^2 u^p_{k}, \chi) + \frac{p+1}{t_k} (\delta_t u^p_{k}, \chi) + B(u^p_{k,1/4}, \chi) = \left( \beta^p_k + \frac{p+1}{t_k} \nu^p_k + \frac{(\Delta t)^2}{4} \mathcal{L} \partial_i^2 u^p_{k}, \chi \right),
\]

for \( 1 \leq k \leq K \) and all \( \chi \in S_h \). Here we used three new notations

\[
\begin{align*}
\beta^p_k & = \delta_t^2 u^p_k - D_t^2 u^p_k, \\
\nu^p_k & = \delta_t u^p_k - D_t u^p_k
\end{align*}
\]

and

\[
B(\phi, \psi) = \int_{\Omega} \sum_{i,j=1}^{N} \left( a_{ij} \frac{\partial \phi}{\partial x_i} \frac{\partial \psi}{\partial x_j} + a_{i0} \phi \psi \right) dx
\]

denotes the bilinear form associated with \( \mathcal{L} \).

The fully discrete solution is defined as a map \( U^p : \{t_k\}_{k=0}^{K} \rightarrow S_h \) such that

\[
(7.3) \quad \partial_t^2 U^p_k + \frac{p+1}{t_k} \delta_t U^p_k + \mathcal{L}_h U^p_{k,1/4} = 0, \quad 1 \leq k \leq K.
\]
Next we introduce an auxiliary function \( W_k^p : \{ t_k \}_{k=0}^K \rightarrow S_h \) defined as the elliptic projection of the exact solution, that is
\[
W_k^p = P_e u_k^p, \quad 0 \leq k \leq K.
\]

Now we denote \( \zeta_k^p = U_k^p - W_k^p \) and \( \eta_k^p = W_k^p - u_k^p \). Then it follows, in view of (7.1), (7.3) and (7.4), that
\[
(\partial_t^2 \zeta_k^p, \chi) + \frac{2p+1}{t_k} (\delta \zeta_k^p, \chi) + B(k_{1/4}, \chi)
\]
\[
= - \left( \partial_t^2 \eta_k^p + \frac{2p+1}{t_k} \delta \eta_k^p + \frac{2p+1}{t_k} \nu_k^p + \frac{(\Delta t)^2}{4} \mathcal{L} \partial_t^2 u_k^p, \chi \right),
\]
for \( 1 \leq k \leq K \) and \( \chi \in S_h \). Set \( \chi = \delta \zeta_k^p \) in (7.5) and sum up from 1 to \( K \), using a "kickback" trick. As a result we obtain
\[
\| \partial_t \zeta_k^{p+1/2} \| + \| \zeta_k^{p+1/2} \| \leq C \left\{ \| \partial_t \zeta_k^{p+1/2} \|^2 + \| \zeta_k^{p+1/2} \|^2 \right\}
\]
\[
+ \Delta t \sum_{k=1}^K \left( t_k \| \partial_t \eta_k^p \|^2 + t_k^{-1} \| \delta \eta_k^p \|^2 + t_k \| \beta_k^p \|^2 \right.
\]
\[
\left. + t_k^{-1} \| \nu_k^p \|^2 + (\Delta t)^4 t_k \| \mathcal{L} \partial_t^2 u_k^p \|^2 \right\}.
\]

In order to estimate the above sum we shall need the following representations, which can be easily proven by integration by parts. First, we have
\[
\| \partial_t \zeta_k^p \|^2 \leq C \sum_{k=1}^K \left( t_k \| \partial_t \eta_k^p \|^2 + t_k^{-1} \| \delta \eta_k^p \|^2 + t_k \| \beta_k^p \|^2 \right.
\]
\[
\left. + t_k^{-1} \| \nu_k^p \|^2 + (\Delta t)^4 t_k \| \mathcal{L} \partial_t^2 u_k^p \|^2 \right\}.
\]

Combining this estimate with (2.8), we immediately deduce
\[
\sum_{k=1}^K t_k \| \partial_t \zeta_k^p \| \Delta t \leq C h^2 \| g \|_{r+1+\varepsilon}^2.
\]

Similarly one can easily compute that
\[
\delta \eta_k^p = (\Delta t)^{-1} \int_{-\Delta t}^{\Delta t} D_t \eta^p(t_k + \tau) \, d\tau,
\]
yielding as a consequence the following inequalities,
\[
t_k^{-1} \| \delta \eta_k^p \|^2 \leq C h^2 \| g \|_{r+1+\varepsilon}^2.
\]

Combining this estimate with (2.8), we immediately deduce
\[
\sum_{k=1}^K t_k \| \delta \eta_k^p \| \Delta t \leq C h^2 \| g \|_{r+1+\varepsilon}^2.
\]

Combining this estimate with (2.8), we immediately deduce
\[
\sum_{k=1}^K t_k \| \delta \eta_k^p \| \Delta t \leq C h^2 \| g \|_{r+1+\varepsilon}^2.
\]
Proceeding further, we have

\[ \beta_k^p = \frac{1}{12} \int_{-\Delta t}^{\Delta t} (|t| - |t|) \left[ 3 - 2 \left( 1 - \frac{|t|}{\Delta t} \right)^2 \right] D^k_t (t_k + \tau) \ d\tau, \]

which result together with (2.12) implies

\[ t_k \| \beta_k^p \|^2 \leq C (\Delta t)^3 \int_{-\Delta t}^{\Delta t} (t_k + \tau)^{-1} \left\{ \| D^2_t u^p (t_k + \tau) \|^2 + \| D^2_t u^{p+1} (t_k + \tau) \|^2 \right\} \ d\tau, \]

\[ \sum_{k=1}^{K} t_k \| \beta_k^p \|^2 \Delta t \equiv C (\Delta t)^4 \| g \|_{3+\varepsilon}^2. \]

Next, observe that

\[ \nu_k^p = (4 \Delta t)^{-1} \int_{-\Delta t}^{\Delta t} (|t|)^2 D^2_t u^p (t_k + \tau) \ d\tau, \]

in view of which we can easily deduce

\[ t_k^{-1} \| \nu_k^p \|^2 \leq C (\Delta t)^3 \int_{-\Delta t}^{\Delta t} (t_k + \tau)^{-1} \left\{ \| D^2_t u^p (t_k + \tau) \|^2 + \| D^2_t u^{p+1} (t_k + \tau) \|^2 \right\} \ d\tau, \]

\[ \sum_{k=1}^{K} t_k^{-1} \| \nu_k^p \|^2 \Delta t \equiv C (\Delta t)^4 \| g \|_{3+\varepsilon}^2. \]

Finally we use the following representation,

\[ \mathcal{L} \partial_t^2 u^p_k = (\Delta t)^{-2} \int_{-\Delta t}^{\Delta t} (|t|)^2 \mathcal{L} D^2_t u^p (t_k + \tau) \ d\tau, \]

which allows us to conclude that

\[ t_k \| \mathcal{L} \partial_t^2 u^p_k \|^2 \leq C (\Delta t)^{-1} \int_{-\Delta t}^{\Delta t} (t_k + \tau) \| D^2_t u^p (t_k + \tau) \|^2 \ d\tau, \]

\[ \sum_{k=1}^{K} t_k \| \mathcal{L} \partial_t^2 u^p_k \| \Delta t \equiv C \| g \|_{3+\varepsilon}^2. \]

Estimating the right-hand side of (7.6) by (7.9), (7.12), (7.15), (7.18) and (7.21), we arrive at the following technical result.

**Lemma 7.1.** Assume that \( g \in H_{r+1+\varepsilon} \), where \( 2 \leq r \in \mathbb{Z} \) and \( 0 < \varepsilon < 1 \). Then there is a constant \( C > 0 \), depending only on \( t^*, \varepsilon \) and \( p \), such that

\[ \| \partial \xi_k^{p+1/2} \|^2 + \| \xi_k^{p+1/2} \|_2^2 \leq C \{ \| \partial \xi_k^p \|^2 + \| \xi_k^p \|_2^2 + (h^{2r} + (\Delta t)^4) \| g \|_{r+1+\varepsilon}^2 \}. \]

In the next step note that both

\[ \| \partial \eta_k \| \equiv Ch' \| g \|_{r+1}, \]

\[ \| \eta_k \| \equiv Ch' \| g \|, \]

hold for any \( 1 \leq k \leq K \). Analogously with § 3 we note that \( U_k^p - u_k^p = \xi_k^p + \eta_k^p \), which fact in combination with Lemma 7.1 and (7.22)--(7.23) gives us the following \( L_2 \) estimate of the error.
Lemma 7.2. Under the assumptions of Lemma 7.1 there exists a constant $C > 0$ such that

\[
\max \left\{ \frac{\partial_t (U_{k+1/2}^p - u_{k+1/2}^p)}{2} + \max_{0 \leq k \leq K} \left\| U_{k+1/2}^p - u_{k+1/2}^p \right\| \right\} \leq C \left\{ \left\| \partial_t \xi_{1/2}^p \right\| + \left\| \xi_{1/2}^p \right\| + (h^r + (\Delta t)^2) \left\| g \right\|_{r+1+\epsilon} \right\}.
\]

Now we intend to discuss the choice of starting values of $U^p$, if we are to preserve the optimal rate of convergence. First, it is clear from Remark 3.2 that at the time level $t_0 = 0$ one can assume that $U^0 = P\xi$, for then $\xi^0 = 0$. Then recalling that $D_t u^p |_{t=0} = 0$ and using Taylor series expansion, we obtain

\[
(7.24) \quad u_t^p = g + \frac{(\Delta t)^2}{2} D_t^2 u^p(0) + \frac{(\Delta t)^3}{6} D_t^3 u^p(0), \quad 0 < \theta < \Delta t,
\]

which prompts us to consider

\[
(7.25) \quad \tilde{u}_t^p = g + \frac{(\Delta t)^2}{2} D_t^2 u^p(0)
\]

as a suitable approximation to $u^p$ at $t_1 = \Delta t$. Thus, if we define

\[
(7.26) \quad U_t^p = P\tilde{u}_t^p = P_e u_t^p + P_e (\tilde{u}_t^p - u_t^p),
\]

it would easily follow that

\[
(7.27) \quad \xi_t^p = P_e (\tilde{u}_t^p - u_t^p) = \frac{(\Delta t)^3}{6} P_e D_t^3 u^p(0), \quad 0 < \theta < \Delta t.
\]

This implies that for any $0 < \theta < t^*$

\[
(7.28) \quad \left\| \xi_t^p \right\|_1 \leq C(\Delta t)^3 \left\| D_t^3 u^p(0) \right\|_1 \leq C(\Delta t)^3 \left\| g \right\|_4.
\]

Altogether the definition of $\xi_{1/2}^p$ and the triangle inequality imply

\[
(7.29) \quad \left\| \xi_{1/2}^p \right\|_1 \leq \frac{1}{2} \left\| \xi_t^p \right\|_1 \leq C(\Delta t)^3 \left\| g \right\|_4.
\]

Furthermore, by virtue of (7.27)

\[
(7.30) \quad \partial_t \xi_{1/2}^p = -\frac{(\Delta t)^2}{6} P_e D_t^3 u^p(0), \quad 0 < \theta < \Delta t,
\]

from which we deduce at once that

\[
(7.31) \quad \left\| \partial_t \xi_{1/2}^p \right\| \leq C(\Delta t)^3 \left\| g \right\|_4.
\]

Combining these results with Lemma 7.2, we derive the principal result of this section.

Theorem 7.1. Suppose that $g \in H_s$, where $s = \max \left\{ r + 1 + \epsilon, 4 \right\}$, $2 \leq r \in \mathbb{Z}$ and $0 < \epsilon < 1$. Also assume that $U^0 = P\xi$ and $U^p = P\xi + ((\Delta t)^2/2) P_t D_t^2 u^p(0)$. Then there exists a constant $C > 0$ such that

\[
\max_{0 \leq k \leq K} \left\| \partial_t (U_{k+1/2}^p - u_{k+1/2}^p) \right\| + \max_{0 \leq k \leq K} \left\| U_{k+1/2}^p - u_{k+1/2}^p \right\| \leq C(h^r + (\Delta t)^2) \left\| g \right\|_s.
\]

Remark 7.1. By virtue of (2.12) we can rewrite $U_t^p$ as

\[
U_t^p = P_e \xi - \frac{(\Delta t)^2}{4(p+1)} P_e \xi g.
\]
In conclusion of this section we shall deduce error estimates in the maximum norm for the cases of $N=1, 2$. To this end, recall (cf. [15], [19] and [20])

$$\eta_k^p \leq C h^r \ln \frac{1}{h} \|u_k^p\|_{W^r}, \quad 0 \leq k \leq K,$$

and

$$\xi_k^p \leq C \left( \ln \frac{1}{h} \right)^{N-1} \|\xi_k^p\|_1, \quad 0 \leq k \leq K,$$

which are valid for $N=1, 2$. Substituting these bounds for the inequality in Lemma 7.1, we obtain

$$\xi_{k+1/2}^p \leq C \left( \ln \frac{1}{h} \right)^{N-1} \left\{ \|\delta \xi_{k+1/2}^p\| + \|\xi_{k+1/2}^p\|_1 + (h^r + (\Delta t)^2) \|g\|_{r+1}\right\},$$

for $N=1, 2$ and $1 \leq k \leq K$. Choosing the starting values $U_0^p$ and $U_1^p$ as above, we finally obtain the desired estimate.

**Theorem 7.2.** Under the assumptions of Theorem 10.1 there exists a constant $C > 0$ such that

$$\max_{0 \leq k \leq K} |U_k^{p+1/2} - U_k^{p+1/2}| \leq C \ln \frac{1}{h} \{h^r + (\Delta t)^2\} (\|g\|_r + C_1),$$

for $N=1, 2$ and $C_1 = \sup_{0 \leq t \leq t_*, k} \|u^p(t)\|_{W^r}$, provided $u \in L_\infty(W_{\infty})$.

**Remark 7.2.** Following Dupont [13] we observe that the $L_2$ estimates of the error at knots $t_k = k\Delta t$ can easily be obtained by using $\xi_{k+1}^p = \xi_{k+1/2}^p + \Delta t \delta \xi_{k+1/2}^p$.

**8. Miscellaneous results.** In this section we shall discuss very briefly the extension of certain results in [7] and [8] to the case of the Euler–Poisson–Darboux equation. Since the ideas and techniques we use to derive the interior estimates of the difference quotients and interior superconvergence estimates are intrinsically based upon the above works [7], [8], it seems to be very natural to omit the proofs whenever it is possible. Furthermore, the reader is referred to [8] for definitions of all new pieces of notation we shall subsequently use in this section.

Regarding the $L_\infty$ interior estimates of the difference quotients for nonsmooth data, one can modify a corresponding result proven for parabolic equations [8].

**Theorem 8.1.** Assume that the family $\{S_h\}$, $0 < h < 1$, is r-regular on $\Omega_0$, $g^h = P_0 g$ and $0 \leq m \in \mathbb{Z}$. Also suppose that $Q_h$ approximates $D_x^r$ with accuracy $r$. Then for any $t_0 > 0$ and $\Omega_1 \subset \Omega_0$ there is a constant $C > 0$ such that

$$|Q_h \mathcal{L}_h^{m} u^{h,p}(t) - D_x^r \mathcal{L}_x^{m} u^p(t)|_{\Omega_1} \leq C h^r \|g\|_r, \quad t_0 \leq t \leq t_*,$$

where $0 < \epsilon < 1$, $p \geq \max \{r+2m+5/2+\alpha, \ r+2m+\alpha+N_0\}$, $2 \leq r \in \mathbb{Z}$ and $N_0 = [N/2] + 1$.

**Proof.** The assertion follows by Sobolev’s inequality, Lemma 2.1 and Theorem 4.2 combined with a modified version of [8, Thm. 6.1], if we proceed along the lines of the proof of Theorem 6.3. Q.E.D.

Finally, we would like to present the interior superconvergent estimate similar to that in [8] for parabolic problems.
Theorem 8.2. Suppose that \( \{ S_h \} \) is \( r \)-regular on \( \Omega_0 \) and \( K_h \) is as in [8]. Let \( g^h = P_0 g \) and \( 0 < \varepsilon < 1 \). Then for any \( t_0 > 0 \) and \( \Omega_1 \subset \Omega_0 \) there exists a constant \( C > 0 \) such that

\[
|K_h \ast u^{h,p}(t) - u^p(t)|_{\Omega_1} \leq Ch^{2r-2} \| g \|_{r-2+\varepsilon}, \quad t_0 \leq t \leq t^*,
\]

provided \( p \) is sufficiently large.

Proof. As in the preceding similar cases the result is derived by following along the lines of the proof of Theorem 6.3. Only this time we combine Lemma 2.1, Sobolev's inequality, Theorem 4.2 and Theorem 5.1 with a suitably adapted variant of [8, Thm. 7.1]. Q.E.D.

We shall conclude this work with the remark that a quite nice discussion of the assumptions (1.4)–(1.9), (6.1)–(6.2), etc., can be found in [8]. That paper also contains several conforming and nonconforming examples of semidiscretization that fit into the framework of our investigation.

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