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SOLUTIONS OF A SYSTEM OF INTEGRAL EQUATIONS IN ORLICZ SPACES

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ABSTRACT. We consider the following system of integral equations

$$u_i(t) = \int_0^1 g_i(t,s) f_i(s, u_1(s), u_2(s), \dots, u_n(s))\, ds,$$
$$\text{a.e. } t \in [0,1], \quad 1 \le i \le n.$$

Our aim is to establish criteria such that the above system has a solution $(u_1, u_2, \dots, u_n)$ where $u_i \in L_\phi$ (Orlicz space), $1 \le i \le n$. We further investigate the system

$$u_i(t) = \int_0^1 g_i(t,s) H(s, u_1(s), u_2(s), \dots, u_n(s))\, ds,$$
$$\text{a.e. } t \in [0,1], \quad 1 \le i \le n$$

and establish the existence of *constant-sign* solutions in Orlicz spaces, i.e., for each $1 \le i \le n$, $\theta u_i \ge 0$ and $u_i \in L_\phi$, where $\theta \in \{1, -1\}$ is fixed.

1. Introduction.

Let $x = (x_1, x_2, \dots, x_N)^T$ and $y = (y_1, y_2, \dots, y_N)^T$ be in $\mathbf{R}^N$. Throughout, by $x \ge y$ we shall mean $x_i \ge y_i$ for each $1 \le i \le N$. Similarly, if $x, y \in \mathbf{R}^{N \times N}$ (real $N \times N$ matrices), then $x \ge y$ also means inequality in the componentwise sense.

In this paper we shall consider the following systems of Hammerstein integral equations

$$u_i(t) = \int_0^1 g_i(t,s) f_i(s, u_1(s), u_2(s), \dots, u_n(s))\, ds, \tag{1.1}$$
$$\text{a.e. } t \in [0,1], \quad 1 \le i \le n$$

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and

$$u_i(t) = \int_0^1 g_i(t,s)H(s, u_1(s), u_2(s), \dots, u_n(s))\, ds, \qquad \text{a.e. } t \in [0,1], \quad 1 \le i \le n \tag{1.2}$$

where for each $1 \le i \le n$, $g_i : [0,1] \times [0,1] \to \mathbf{R}^{N\times N}$ is a matrix valued kernel function and $f_i, H : [0,1] \times \mathbf{R}^N \times \mathbf{R}^N \times \cdots \times \mathbf{R}^N \to \mathbf{R}^N$ is a single-valued nonlinear function.

Let $u = (u_1, u_2, \dots, u_n)$. We are interested in establishing the existence of one and more solutions u of the systems (1.1) and (1.2) in Orlicz spaces, i.e., u_i is in an Orlicz space L_ϕ for each $1 \le i \le n$. In particular, for (1.2), we are concerned with the existence of *constant-sign* solutions u in Orlicz spaces, i.e., for each $1 \le i \le n$, in addition to $u_i \in L_\phi$, we have $\theta u_i(t) \ge 0$ for $t \in [0,1]$, where $\theta \in \{1, -1\}$ is fixed. Note that *constant-sign* solutions include *positive* solutions ($\theta = 1$), the usual consideration in the literature. For system (1.2), we shall tackle the case when H is 'nonnegative' and also the case when H can be 'negative' (semi-positone). The main tools employed in this paper are the Leray-Schauder alternative and Krasnosel'skii's fixed point theorem.

In the literature [**1, 9, 10, 12**], mostly solutions of Hammerstein integral equations are sought in $C[0,1]$ and $L^p[0,1]$ with $p > 1$. The more recent work on the existence of solutions of Hammerstein integral equations can be found in [**2–7, 13, 17–19**] where a variety of techniques including Krasnosel'skii's fixed point theorem and fixed point index theory have been used. Those results obtained for $L^p[0,1]$ invariably assume a polynomial type restriction (in y) on the nonlinearity $f(t,y)$. On the other hand, seeking solutions in other Orlicz spaces [**8, 10, 16, 20–24**] will lead to restrictions that are *not* of polynomial type, and hence will allow us to consider new classes of equations. We remark that our present work generalizes all the previous work done on the existence of solutions in Orlicz spaces to (i) *systems*, (ii) *constant-sign* solutions and (iii) *semi-positone* nonlinearity. Moreover, our methodology, especially the application of Krasnosel'skii's fixed point theorem in Orlicz spaces, is entirely *new* in the literature. Hence, our results are new even in the case $n = 1$.

The paper is outlined as follows. Section 2 includes the main tools used. The results for (1.1) and (1.2) are respectively presented in

Sections 3 and 4. Finally, the semi-positone case of (1.2) is investigated in Section 5.

2. Preliminaries. The following two theorems will be needed to establish the main results later. The first theorem is known as the *Leray-Schauder alternative* and the second is usually called *Krasnosel'skii's fixed point theorem in a cone.*

Theorem 2.1 [**1**]. *Let B be a Banach space with $E \subseteq B$ closed and convex. Assume U is a relatively open subset of E with $0 \in U$ and $S : \overline{U} \to E$ is a continuous and compact map. Then either*

(a) *S has a fixed point in $\overline{U}$, or*

(b) *there exist $u \in \partial U$ and $\lambda \in (0,1)$ such that $u = \lambda S u$.*

Theorem 2.2 [**11**]. *Let $B = (B, \| \cdot \|)$ be a Banach space, and let $C \subset B$ be a cone in B. Assume Ω_1 and Ω_2 are open subsets of B with $0 \in \Omega_1$, $\overline{\Omega}_1 \subset \Omega_2$, and let $S : C \cap (\overline{\Omega}_2 \backslash \Omega_1) \to C$ be a continuous and completely continuous operator such that, either*

(a) *$\|Su\| \le \|u\|$, $u \in C \cap \partial\Omega_1$, and $\|Su\| \ge \|u\|$, $u \in C \cap \partial\Omega_2$, or*

(b) *$\|Su\| \ge \|u\|$, $u \in C \cap \partial\Omega_1$, and $\|Su\| \le \|u\|$, $u \in C \cap \partial\Omega_2$.*

Then S has a fixed point in $C \cap (\overline{\Omega}_2 \backslash \Omega_1)$.

3. Results for (1.1). In this section we shall employ the Leray-Schauder alternative (Theorem 2.1) to obtain some existence results for the system (1.1) in Orlicz spaces.

Let B be a Banach space. Let the operator $S : B \to (\mathbf{R}^N)^n$ be defined by

$$Su(t) = (S_1 u(t), S_2 u(t), \dots, S_n u(t)), \quad \text{a.e. } t \in [0,1] \tag{3.1}$$

where

$$S_i u(t) = \int_0^1 g_i(t,s) f_i(s, u(s))\, ds, \quad \text{a.e. } t \in [0,1], \quad 1 \le i \le n. \tag{3.2}$$

Clearly, a fixed point of the operator S is a solution of system (1.1). We observe that the operator S_i can be written as

$$S_i = A_i F_i \tag{3.3}$$

where $F_i : B \to \mathbf{R}^N$ is defined by

$$F_i u(t) = f_i(t, u(t)), \quad t \in [0,1] \tag{3.4}$$

and $A_i : \mathbf{R}^N \to \mathbf{R}^N$ is given by

$$A_i x(t) = \int_0^1 g_i(t,s) x(s)\, ds, \quad \text{a.e. } t \in [0,1]. \tag{3.5}$$

Our first result is a general existence principle in B.

Theorem 3.1. *Let $X = (X, \ |\cdot|_X)$ be a Banach space, and let $X^n = X \times X \times \cdots \times X$ (n times) be equipped with the norm $\|\cdot\|$ where*

$$\|u\| \equiv \max_{1 \le i \le n} |u_i|_X, \quad u \in X^n.$$

Let Y be a Banach space. For each $1 \le i \le n$, suppose

$$F_i : X^n \longrightarrow Y \quad \text{and} \quad A_i : Y \longrightarrow X \tag{3.6}$$

and

$$A_i F_i : X^n \longrightarrow X \text{ is continuous and completely continuous.} \tag{3.7}$$

Moreover, for all $\lambda \in (0,1)$, a positive constant M_0 (independent of λ) exists such that for any solution $u \in X^n$ of the system

$$u_i = \lambda A_i F_i u, \quad \text{a.e. } 1 \le i \le n \tag{3.8$_\lambda$}$$

we have

$$\|u\| \ne M_0. \tag{3.9}$$

Then, (1.1) *has a solution $u^* \in X^n$ with $\|u^*\| \le M_0$.*

Proof. Clearly, a solution of $(3.8)_\lambda$ is a fixed point of the equation $u = \lambda S u$. Now (3.7) guarantees that S is continuous and completely continuous. In the context of Theorem 2.1, let

$$U = \{u \in X^n \mid \|u\| < M_0\}.$$

Suppose that u is a solution of $(3.8)_\lambda$ for some $\lambda \in (0,1)$. Then, $u \notin \partial U$. Thus, case (b) of Theorem 2.1 cannot arise and case (a) of Theorem 2.1 must hold, i.e., system (1.1) has a solution $u^* \in \overline{U}$ with $\|u^*\| \le M_0$. $\square$

We shall now tackle the existence of a solution u of (1.1) with $u_i \in X$, $1 \le i \le n$, where X is an *Orlicz space*.

To begin, let P and Q be complementary N-functions [**13**]. The *Orlicz class*, denoted by $\mathcal{O}_P$, consists of measurable functions $y : [0,1] \to \mathbf{R}^N$ for which

$$\rho(y;P) = \int_0^1 P(y(x))\,dx < \infty.$$

We shall denote by $L_P([0,1],\mathbf{R}^N)$ the *Orlicz space* of all measurable functions $y : [0,1] \to \mathbf{R}^N$ for which

$$|y|_P = \sup_{\substack{\rho(v;Q)\le 1\\ v\in\mathcal{O}_Q}} \left| \int_0^1 y(x)\cdot v(x), dx \right| < \infty.$$

Note also Hölder's inequality [**16**, page 74] which says

$$\left| \int_0^1 y(x)\cdot v(x\,dx \right| \le |y|_P \cdot |v|_Q.$$

It is known that $(L_P([0,1],\mathbf{R}^N),\ |\cdot|_P)$ is a Banach space [**13**]. Let $E_P([0,1],\mathbf{R}^N)$ be the closure in $L_P([0,1],\mathbf{R}^N)$ of the set of all bounded functions. Note that $E_P \subseteq L_P \subseteq \mathcal{O}_P$. We have $E_P = L_P = \mathcal{O}_P$ if P satisfies the $(\triangle_2)$ condition, which is $(\triangle_2)$ there exist ω, $y_0 \ge 0$, such that for $y \ge y_0$, we have $P(2y) \le \omega P(y)$.

For a discussion of the $(\triangle_2)$ condition, we refer the reader to [**16**, pages 23–29]. For example, if P grows faster than a power, then Q satisfies the $(\triangle_2)$ condition.

Using the ideas of [**8, 13**] we can present many existence principles in an Orlicz space. One such result is as follows.

Theorem 3.2. *Let P and Q be complementary N-functions. Suppose that*

$$\begin{cases} \phi \text{ and } \psi \text{ are complementary } N\text{-functions, and the functions} \\ Q \text{ and } \phi \text{ satisfy the } (\triangle_2) \text{ condition,} \end{cases} \tag{3.10}$$

$$\begin{cases} \text{for each } 1 \le i \le n,\ g_i(t,\cdot) \in E_P \text{ for a.e. } t \in [0,1], \\ \text{and the function } t \mapsto |g_i(t,\cdot)|_P \text{ belongs to } E_\phi, \end{cases} \tag{3.11}$$

$$\begin{cases} \text{for each } 1 \le i \le n,\ f_i \text{ is a Carathéodory function, i.e.,} \\ \text{(i)} t \mapsto f_i(t,u) \text{ is measurable for every } u \in (\mathbf{R}^N)^n \\ \text{(ii)} u \mapsto f_i(t,u) \text{ is continuous for a.e. } t \in [0,1] \end{cases} \tag{3.12}$$

and

$$\begin{cases} \text{for each } r > 0 \text{ and } 1 \le i \le n, \\ \text{there exists an} \eta_{r,i} \in L_Q([0,1],\mathbf{R}) \text{ and } K_{r,i} \ge 0 \\ \text{such that } |f_i(t,u)| \le \eta_{r,i}(t) + K_{r,i}\ Q^{-1}\left(\phi\left(u_i/r\right)\right) \\ \text{for a.e. } t \in [0,1] \text{ and every } u \in (\mathbf{R}^N)^n. \end{cases} \tag{3.13}$$

Moreover, assume that there is a positive constant M_0, independent of λ, with

$$\|u\|_\phi \equiv \max_{1\le i\le n} |u_i|_\phi \ne M_0 \tag{3.14}$$

for any solution u of $(3.8)_\lambda$. Then, (1.1) has a solution $u^* \in (L_\phi([0,1],\mathbf{R}^N))^n$ with $\|u^*\|_\phi \le M_0$.

Proof. It follows immediately from Lemma 16.3 and Theorem 16.3 (take $M_1 = Q,\ M_2 = \phi$ and $N_1 = P$) of [**13**] that $A_i : E_Q = L_Q \to E_\phi = L_\phi$ is continuous and completely continuous. Let

$$U = \{u \in (L_\phi)^n \mid \|u\|_\phi < M_0\}. \tag{3.15}$$

Applying Theorem 17.6 in [**13**], we deduce that $F_i : \overline{U} \to L_Q$ is continuous and F_i maps bounded sets into bounded sets. Thus $A_iF_i : \overline{U} \to L_\phi$ is continuous (A_i is continuous and F_i is also continuous) and completely continuous (A_i is completely continuous and F_i maps bounded sets into bounded sets). With $X = L_\phi$ and $Y = L_Q$, the result now follows from Theorem 3.1. □

Remark 3.1. By placing other conditions on g_i and f_i (see [**16**, Sections 15, 16, 17]) we may deduce other existence results in an Orlicz space.

Our next result uses Theorem 3.2.

Theorem 3.3. *Let P and Q be complementary N-functions. Suppose* (3.10–3.13) *hold. Moreover, assume, for each $1 \le i \le n$,*

$$\sup_{r\in(0,\infty)} \frac{r}{|q_i|_\phi \cdot |\eta_{r,i}|_Q + 2K_{r,i}|q_i|_\phi} > 1 \tag{3.16}$$

where $q_i(t) = |g_i(t,\cdot)|_P$. Then, (1.1) *has a solution $u^* \in (L_\phi([0,1], \mathbf{R}^N))^n$.*

Proof. In view of (3.16), for any $1 \le i \le n$ there exists a positive constant M_0 such that

$$\frac{M_0}{|q_i|_\phi \cdot |\eta_{M_0,i}|_Q + 2K_{M_0,i}|q_i|_\phi} > 1. \tag{3.17}$$

Let u be a solution of $(3.8)_\lambda$ for some $\lambda \in (0,1)$ with $\|u\|_\phi = M_0$. Then, there exists some $j \in \{1,2,\dots,n\}$ such that $|u_j|_\phi = M_0$.

By using [**16**, Theorem 10.5 with $k=1$], we have

$$\left|Q^{-1}\left(\phi\left(\frac{u_i}{M_0}\right)\right)\right|_Q \le 1 + \int_0^1 \phi\left(\frac{u_i(s)}{M_0}\right) ds,\ 1 \le i \le n. \tag{3.18}$$

Now, applying Hölder's inequality, we get for $t \in [0,1]$,

$$\begin{aligned} |u_j(t)| &= \left|\int_0^1 g_j(t,s) f_j(s,u(s))ds\right| \le |g_j(t,\cdot)|_P\ |f_j(\cdot,u(\cdot))|_Q \\ &= q_j(t)\ |f_j(\cdot,u(\cdot))|_Q. \end{aligned} \tag{3.19}$$

Hence, using (3.13) (when $r = M_0$) and (3.18) in (3.19), we find

$$
\begin{aligned}
|u_j|_\phi &\le |q_j|_\phi \; |f_j(\cdot, u(\cdot))|_Q \\
&\le |q_j|_\phi \left| \eta_{M_0,j}(\cdot) + K_{M_0,j} \; Q^{-1}\left(\phi\left(\frac{u_j(\cdot)}{M_0}\right)\right)\right|_Q \\
&\le |q_j|_\phi \left\{ |\eta_{M_0,j}|_Q + K_{M_0,j} \left| Q^{-1}\left(\phi\left(\frac{u_j}{M_0}\right)\right)\right|_Q \right\} \\
&\le |q_j|_\phi \left\{ |\eta_{M_0,j}|_Q + K_{M_0,j}\left(1 + \int_0^1 \phi\left(\frac{u_j(s)}{M_0}\right) ds\right)\right\}.
\end{aligned} \tag{3.20}
$$

Note that Lemma 9.2 in [**13**] provides

$$\int_0^1 \phi\left(\frac{u_j(s)}{M_0}\right) ds \le \frac{|u_j|_\phi}{M_0} = \frac{M_0}{M_0} = 1. \tag{3.21}$$

Substituting (3.21) into (3.20) immediately leads to

$$M_0 \le |q_j|_\phi \left(|\eta_{M_0,j}|_Q + 2K_{M_0,j}\right)$$

or

$$\frac{M_0}{|q_j|_\phi \cdot |\eta_{M_0,j}|_Q + 2K_{M_0,j}|q_j|_\phi} \le 1,$$

which contradicts (3.17).

Hence, any solution u of $(3.8)_\lambda$ must satisfy $\|u\|_\phi \neq M_0$; thus, we have condition (3.14). The conclusion is now immediate from Theorem 3.2. □

Remark 3.2. Let p (> 1) and q be integers such that $1/p + 1/q = 1$. When $n = 1$, in [**21**, Theorem 3.6] the existence of a solution in $L^p[0,1]$ is established using the conditions

$$|f(t,u)| \le \eta(t) + M|u|^{p/q} \tag{3.22}$$

and

$$\sup_{r\in(0,\infty)} \left(\frac{r}{a_0 + a_1\, r^{p/q}}\right) > 1 \tag{3.23}$$

where a_0 and a_1 are some fixed constants. We remark that our Theorem 3.3 (existence of a solution in *Orlicz space*) is 'analogous' to [**21**, Theorem 3.6] in the sense that if we let $P(x) = \phi(x) = |x|^p/p$, then $Q(x) = |x|^q/q$ and so $Q^{-1}(x) = (qx)^{1/q}$ for $x \geq 0$. Then, (3.13) (when $n = 1$) with

$$K_r = M \cdot r^{p/q} \left(\frac{p}{q}\right)^{1/q}$$

reduces to (3.22), since with this K_r we have

$$K_r\ Q^{-1}\left(\phi\left(\frac{u}{r}\right)\right) = M|u|^{p/q}.$$

Moreover, condition (3.16) is 'parallel' to (3.23).

Remark 3.3. It is also possible to prove Theorem 3.3 using Schauder's fixed point theorem.

Remark 3.4. Of course (see Theorem 17.6 in [**13**]) one could replace (3.13) in Theorem 3.3 with the following condition:

$$(3.24)\qquad \begin{cases} \text{there exists } \varepsilon > 0 \text{ such that for each } 0 < r \leq M_0 + \varepsilon \\ \text{and } 1 \leq i \leq n, \\ \text{there exists } \eta_{r,i} \in L_Q([0,1], \mathbf{R}) \text{ and } K_{r,i} \geq 0 \text{ such that} \\ |f_i(t,u)| \leq \eta_{r,i}(t) + K_{r,i}\ Q^{-1}\left(\phi\left(u_i/r\right)\right) \\ \text{for a.e. } t \in [0,1] \text{ and every } u \in (\mathbf{R}^N)^n, \end{cases}$$

where M_0 is as in (3.17). In fact if we want to be more precise we only need the inequality in (3.24) to hold at $r = M_0 + \varepsilon$ (to apply Theorem 17.6 in [13]) and $r = M_0$ (for the argument in (3.20)). Note also if (3.24) holds at $r = M_0 + \varepsilon$, then from the monotonicity of ϕ and Q^{-1}, one has an inequality of type (3.24) at $r = M_0$.

Remark 3.5. All the results in this section hold for system (1.2), with f_i replaced by H in the conditions.

Example 3.1. Suppose $n = N = 1$, $P(x) = x^2/2$ (so $Q(x) = x^2/2$), $\phi(x) = |x|^2[|\ln|x|| + 1]$. Assume that (3.11) and (3.12) hold with $g_i = g$

and $f_i = f$. In addition suppose there exist $\eta \in L_Q([0,1],\mathbf{R})$ and $b > 0$ with

$$|f(t,u)| \le \eta(t) + b|u|\sqrt{|\ln|u|| + 1}$$
$$\text{for a.e. } t \in [0,1] \quad \text{and} \quad u \in \mathbf{R}.$$

Finally assume there exist $\varepsilon > 0$ and $M_0 > 0$ with

$$M_0 > |q|_\phi |\eta|_Q + \sqrt{2}|q|_\phi b\sqrt{M_0(M_0+\varepsilon)[|\ln|M_0+\varepsilon|| + 1]}. \tag{3.25}$$

Then, (1.1) has a solution in $L_\phi([0,1],\mathbf{R})$.

To see this we will apply Theorem 3.3 (with Remark 3.4). Firstly notice (3.10) is immediate and (3.17) (so (3.16)) also follows from (3.25) once we show

$$K_r = \frac{b}{\sqrt{2}}\sqrt{r(M_0+\varepsilon)[|\ln|M_0+\varepsilon|| + 1]}$$

if $0 < r \le M_0 + \varepsilon$. To see this and (3.24) at the same time notice that if $0 < r \le M_0 + \varepsilon$ then for $u \in \mathbf{R}$ we have

$$\begin{aligned}
\phi(u) = \phi\left(\frac{r}{M_0+\varepsilon}(M_0+\varepsilon)\frac{u}{r}\right) &\le \frac{r}{M_0+\varepsilon}\phi\left((M_0+\varepsilon)\frac{u}{r}\right) \\
&\le r(M_0+\varepsilon)\left|\frac{u}{r}\right|^2\left[\left|\ln\left|\frac{u}{r}\right|\right| + |\ln|M_0+\varepsilon|| + 1\right] \\
&\le r(M_0+\varepsilon)\,[|\ln|M_0+\varepsilon|| + 1]\,\phi\left(\frac{u}{r}\right)
\end{aligned}$$

and so (note $Q^{-1}(x) = \sqrt{2x}$ for $x > 0$)

$$\begin{aligned}
b|u|\sqrt{|\ln|u|| + 1} &= \frac{b}{\sqrt{2}}\; Q^{-1}(\phi(u)) \\
&\le \frac{b}{\sqrt{2}}\sqrt{2r(M_0+\varepsilon)\,[|\ln|M_0+\epsilon|| + 1]\,\phi\left(\frac{u}{r}\right)} \\
&= \frac{b}{\sqrt{2}}\sqrt{r(M_0+\varepsilon)\,[|\ln|M_0+\varepsilon|| + 1]}\; Q^{-1}\left(\phi\left(\frac{u}{r}\right)\right) \\
&= K_r\; Q^{-1}\left(\phi\left(\frac{u}{r}\right)\right).
\end{aligned}$$

Thus, (3.24) and (3.16) hold and the result follows from Theorem 3.3 (with Remark 3.4).

Remark 3.6. It is easy to generalize the above example by considering $P(x) = |x|^p/p$, $p > 1$ and $\phi(x) = |x|^a[|\ln|x|| + 1]$, $a > 1$. Indeed other ϕ, etc., could be considered (see [**16**, page 219]).

4. Results for (1.2). In this section, the Krasnosel'skii's fixed point theorem (Theorem 2.2) will be used to yield some existence results for system (1.2) in Orlicz spaces. In particular, we are interested in the existence of *constant-sign* solutions in Orlicz spaces.

Let P and Q be complementary N-functions. Suppose ϕ and ψ are complementary N-functions, and the functions Q and ϕ satisfy the $(\triangle_2)$ condition. Let the Banach space

$$B = \left\{ u \mid u \in \left(L_\phi([0,1], \mathbf{R}^N) \right)^n \right\}$$

be equipped with the norm

$$\|u\|_\phi = \max_{1 \le i \le n} |u_i|_\phi$$

where

$$|u_i|_\phi = \sup_{\substack{\rho(v;\psi) \le 1 \\ v \in \mathcal{O}_\psi}} \left| \int_0^1 u_i(x) \cdot v(x)\, dx \right|, \quad 1 \le i \le n.$$

Define the operator $S : B \to (\mathbf{R}^N)^n$ by

$$Su(t) = (S_1 u(t), S_2 u(t), \dots, S_n u(t)), \quad \text{a.e. } t \in [0,1], \tag{4.1}$$

where

$$S_i u(t) = \int_0^1 g_i(t,s) H(s, u(s))\, ds, \quad \text{a.e. } t \in [0,1], \quad 1 \le i \le n. \tag{4.2}$$

Clearly, a fixed point of the operator S is a solution of system (1.2).

Our first result gives the existence of a *constant-sign* solution in $(L_\phi([0,1], \mathbf{R}^N))^n$.

Theorem 4.1. *Let $\theta \in \{1, -1\}$ be fixed, and let P and Q be complementary N-functions. Suppose* (3.10) *and* (3.11) *hold. Moreover, assume*

$$\text{(4.3)} \qquad H \text{ is a Carathéodory function},$$

$$\text{(4.4)} \qquad \begin{cases} \text{for each } r > 0 \text{ and } 1 \le i \le n, \\ \text{there exists } \eta_{r,i} \in L_Q([0,1], \mathbf{R}) \text{ and } K_{r,i} \ge 0 \\ \text{such that } |H(t,u)| \le \eta_{r,i}(t) + K_{r,i}\ Q^{-1}\left(\phi\left(u_i/r\right)\right) \\ \text{for a.e. } t \in [0,1] \text{ and every } u \in (\mathbf{R}^N)^n, \end{cases}$$

$$\text{(4.5)} \quad \theta H(t,u) \ge 0 \text{ for } (t,u) \in [0,1] \times A, \quad \text{where } A = \begin{cases} \left((0,\infty)^N\right)^n & \text{if } \theta = 1 \\ \left((-\infty,0)^N\right)^n & \text{if } \theta = -1 \end{cases}$$

(4.6)
$$\begin{cases} \text{for } 0 \le x \le |u_j| \le y,\ 1 \le j \le n \text{ and a.e. } t \in [0,1], \\ \theta H(t, u_1, u_2, \dots, x, \dots, u_n) \le \theta H(t, u_1, u_2, \dots, u_j, \dots, u_n) \\ \quad \le \theta H(t, u_1, u_2, \dots, y, \dots, u_n), \end{cases}$$

$$\text{(4.7)} \qquad \begin{cases} \text{there exist a constant } 0 < M \le 1, \\ \ \text{and nonnegative functions } a, b \\ \text{with } a : [0,1] \to \mathbf{R}^N,\ b : [0,1] \to (\mathbf{R}^N)^T,\ a(t), b(t) > 0, \\ \text{a.e. } t \in [0,1], \\ \text{and } a \in L_\phi([0,1], \mathbf{R}^N) \text{ such that for each } 1 \le i \le n, \\ Ma(t)b(s) \le g_i(t,s) \le a(t)b(s),\ t \in [0,1],\ \text{a.e. } s \in [0,1], \end{cases}$$

$$\text{(4.8)} \quad \text{there exists } \delta > 0 \text{ such that } \phi(xy) \le \frac{1}{\delta}\ \phi(x)\phi(y) \text{ for } x, y \ge 0,$$

(4.9)
$$\begin{cases} \text{there exists } \beta > 0 \text{ such that} \\ \phi^{-1}\Big(\delta \int_0^1 \phi\Big(M\theta a(s) \int_0^1 b(\tau) H(\tau, \beta\gamma(\tau), \beta\gamma(\tau), \dots, \beta\gamma(\tau)) d\tau\Big) ds\Big) \ge \beta, \\ \text{where } \gamma(\tau) = M(a(\tau)/|a|_\phi), \text{ of course we also assume the above} \\ \text{integral exists} \end{cases}$$

and

$$\text{(4.10)}\qquad \begin{cases} \text{there exists an } \alpha > 0 \text{ such that} \\ |q_i|_\phi \left(|\eta_{\alpha,i}|_Q + 2K_{\alpha,i}\right) \le \alpha \text{ for each } 1 \le i \le n, \\ \text{where } q_i(t) = |g_i(t,\cdot)|_P. \end{cases}$$

Then, (1.2) *has a constant-sign solution* $u^* \in (L_\phi([0,1],\mathbf{R}^N))^n$ *such that*

(a) *if* $\alpha < \beta$, *then* $\alpha \le \|u^*\|_\phi \le \beta$ *and for each* $1 \le i \le n$, $\theta u_i^*(t) \ge \gamma(t)\alpha$, *almost everywhere* $t \in [0,1]$;

(b) *if* $\beta < \alpha$, *then* $\beta \le \|u^*\|_\phi \le \alpha$ *and for each* $1 \le i \le n$, $\theta u_i^*(t) \ge \gamma(t)\beta$, *almost everywhere* $t \in [0,1]$.

Remark 4.1. A typical example of a $(\triangle_2)$ function ϕ which satisfies (4.8) is $\phi(x) = |x|^2[|\ln|x|| + 1]$. Note that (4.8) is immediate with $\delta = 1$ since

$$\begin{aligned}\phi(xy) &\le x^2y^2[|\ln|x|| + |\ln|y|| + 1] \le x^2y^2[|\ln|x|| + 1][|\ln|y|| + 1] \\ &= \phi(x)\phi(y).\end{aligned}$$

Proof of Theorem 4.1. To begin, we define a cone C in B by

$$\text{(4.11)}\qquad C = \Big\{ u \in \left(L_\phi([0,1],\mathbf{R}^N)\right)^n \;\Big|\; \text{for each } 1 \le i \le n,\ \theta u_i(t) \ge \gamma(t)\|u\|_\phi \text{ for a.e. } t \in [0,1] \Big\},$$

where $\gamma(\cdot)$ is defined in (4.9). It is clear that a fixed point of S in C is a constant-sign solution of (1.2).

Moreover, let Ω_α and Ω_β be open subsets of $(L_\phi([0,1],\mathbf{R}^N))^n$ defined by

$$\Omega_\alpha = \left\{ u \in \left(L_\phi([0,1],\mathbf{R}^N)\right)^n \mid \|u\|_\phi < \alpha \right\}$$

and

$$\Omega_\beta = \left\{ u \in \left(L_\phi([0,1],\mathbf{R}^N)\right)^n \mid \|u\|_\phi < \beta \right\}.$$

First, using a similar argument as in the proof of Theorem 3.2, we see that $S_i : C \cap \overline{\Omega}_{\max\{\alpha,\beta\}} \to L_\phi$ is continuous and completely continuous for each $1 \le i \le n$. Thus, $S : C \cap \overline{\Omega}_{\max\{\alpha,\beta\}} \to (L_\phi)^n$ is also continuous and completely continuous.

Next, we shall show that $S : C \cap \overline{\Omega}_{\max\{\alpha,\beta\}} \to C$. Let $u \in C \cap \overline{\Omega}_{\max\{\alpha,\beta\}}$. By (4.5) and (4.7), we have

$$\theta S_i u(t) \le \int_0^1 a(t)b(s)\theta H(s, u(s))\, ds, \quad \text{a.e. } t \in [0,1],\ 1 \le i \le n \tag{4.12}$$

which leads to

$$|S_i u|_\phi \le |a|_\phi \int_0^1 b(s)\theta H(s, u(s))\, ds, \quad 1 \le i \le n.$$

Hence, it follows that

$$\|Su\|_\phi = \max_{1\le i\le n} |S_i u|_\phi \le |a|_\phi \int_0^1 b(s)\theta H(s, u(s))\, ds. \tag{4.13}$$

Also, from (4.5) and (4.7) we get

$$\theta S_i u(t) \ge M \int_0^1 a(t)b(s)\theta H(s, u(s))\, ds \ge 0, \quad \text{a.e. } t \in [0,1],\ 1 \le i \le n. \tag{4.14}$$

Substituting (4.13) into (4.14) gives

$$\theta S_i u(t) \ge Ma(t)\, \frac{\|Su\|_\phi}{|a|_\phi} = \gamma(t)\|Su\|_\phi, \quad \text{a.e. } t \in [0,1],\ 1 \le i \le n.$$

This shows that $S : C \cap \overline{\Omega}_{\max\{\alpha,\beta\}} \to C$.

We shall now prove that $\|Su\|_\phi \ge \|u\|_\phi$ for $u \in C \cap \partial\Omega_\beta$. Let $u \in C \cap \partial\Omega_\beta$. Then, $\|u\|_\phi = \beta$. Let $1 \le j \le n$ be fixed. Since $\phi(x)$ is increasing for $x \ge 0$, we have, in view of (4.14) and (4.6),

$$\begin{aligned} &\int_0^1 \phi(\theta S_j u(s))\, ds \\ &\ge \int_0^1 \phi\left(Ma(s)\int_0^1 b(\tau)\theta H(\tau, u(\tau))\, d\tau\right) ds \\ &\ge \int_0^1 \phi\left(Ma(s)\int_0^1 b(\tau)\theta H(\tau, \beta\gamma(\tau), \beta\gamma(\tau), \dots, \beta\gamma(\tau))\, d\tau\right) ds. \end{aligned} \tag{4.15}$$

Now, Lemma 9.2 in [**13**] provides

$$\int_0^1 \phi\left(\frac{\theta S_j u(s)}{\|Su\|_\phi}\right) ds \le \frac{|S_j u|_\phi}{\|Su\|_\phi} \le 1. \tag{4.16}$$

On the other hand, using (4.8) we have

$$\int_0^1 \phi\left(\frac{\theta S_j u(s)}{\|Su\|_\phi}\right) ds \ge \delta \int_0^1 \frac{\phi(\theta S_j u(s))}{\phi(\|Su\|_\phi)}\, ds. \tag{4.17}$$

Coupling (4.16) and (4.17) yields

$$\delta \int_0^1 \frac{\phi(\theta S_j u(s))}{\phi(\|Su\|_\phi)}\, ds \le 1 \tag{4.18}$$

or

$$\begin{aligned}
&\phi(\|Su\|_\phi)\\
&\quad \ge \delta \int_0^1 \phi(\theta S_j u(s))\, ds\\
&\quad \ge \delta \int_0^1 \phi\left(Ma(s) \int_0^1 b(\tau)\theta H(\tau, \beta\gamma(\tau), \beta\gamma(\tau), \dots, \beta\gamma(\tau))\, d\tau\right) ds
\end{aligned}$$

where we have used (4.15) in the second inequality. This implies, noting (4.9),

$$\begin{aligned}
\|Su\|_\phi \ge \phi^{-1}\Bigg(\delta \int_0^1 \phi\Bigg(Ma(s) \int_0^1 b(\tau)\theta H(\tau, \beta\gamma(\tau), \beta\gamma(\tau), \dots,&\\
\beta\gamma(\tau))\, d\tau\Bigg) ds\Bigg)&\\
\ge \beta = \|u\|_\phi.&
\end{aligned}$$

We have thus shown that $\|Su\|_\phi \ge \|u\|_\phi$ for $u \in C \cap \partial\Omega_\beta$.

Finally, it remains to prove that $\|Su\|_\phi \le \|u\|_\phi$ for $u \in C \cap \partial\Omega_\alpha$. Let $u \in C \cap \partial\Omega_\alpha$. Then, $\|u\|_\phi = \alpha$. Applying Hölder's inequality, we get for $t \in [0,1]$,

$$\begin{aligned}
|S_i u(t)| &= \left|\int_0^1 g_i(t,s) H(s, u(s))\, ds\right|\\
&\le |g_i(t,\cdot)|_P |H(\cdot, u(\cdot))|_Q = q_i(t) |H(\cdot, u(\cdot))|_Q.
\end{aligned} \tag{4.19}$$

Hence, using (4.4) (when $r=\alpha$) and (3.18) in (4.19), we find

$$
\begin{aligned}
|S_i u|_\phi &\le |q_i|_\phi \; |H(\cdot, u(\cdot))|_Q \\
&\le |q_i|_\phi \left| \eta_{\alpha,i}(\cdot) + K_{\alpha,i} \; Q^{-1}\left(\phi\left(\frac{u_i(\cdot)}{\alpha}\right)\right)\right|_Q \\
&\le |q_i|_\phi \left\{ |\eta_{\alpha,i}|_Q + K_{\alpha,i} \left| Q^{-1}\left(\phi\left(\frac{u_i}{\alpha}\right)\right)\right|_Q \right\} \\
&\le \; |q_i|_\phi \left\{ |\eta_{\alpha,i}|_Q + K_{\alpha,i}\left(1 + \int_0^1 \phi\left(\frac{u_i(s)}{\alpha}\right) ds\right)\right\}.
\end{aligned} \tag{4.20}
$$

Once again, from Lemma 9.2 in [**13**] we have

$$
\int_0^1 \phi\left(\frac{u_i(s)}{\alpha}\right) ds \le \frac{|u_i|_\phi}{\alpha} \le 1. \tag{4.21}
$$

Substituting (4.21) into (4.20) immediately leads to

$$
|S_i u|_\phi \le \max_{1\le j\le n} |q_j|_\phi l(|\eta_{\alpha,j}|_Q + 2K_{\alpha,j}\mathbf{R}),\ 1\le i\le n,
$$

which yields, noting (4.10),

$$
\|Su\|_\phi \le \max_{1\le j\le n} |q_j|_\phi l(|\eta_{\alpha,j}|_Q + 2K_{\alpha,j}\mathbf{R}) \le \alpha = \|u\|_\phi.
$$

Thus, we have proved that $\|Su\|_\phi \le \|u\|_\phi$ for $u\in C\cap\partial\Omega_\alpha$.

We conclude by Theorem 2.1 that S has a fixed point

$$
u^* \in C\cap(\overline{\Omega}_{\max\{\alpha,\beta\}}\backslash\Omega_{\min\{\alpha,\beta\}}).
$$

Hence, u^* is of constant sign and satisfies

$$
\min\{\alpha,\beta\} \le \|u^*\|_\phi \le \max\{\alpha,\beta\}
$$

and

$$
\theta u_i^*(t) \ge \gamma(t)\|u^*\|_\phi \ge \gamma(t)\min\{\alpha,\beta\}, \quad \text{a.e. } t\in[0,1], \quad 1\le i\le n.
$$

The proof is complete. □

Remark 4.2. In (4.10) if we have strict inequality instead, i.e.,

$$\text{(4.10)}' \qquad \text{there exists } \alpha > 0 \text{ such that } |q_i|_\phi l(|\eta_{\alpha,i}|_Q + 2K_{\alpha,i}\mathbf{R}) < \alpha \quad \text{for each } 1 \le i \le n,$$

then from the latter part of the proof of Theorem 4.1 we see that a fixed point u^* of S must satisfy $\|u^*\| \neq \alpha$. Hence, with (4.10) replaced by (4.10)$'$, the conclusion of Theorem 4.1 becomes: (1.2) has a constant-sign solution $u^* \in l(L_\phi([0,1],\mathbf{R}^N)\mathbf{R})^n$ such that

(a) if $\alpha < \beta$, then $\alpha < \|u^*\|_\phi \le \beta$ and for each $1 \le i \le n$, $\theta u_i^*(t) > \gamma(t)\alpha$, almost everywhere $t \in [0,1]$;

(b) if $\beta < \alpha$, then $\beta \le \|u^*\|_\phi < \alpha$ and for each $1 \le i \le n$, $\theta u_i^*(t) \ge \gamma(t)\beta$, almost everywhere $t \in [0,1]$.

Remark 4.3. Of course (4.4) can be replaced by (3.24) with f_i replaced by H and M_0 replaced by $\max\{\alpha,\beta\}$.

The next result gives the existence of *two* solutions in $(L_\phi([0,1],\mathbf{R}^N))^n$.

Theorem 4.2. *Let $\theta \in \{1,-1\}$ be fixed, and let P and Q be complementary N-functions. Suppose* (3.10), (3.11), (4.3)–(4.9) *and* (4.10)$'$ *hold with $\alpha < \beta$. Then,* (1.2) *has* (*at least*) *two solutions $u^1, u^2 \in (L_\phi([0,1],\mathbf{R}^N))^n$ such that*

$$0 \le \|u^1\|_\phi < \alpha < \|u^2\|_\phi \le \beta \quad \textit{and}$$
$$\theta u_i^2(t) > \gamma(t)\alpha, \;\; \textit{a.e. } t \in [0,1], \;\; 1 \le i \le n.$$

Moreover, u^2 is of constant sign.

Proof. The existence of u^1 follows from Theorem 3.3 (let $M_0 = \alpha$ in the proof), while that of u^2 is guaranteed by Theorem 4.1 (see Remark 4.2). □

In Theorem 4.2 it is possible to have $\|u^1\|_\phi = 0$. The next result guarantees the existence of *two nontrivial constant-sign* solutions.

Theorem 4.3. *Let $\theta \in \{1,-1\}$ be fixed, and let P and Q be complementary N-functions. Suppose* (3.10), (3.11), (4.3)–(4.10) *and* *also* (4.9) $|_{\beta=\widetilde{\beta}}$ *hold, where* $0<\widetilde{\beta}<\alpha<\beta$. *Then,* (1.2) *has (at least) two constant-sign solutions* $u^1, u^2 \in (L_\phi([0,1],\mathbf{R}^N))^n$ *such that*

$$0<\widetilde{\beta}\le \|u^1\|_\phi \le \alpha \le \|u^2\|_\phi \le \beta$$

and

$$\theta u_i^1(t)\ge \gamma(t)\widetilde{\beta},\quad \theta u_i^2(t)\ge \gamma(t)\alpha,\quad a.e.\ t\in[0,1],\quad 1\le i\le n.$$

Proof. This follows from Theorem 4.1. □

Finally, we give the existence of *multiple* solutions of (1.2) in $(L_\phi([0,1],\mathbf{R}^N))^n$.

Theorem 4.4. *Let* $\theta\in\{1,-1\}$ *be fixed, and let* P *and* Q *be complementary* N*-functions. Suppose* (3.10), (3.11) *and* (4.3)–(4.8) *hold. Let* (4.9) *be satisfied for* $\beta=\beta_\ell$, $\ell=1,2,\dots,m$.

(a) *Let* (4.10) *be satisfied for* $\alpha=\alpha_\ell$, $\ell=1,2,\dots,k$.

(i) *If* $m=k+1$ *and* $0<\beta_1<\alpha_1<\dots<\beta_k<\alpha_k<\beta_{k+1}$, *then* (1.2) *has* (*at least*) $2k$ *constant-sign solutions* $u^1,\dots,u^{2k}\in(L_\phi([0,1],\mathbf{R}^N))^n$ *such that*

$$0<\beta_1\le\|u^1\|_\phi\le\alpha_1\le\|u^2\|_\phi\le\beta_2\le\dots\le\alpha_k\le\|u^{2k}\|_\phi\le\beta_{k+1}.$$

(ii) *If* $m=k$ *and* $0<\beta_1<\alpha_1<\dots<\beta_k<\alpha_k$, *then* (1.2) *has* (*at least*) $2k-1$ *constant-sign solutions* $u^1,\dots,u^{2k-1}\in(L_\phi([0,1],\mathbf{R}^N))^n$ *such that*

$$0<\beta_1\le\|u^1\|_\phi\le\alpha_1\le\|u^2\|_\phi\le\beta_2\le\dots\le\beta_k\le\|u^{2k-1}\|_\phi\le\alpha_k.$$

(b) *Let* $(4.10)'$ *be satisfied for* $\alpha=\alpha_1$, *and let* (4.10) *be satisfied for* $\alpha=\alpha_\ell$, $\ell=2,\dots,k$.

(i) *If* $k = m + 1$ *and* $0 < \alpha_1 < \beta_1 < \cdots < \alpha_m < \beta_m < \alpha_{m+1}$, *then* (1.2) *has* (*at least*) $2m+1$ *solutions* $u^0, \dots, u^{2m} \in (L_\phi([0,1], \mathbf{R}^N))^n$ *such that*

$$0 \le \|u^0\|_\phi < \alpha_1 < \|u^1\|_\phi \le \beta_1 \le \|u^2\|_\phi \le \alpha_2 \le \cdots \\ \le \beta_m \le \|u^{2m}\|_\phi \le \alpha_{m+1}.$$

Moreover, $u^1, \dots, u^{2m}$ *are of constant sign.*

(ii) *If* $k = m$ *and* $0 < \alpha_1 < \beta_1 < \cdots < \alpha_k < \beta_k$, *then* (1.2) *has* (*at least*) $2k$ *solutions* $u^0, \dots, u^{2k-1} \in (L_\phi([0,1], \mathbf{R}^N))^n$ *such that*

$$0 \le \|u^0\|_\phi < \alpha_1 < \|u^1\|_\phi \le \beta_1 \le \|u^2\|_\phi \le \alpha_2 \le \cdots \\ \le \alpha_k < \|u^{2k-1}\|_\phi \le \beta_k.$$

Moreover, $u^1, \dots, u^{2k-1}$ *are of constant sign.*

Proof. In (a), we just apply Theorem 4.3 repeatedly. In (b), Theorem 3.3 is used to obtain the existence of $u^0 \in (L_\phi([0,1], \mathbf{R}^N))^n$ with $0 \le \|u^0\| < \alpha_1$. The results then follow by repeated use of Theorem 4.3. □

Remark 4.4. If (4.10) is replaced by (4.10)$'$ in Theorems 4.3 and 4.4, then noting Remark 4.2 those inequalities involving αs in the conclusions must be *strict*.

5. Semi-positone (semi-bounded) case of (1.2). In Section 4, the nonlinearity H considered is '*nonnegative*' in the sense of condition (4.5). We shall now tackle the '*semi-positone*' (semi-bounded) case, i.e., when θH can take *negative* values.

Our first result gives the existence of a *constant-sign* solution in $(L_\phi([0,1], \mathbf{R}^N))^n$.

Theorem 5.1. *Let* $\theta \in \{1, -1\}$ *be fixed, and let* P *and* Q *be complementary* N*-functions. Suppose* (3.10), (3.11), (4.3), (4.7) *and*

(4.8) *hold. Moreover, assume*

$$(5.1)\quad \begin{cases} \text{there exists } K \in (0,\infty)^N \text{ such that } \theta H(t,u) + K \geq \rho(u) \\ \text{ for a.e. } t \in [0,1],\ u \in A, \\ \text{where } A = \begin{cases} \left((0,\infty)^N\right)^n & \text{if } \theta = 1, \\ \left((-\infty,0)^N\right)^n & \text{if } \theta = -1, \end{cases} \\ \rho : A \to [0,\infty)^N \text{ is continuous, and if} \\ 0 \leq \theta x \leq \theta u_j \leq \theta y \qquad\qquad 1 \leq j \leq n, \\ \text{then } \rho(u_1,\dots,x,\dots,u_n) \leq \rho(u_1,\dots,u_j,\dots,u_n) \\ \leq \rho(u_1,\dots,y,\dots,u_n), \end{cases}$$

$$(5.2)\quad \int_0^1 b(s)K\,ds < \infty,$$

$$(5.3)\quad \begin{cases} \theta H(t,u) + K \leq \psi(u) \text{ for a.e. } t \in [0,1],\ u \in A, \\ \text{where } \psi : A \to [0,\infty)^N \text{ is continuous, and if} \\ 0 \leq \theta x \leq \theta u_j \leq \theta y,\ 1 \leq j \leq n, \\ \text{then } \psi(u_1,\dots,x,\dots,u_n) \leq \psi(u_1,\dots,u_j,\dots,u_n) \\ \quad \leq \psi(u_1,\dots,y,\dots,u_n), \end{cases}$$

(5.4)
$$\begin{cases} \text{there exists } \Gamma \in C[0,\infty) \text{ such that for } u \in \left(L_\phi([0,1],\mathbf{R}^N)\right)^n, \\ |\psi(u)|_Q \leq \Gamma(\|u\|_\phi), \end{cases}$$

(5.5)
$$\begin{cases} \text{there exists } \beta \geq (|a|_\phi/M)\int_0^1 b(s)K\,ds > 0 \text{ such that} \\ \phi^{-1}\left(\delta\int_0^1 \phi\Big(Ma(s)\int_0^1 b(\tau)\rho(\theta\eta(\tau),\theta\eta(\tau),\dots,\theta\eta(\tau))\,d\tau\Big)\,ds\right) \geq \beta, \\ \text{where } \eta(\tau) = a(\tau)\left[(M\beta/|a|_\phi) - \int_0^1 b(s)K\,ds\right], \\ \text{of course we also assume the above integral exists,} \end{cases}$$

(5.6)
$$\begin{cases} \text{there exists } \alpha \geq (|a|_\phi/M)\int_0^1 b(s)K\,ds > 0 \text{ such that} \\ |q_i|_\phi \cdot \Gamma(\alpha) \leq \alpha \text{ for each } 1 \leq i \leq n,\ \text{ where } q_i(t) = |g_i(t,\cdot)|_P \end{cases}$$

and

(5.7)

$$\begin{cases} \textit{let } \xi_i(t) = \theta \int_0^1 g_i(t,s)K\,ds, \ 1 \le i \le n \textit{ and} \\ \textit{for each } r > 0 \textit{ and } 1 \le i \le n, \textit{ there exists } \eta_{r,i} \in L_Q([0,1], \mathbf{R}) \\ \textit{and } K_{r,i} \ge 0 \\ \textit{such that } |H(t, u-\xi) + \theta K| \le \eta_{r,i}(t) + K_{r,i}\ Q^{-1}\left(\phi\left(u_i - \xi_i/r\right)\right) \\ \textit{for a.e. } t \in [0,1] \textit{ and every } u \in (\mathbf{R}^N)^n. \end{cases}$$

Then, (1.2) *has a constant-sign solution* $u^* \in (L_\phi([0,1], \mathbf{R}^N))^n$ *such that for* $1 \le i \le n$,

$$\begin{aligned} \theta u_i^*(t) &\ge 0, \quad t \in [0,1] \textit{ and} \\ \theta u_i^*(t) &> 0, \quad t \in I \equiv \{t \in [0,1] \mid a(t) > 0\}. \end{aligned} \tag{5.8}$$

Moreover,

$$\min\{\alpha, \beta\} - \|\xi\|_\phi \le \|u^*\|_\phi \le \max\{\alpha, \beta\} + \|\xi\|_\phi. \tag{5.9}$$

Proof. To show that (1.2) has a constant-sign solution, we consider the system

(5.10)

$$y_i(t) = \int_0^1 g_i(t,s) H^*(s, y_1(s) - \xi_1(s), y_2(s) - \xi_2(s), \ldots, y_n(s) - \xi_n(s))\, ds,$$
$$\text{a.e. } t \in [0,1],\ 1 \le i \le n$$

where for each $1 \le i \le n$,

$$\xi_i(t) = \theta \int_0^1 g_i(t,s)K\,ds, \quad t \in [0,1]$$

and

$$\begin{aligned} H^*(t, v_1, v_2, \ldots, v_n) = H(t, v_1, v_2, \ldots, v_n) + \theta K, \\ \text{if } \theta v_j \ge 0, \quad 1 \le j \le n. \end{aligned} \tag{5.11}$$

We shall show that system (5.10) has a constant-sign solution $y^* \in (L_\phi([0,1], \mathbf{R}^N))^n$ satisfying

$$\theta y_i^*(t) \ge \theta \xi_i(t), \quad t \in [0,1],\ 1 \le i \le n. \tag{5.12}$$

It is then clear that $u^* = y^* - \xi = (y_1^* - \xi_1, y_2^* - \xi_2, \dots, y_n^* - \xi_n)$ is a *constant-sign solution* of (1.2). Moreover, u^* fulfills (5.8) since for $t \in I$,

$$\theta y_i^*(t) \geq \theta \xi_i(t) \geq \int_0^1 M a(t) b(s) K \, ds > 0, \quad 1 \leq i \leq n$$

(use (4.7)). Note that $\xi \in (L_\phi([0,1], \mathbf{R}^N))^n$, since in view of (4.7) and (5.2) we have

$$\|\xi\|_\phi = \max_{1 \leq i \leq n} |\xi_i|_\phi \leq \left| \int_0^1 a(t) b(s) K \, ds \right|_\phi = |a|_\phi \cdot \int_0^1 b(s) K \, ds < \infty.$$

Thus, $u^* = y^* - \xi$ is a constant-sign solution of (1.2) in $(L_\phi([0,1], \mathbf{R}^N))^n$.

Without loss of generality, assume that $\beta > \alpha$. To proceed, we introduce the cone C

(5.13)
$$C = \Big\{ y \in \big(L_\phi([0,1], \mathbf{R}^N)\big)^n \ \Big| \ \text{for each } 1 \leq i \leq n, \ \theta y_i(t) \geq \gamma(t) \|y\|_\phi \quad \text{for a.e. } t \in [0,1] \Big\}$$

where $\gamma(t) = M(a(t)/|a|_\phi)$. Moreover, we define open subsets Ω_α and Ω_β of $(L_\phi([0,1], \mathbf{R}^N))^n$ by

$$\Omega_\alpha = \Big\{ y \in \big(L_\phi([0,1], \mathbf{R}^N)\big)^n \ \Big| \ \|y\|_\phi < \alpha \Big\}$$

and

$$\Omega_\beta = \Big\{ y \in \big(L_\phi([0,1], \mathbf{R}^N)\big)^n \ \Big| \ \|y\|_\phi < \beta \Big\}.$$

Let the operator $T : C \cap (\overline{\Omega}_\beta \backslash \Omega_\alpha) \to (L_\phi([0,1], \mathbf{R}^N))^n$ be defined by

$$Ty(t) = (T_1 y(t), T_2 y(t), \dots, T_n y(t)), \quad \text{a.e. } t \in [0,1] \tag{5.14}$$

where

(5.15)
$$T_i y(t) = \int_0^1 g_i(t,s) H^*(s, y(s) - \xi(s)) \, ds, \quad \text{a.e. } t \in [0,1], \ 1 \leq i \leq n.$$

Clearly, a fixed point of the operator T is a solution of system (5.10). Indeed, a fixed point of T obtained in C will be a *constant-sign solution* of system (5.10).

First, we shall show that T is well defined. Let $y \in C \cap (\overline{\Omega}_\beta \backslash \Omega_\alpha)$. Then, $\alpha \le \|y\|_\phi \le \beta$. Using (4.7), we obtain for $t \in [0,1]$ and $1 \le i \le n$,

$$\begin{aligned} \theta[y_i(t) - \xi_i(t)] &= \theta y_i(t) - \int_0^1 g_i(t,s) K\,ds \\ &\ge \gamma(t)\|y\|_\phi - \int_0^1 a(t)b(s)K\,ds \\ &= a(t)\left[\frac{M\|y\|_\phi}{|a|_\phi} - \int_0^1 b(s)K\,ds\right]. \end{aligned} \tag{5.16}$$

Since $\|y\|_\phi \ge \alpha$, it follows that

$$\theta[y_i(t) - \xi_i(t)] \ge a(t)\left[\frac{M\alpha}{|a|_\phi} - \int_0^1 b(s)K\,ds\right] \ge 0, \\ t \in [0,1], \quad 1 \le i \le n$$

(use (5.6)), or

$$y(t) - \xi(t) \in A, \quad t \in [0,1]. \tag{5.17}$$

Hence, noting (5.11) we have

$$H^*(t, y(t) - \xi(t)) = H(t, y(t) - \xi(t)) + \theta K, \quad t \in [0,1]. \tag{5.18}$$

Also, since $\theta\xi_i \ge 0$, it is obvious that

$$\theta[y_i(t) - \xi_i(t)] \le \theta y_i(t), \quad t \in [0,1],\ 1 \le i \le n. \tag{5.19}$$

Now, we apply (5.18), (5.19) and (5.3) to get

(5.20)
$$\begin{aligned} \theta T_i y(t) &= \int_0^1 g_i(t,s)[\theta H(s, y(s) - \xi(s)) + K]\,ds \\ &\le \int_0^1 g_i(t,s)\psi(y(s) - \xi(s))\,ds \\ &\le \int_0^1 g_i(t,s)\psi(y(s))\,ds, \quad t \in [0,1],\ 1 \le i \le n. \end{aligned}$$

By Lemma 16.3(a) of [**13**] (with $M_2 = \phi$, $N_1 = P$, $M_1 = Q$), we have

$$\left| \int_0^1 g_i(t,s)v(s)\,ds \right|_\phi \leq |q_i|_\phi \cdot |v|_Q, \quad 1 \leq i \leq n.$$

Therefore, using the above inequality and (5.4) leads to

$$\begin{aligned} |T_i y|_\phi &\leq \left| \int_0^1 g_i(t,s)\psi(y(s))\,ds \right|_\phi \\ &\leq |q_i|_\phi \cdot |\psi(y)|_Q \leq |q_i|_\phi \cdot \Gamma(\|y\|_\phi) < \infty, \quad 1 \leq i \leq n. \end{aligned} \tag{5.21}$$

This shows that $T_i y \in L_\phi([0,1],\mathbf{R}^N)$ for each $1 \leq i \leq n$, i.e., $Ty \in (L_\phi([0,1],\mathbf{R}^N))^n$. Hence, $T : C \cap (\overline{\Omega}_\beta \backslash \Omega_\alpha) \to (L_\phi([0,1],\mathbf{R}^N))^n$ is well defined.

Next, we claim that $T : C \cap (\overline{\Omega}_\beta \backslash \Omega_\alpha) \to (L_\phi([0,1],\mathbf{R}^N))^n$ is continuous and completely continuous. We observe that the operator T_i, $1 \leq i \leq n$, can be written as

$$T_i = A_i F^*, \tag{5.22}$$

where $F^* : C \cap (\overline{\Omega}_\beta \backslash \Omega_\alpha) \to \mathbf{R}^N$ is defined by

$$F^* y(t) = H^*(t, y(t) - \xi(t)), \quad t \in [0,1] \tag{5.23}$$

and $A_i : \mathbf{R}^N \to \mathbf{R}^N$ is given by (3.5). As in the proof of Theorem 3.2, we can show that T_i, $1 \leq i \leq n$, is continuous and completely continuous, and hence so is T.

Now, we shall show that T maps $C \cap (\overline{\Omega}_\beta \backslash \Omega_\alpha)$ into C. Let $y \in C \cap (\overline{\Omega}_\beta \backslash \Omega_\alpha)$. We already have $Ty \in (L_\phi([0,1],\mathbf{R}^N))^n$. Next, in view of (5.20), (5.1) and (4.7), it follows that

$$\begin{aligned} \theta T_i y(t) \leq \int_0^1 a(t)b(s)[\theta H(s, y(s) - \xi(s)) + K]\,ds, \\ \text{a.e. } t \in [0,1],\ 1 \leq i \leq n \end{aligned} \tag{5.24}$$

which leads to

$$|T_i y|_\phi \leq |a|_\phi \int_0^1 b(s)[\theta H(s, y(s) - \xi(s)) + K]\,ds, \quad 1 \leq i \leq n.$$

Hence, it follows that

$$\|Ty\|_\phi = \max_{1\le i\le n} |T_i u|_\phi \le |a|_\phi \int_0^1 b(s)[\theta H(s, y(s)-\xi(s))+K]\, ds. \tag{5.25}$$

Also, from (5.20), (5.1) and (4.7) we obtain

$$\begin{aligned} \theta T_i y(t) \ge M \int_0^1 a(t)b(s)[\theta H(s, y(s)-\xi(s))+K] \ge 0,\\ \text{a.e. } t\in[0,1], \quad 1\le i\le n. \end{aligned} \tag{5.26}$$

Substituting (5.25) into (5.26) yields

$$\theta T_i y(t) \ge Ma(t)\,\frac{\|Ty\|_\phi}{|a|_\phi} = \gamma(t)\|Ty\|_\phi, \text{ a.e. } t\in[0,1],\ 1\le i\le n.$$

We have proved that $T : C\cap(\overline{\Omega}_\beta\backslash\Omega_\alpha)\to C$.

Next, we shall verify that $\|Ty\|_\phi \ge \|y\|_\phi$ for $y\in C\cap\partial\Omega_\beta$. Let $y\in C\cap\partial\Omega_\beta$. Then, $\|y\|_\phi=\beta$. From (5.16) and (5.5), we have

$$\begin{aligned} \theta[y_i(t)-\xi_i(t)] &\ge a(t)\left[\frac{M\|y\|_\phi}{|a|_\phi} - \int_0^1 b(s)K\, ds\right]\\ &= a(t)\left[\frac{M\beta}{|a|_\phi} - \int_0^1 b(s)K\, ds\right]\\ &= \eta(t)\ge 0, \quad t\in[0,1],\ 1\le i\le n. \end{aligned} \tag{5.27}$$

Let $1\le j\le n$ be fixed. Noting (5.20), (5.1), (4.7), (5.27) and (5.1), we find

(5.28)

$$\begin{aligned} \theta T_j y(t) &= \int_0^1 g_i(t,s)[\theta H(s, y(s)-\xi(s))+K]\, ds\\ &\ge \int_0^1 Ma(t)b(s)\rho(y(s)-\xi(s))\, ds\\ &\ge Ma(t)\int_0^1 b(s)\rho(\theta\eta(s),\theta\eta(s),\dots,\theta\eta(s))\, ds \ge 0,\ t\in[0,1]. \end{aligned}$$

Since $\phi(x)$ is increasing for $x\ge 0$, we have, in view of (5.28),

$$\begin{aligned} &\int_0^1 \phi(\theta T_j y(s))\, ds\\ &\quad\ge \int_0^1 \phi\left(Ma(s)\int_0^1 b(\tau)\rho(\theta\eta(\tau),\theta\eta(\tau),\dots,\theta\eta(\tau))\, d\tau\right) ds. \end{aligned} \tag{5.29}$$

Now, Lemma 9.2 in [**13**] provides

$$\int_0^1 \phi\bigg(\frac{\theta T_j y(s)}{\|Ty\|_\phi}\bigg)\,ds \le \frac{|T_j y|_\phi}{\|Ty\|_\phi} \le 1. \tag{5.30}$$

On the other hand, using (4.8) gives

$$\int_0^1 \phi\bigg(\frac{\theta T_j y(s)}{\|Ty\|_\phi}\bigg)\,ds \ge \delta \int_0^1 \frac{\phi(\theta T_j y(s))}{\phi(\|Ty\|_\phi)}\,ds. \tag{5.31}$$

A combination of (5.30) and (5.31) yields

$$\delta \int_0^1 \frac{\phi(\theta T_j y(s))}{\phi(\|Ty\|_\phi)}\,ds \le 1$$

or, together with (5.28),

$$\begin{aligned}\phi(\|Ty\|_\phi)& \\ &\ge \delta \int_0^1 \phi(\theta T_j y(s))\,ds \\ &\ge \delta \int_0^1 \phi\left(Ma(s)\int_0^1 b(\tau)\rho(\theta\eta(\tau),\theta\eta(\tau),\dots,\theta\eta(\tau))\,d\tau\right)\,ds.\end{aligned}$$

Hence, noting (5.5) we get

$$\begin{aligned}\|Ty\|_\phi \ge \phi^{-1}\bigg(\delta\int_0^1 \phi\bigg(Ma(s)\int_0^1 b(\tau)\rho(\theta\eta(\tau),\theta\eta(\tau),& \\ \dots,\theta\eta(\tau))\,d\tau\bigg)\,ds\bigg) \ge \beta = \|y\|_\phi.&\end{aligned}$$

We have thus shown that $\|Ty\|_\phi \ge \|y\|_\phi$ for $y \in C \cap \partial\Omega_\beta$.

We shall now prove that $\|Ty\|_\phi \le \|y\|_\phi$ for $y \in C \cap \partial\Omega_\alpha$. Let $y \in C \cap \partial\Omega_\alpha$. Then, $\|y\|_\phi = \alpha$. From (5.21) and (5.6), we find

$$|T_i y|_\phi \le |q_i|_\phi \cdot \Gamma(\|y\|_\phi) = |q_i|_\phi \cdot \Gamma(\alpha) \le \alpha, \quad 1 \le i \le n,$$

which implies

$$\|Ty\|_\phi \le \alpha = \|y\|_\phi.$$

Hence, $\|Ty\|_\phi \le \|y\|_\phi$ for $y \in C \cap \partial\Omega_\alpha$.

We conclude by Theorem 2.1 that T has a fixed point $y^* \in C \cap (\overline{\Omega}_\beta \backslash \Omega_\alpha)$. Hence, y^* is of constant sign and $\alpha \le \|y^*\|_\phi \le \beta$. Since $u^* = y^* - \phi$, we obtain (5.9).

It remains to show that y^* satisfies (5.12). This is clear since from (5.6) and (4.7), we get for $t \in [0,1]$ and $1 \le i \le n$,

$$\begin{aligned}\theta y_i^*(t) &\ge \gamma(t)\|y^*\|_\phi \ge \gamma(t)\alpha \\ &\ge \gamma(t)\frac{|a|_\phi}{M}\int_0^1 b(s)K\,ds \\ &= \int_0^1 a(t)b(s)K\,ds \\ &\ge \int_0^1 g_i(t,s)K\,ds \\ &= \theta\xi_i(t).\end{aligned}$$

The proof is complete. □

Remark 5.1. A remark similar to Remark 4.3 applies here in relation to (5.7).

Our next result guarantees the existence of *two constant-sign* solutions in $(L_\phi([0,1],\mathbf{R}^N))^n$.

Theorem 5.2. *Let* $\theta \in \{1,-1\}$ *be fixed, and let* P *and* Q *be complementary* N*-functions. Suppose* (3.10), (3.11), (4.3), (4.7), (4.8), (5.1)–(5.7) *and also* $(5.5)|_{\beta=\widetilde{\beta}}$ *hold, where* $0 < \widetilde{\beta} < \alpha < \beta$. *Then,* (1.2) *has* (*at least*) *two constant-sign solutions* $u^1, u^2 \in (L_\phi([0,1],\mathbf{R}^N))^n$ *such that for* $k = 1,2$ *and* $1 \le i \le n$,

$$\theta u_i^k(t) \ge 0, \quad t \in [0,1]$$

and

$$\theta u_i^k(t) > 0, \quad t \in I \equiv \{t \in [0,1] \mid a(t) > 0\}.$$

Moreover,

$$u^1 = y^1 - \xi \text{ and } u^2 = y^2 - \xi,$$

where $\xi_i(t) = \theta \int_0^1 g_i(t,s)K\,ds$, $1 \le i \le n$, y^1, $y^2 \in (L_\phi([0,1], \mathbf{R}^N))^n$ *are of constant sign and satisfy*

$$0 < \widetilde{\beta} \le \|y^1\|_\phi \le \alpha \le \|y^2\|_\phi \le \beta$$

and

$$\theta y_i^1(t) \ge \gamma(t)\widetilde{\beta}, \quad \theta y_i^2(t) \ge \gamma(t)\alpha, \text{ a.e. } t \in [0,1], \ 1 \le i \le n.$$

Proof. This follows from Theorem 5.1. □

Finally, by applying Theorem 5.1 repeatedly we obtain the existence of *multiple constant-sign* solutions of (1.2) in $(L_\phi([0,1], \mathbf{R}^N))^n$.

Theorem 5.3. *Let* $\theta \in \{1, -1\}$ *be fixed, and let* P *and* Q *be complementary* N*-functions. Suppose* (3.10), (3.11), (4.3), (4.7), (4.8), (5.1)–(5.4) *and* (5.7) *hold. Let* (5.5) *be satisfied for* $\beta = \beta_\ell$, $\ell = 1, 2, \dots, m$ *and let* (5.6) *be satisfied for* $\alpha = \alpha_\ell$, $\ell = 1, 2, \dots, k$.

(a) *If* $m = k+1$ *and* $0 < \beta_1 < \alpha_1 < \cdots < \beta_k < \alpha_k < \beta_{k+1}$, *then* (1.2) *has* (*at least*) $2k$ *constant-sign solutions* $u^1, \dots, u^{2k} \in (L_\phi([0,1], \mathbf{R}^N))^n$ *such that for* $j = 1, 2, \dots, 2k$ *and* $1 \le i \le n$,

$$\theta u_i^j(t) \ge 0, \quad t \in [0,1]$$

and

$$\theta u_i^j(t) > 0, \quad t \in I \equiv \{t \in [0,1] \mid a(t) > 0\}.$$

Moreover,

$$u^j = y^j - \xi, \quad 1 \le j \le 2k,$$

where $\xi_i(t) = \theta \int_0^1 g_i(t,s)K\,ds$, $1 \le i \le n$, $y^j \in (L_\phi([0,1], \mathbf{R}^N))^n$ *is of constant sign and satisfies*

$$0 < \beta_1 \le \|y^1\|_\phi \le \alpha_1 \le \|y^2\|_\phi \le \beta_2 \le \cdots \le \alpha_k \le \|y^{2k}\|_\phi \le \beta_{k+1}.$$

(b) *If* $m = k$ *and* $0 < \beta_1 < \alpha_1 < \cdots < \beta_k < \alpha_k$*, then* (1.2) *has* (*at least*) $2k-1$ *constant-sign solutions* $u^1, \dots, u^{2k-1} \in (L_\phi([0,1], \mathbf{R}^N))^n$ *such that for* $j = 1, 2, \dots, 2k-1$ *and* $1 \le i \le n$,

$$\theta u_i^j(t) \ge 0, \quad t \in [0,1]$$

and

$$\theta u_i^j(t) > 0, \quad t \in I \equiv \{t \in [0,1] \mid a(t) > 0\}.$$

Moreover,

$$u^j = y^j - \xi, \quad 1 \le j \le 2k-1$$

where $\xi_i(t) = \theta \int_0^1 g_i(t,s) K\, ds$, $1 \le i \le n$, $y^j \in (L_\phi([0,1], \mathbf{R}^N))^n$ *is of constant sign and satisfies*

$$0 < \beta_1 \le \|y^1\|_\phi \le \alpha_1 \le \|y^2\|_\phi \le \beta_2 \le \cdots \le \beta_k \le \|y^{2k-1}\|_\phi \le \alpha_k.$$

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