Perturbations of gravitational instantons

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Ashtekar's spinorial formulation of general relativity is used to study perturbations of gravitational instantons corresponding to finite-action solutions of the Euclidean Einstein equations (with a nonzero cosmological constant) possessing an anti-self-dual Weyl curvature tensor. It is shown that, with an appropriate "on-shell" form of infinitesimal gauge transformations, the space of solutions to the linearized instanton equation can be described in terms of an elliptic complex; the cohomology of the complex defines gauge-inequivalent perturbations. Using this elliptic complex we prove that there are no nontrivial solutions to the linearized instanton equation on conformally anti-self-dual Einstein spaces with a positive cosmological constant. Thus, the space of gravitational instantons is discrete when the cosmological constant is positive; i.e., the dimension of the gravitational moduli space in this case is zero. We discuss the issue of linearization stability as well as the feasibility of using the Atiyah-Singer index theorem to compute the dimension of the gravitational moduli space when the cosmological constant is negative.

I. INTRODUCTION

In the functional-integral approach to quantum gravity it is common to perform integrations over classes of Euclidean geometries in order to give a tractable definition to the integral over four-geometries,[1] to define a "ground state,"[2] to study statistical mechanics and/or thermodynamics,[1,3] etc. In a semiclassical approximation, the Euclidean functional integral is dominated by finite-action Euclidean signature solutions to Einstein's equations—the "gravitational instantons." While many such instantons have been found, the detailed structure of the space of gravitational instantons is not nearly as well understood as that of, say, the instantons of non-Abelian gauge theory. From a classical perspective gravitational instantons, especially those with self-dual curvature, are important because they shed light on the rather complicated structure of the Einstein equations themselves.[4]

It is natural to ask whether the variety of techniques used to study self-dual solutions to the field equations of non-Abelian gauge theory[5,6] could be fruitfully employed also in general relativity. At first sight, the connection seems difficult to make because the two theories are rather different both physically and geometrically; however, as we shall see, progress can be made by using the spinorial variables introduced by Ashtekar.[7] In terms of these variables, the resulting form of Einstein's theory parallels that of a non-Abelian gauge theory with, in the Euclidean case, an SU(2) gauge group. While most work using these new variables has been devoted to the canonical formulation of general relativity, a covariant four-dimensional formulation also exists,[8] and in this covariant formulation it has been shown[9] that there is a self-duality ansatz analogous to that used to find instantons in gauge theories. In the gravitational case this ansatz leads, in particular, to all finite-action solutions (on a compact manifold) of the Euclidean Einstein equations with a cosmological constant (Einstein spaces) that possess an anti-self-dual Weyl tensor. The ansatz has since been shown[10] to be equivalent to a metric-independent, *quadratic* condition on the curvature of an SU(2) connection on the spacetime manifold so, at least as far as conformally self-dual instantons are concerned,[11] the metric can be taken completely out of the picture.

In this paper we will show that at least one method of analysis that proved useful in analyzing self-dual solutions of non-Abelian gauge theories can be carried over to general relativity in terms of the Ashtekar variables. In particular, following Ref. 5, we will study perturbations of conformally (anti-)self-dual Einstein spaces thereby obtaining a "tangent space" approximation to the space of solutions of the ansatz given in Refs. 9 and 10. As we shall see, the perturbations can be described in terms of an elliptic complex, the cohomology of which defines the space of solutions to the linearized equations modulo the action of the gauge group, which is the semidirect product of a local SU(2) group and the spacetime diffeomorphism group. The equivalence classes of such solutions turn out to be defined by a surprisingly simple elliptic differential equation. Indeed, using this equation we show that there are no nontrivial solutions to the linearized equations on compact manifolds with a positive cosmological constant. Therefore, in this case, the space of (anti-)self-dual instantons is discrete; i.e., the gravitational moduli space is trivial. Moreover, when the cosmological constant is negative, the dimension of the moduli space can be determined by an application of the Atiyah-Singer index theorem.

We organize this paper as follows. Section II gives the key equations that summarize the results of Refs. 9 and 10. In Sec. III we present the linearized equations and study the influence of the gauge group on their mathematical structure. Section IV deals with the construction of an elliptic complex that describes gauge-inequivalent solutions to the linearized equations. There we prove that the gravitational moduli space is zero dimensional when the cosmological constant is positive. In Sec. V we digress a bit to comment on the relationship between the form of infinitesimal gauge transformations we need to define the complex and a more conventional form of the transformations. In Secs. VI and VII we discuss issues which are important for understanding the space of instantons when the cosmological constant is negative. In particular, in Sec. VI, we confront the well-known fact that solutions to linearized equations of motion are not necessarily in one-to-one correspondence with solutions of the full, nonlinear equations; this is the issue of linearization stability. Section VI formulates the stability question in terms of the framework we have developed but, unfortunately, does not answer it. We discuss in Sec. VII the possibility of applying the Atiyah-Singer index theorem to compute the analytical index of instanton perturbations. As in non-Abelian gauge theory this theorem, when combined with a linearization stability argument, allows one to compute the dimension of the space of solutions to the full, nonlinear instanton equation (for negative cosmological constant). We conclude in Sec. VIII with a discussion of the results obtained and work left to be done.

II. PRELIMINARIES

We begin with the spinorial form of the Einstein equations for a spacetime of Euclidean signature.[9] They can be expressed in terms of a soldering form $\gamma_a^{AA'}$, which is a nondegenerate map between vector fields and SU(2) $\times$SU(2) spinors,[12] and an SU(2) spin connection A_a^{AB}. The spacetime metric is obtained via

$$g_{ab}=\gamma_a^{AA'}\gamma_{bAA'}\ , \tag{2.1}$$

while the curvature of the spin connection is given by

$$F_{ab}{}^{AB}=2(\partial_{[a}A_{b]}{}^{AB}+A_{[a}{}^{AC}A_{b]C}{}^{B})\ . \tag{2.2}$$

If we define the self-dual two-forms

$$\Sigma_{ab}{}^{AB}:=2\gamma_{[a}{}^{AA'}\gamma_{b]A'}{}^{B}\ , \tag{2.3}$$

then the vacuum Einstein equations with cosmological constant λ can be written as

$$\begin{aligned} &D_{[a}\Sigma_{bc]}{}^{AB}=0\ ,\\ &\gamma_{[a}{}^{AA'}F_{bc]A}{}^{B}+\tfrac{1}{6}\lambda\gamma_{[a}{}^{AA'}\Sigma_{bc]A}{}^{B}=0\ , \end{aligned} \tag{2.4}$$

where D_a is the derivative operator built from A_a. (Notice that our cosmological constant is -6 times that of Ref. 9. Our conventions are consistent with the Einstein equations taking the form $R_{ab}=\lambda g_{ab}$.)

The observation of Samuel,[9] as modified by Capovilla *et al.*,[10] is that (2.4) will be solved by any SU(2) curvature satisfying

$$\tfrac{1}{4}F_{[ab}{}^{(AB}F_{cd]}{}^{CD)}=0\ . \tag{2.5}$$

The idea behind (2.5) is that if a curvature satisfies (2.5) then it can be written as the exterior product of two (possibly degenerate) soldering forms; this product is then taken to *define* Σ via

$$F_{ab}{}^{AB}=-\tfrac{1}{6}\lambda\Sigma_{ab}{}^{AB}\ , \tag{2.6}$$

where λ, a constant of dimension $(\text{length})^{-2}$, is needed for dimensional consistency. If we demand that the soldering form so defined is nondegenerate and yields a metric [through (2.1)] of Euclidean signature, then it is simple to verify that (2.5) and/or (2.6) provides a solution of (2.4) corresponding to a compact Riemannian manifold. As the two-forms Σ are self-dual (with respect to the metric built from γ), so too are the SU(2) field strengths; it can be shown that the solutions generated in this manner have anti-self-dual Weyl tensor. Thus Eq. (2.5), which we shall refer to as the *instanton equation*, leads to conformally anti-self-dual Einstein spaces. It has been shown[13] that all such spacetimes arise from solutions to (2.5) [or (2.6)].

Because Eqs. (2.4), (2.5), and (2.6) are polynomial in all basic variables, it is permissible to allow the soldering form, and hence also the metric, to be degenerate, although in such a case the resulting spacetime geometry is not Riemannian. In consideration of the loss of mathematical control over the various differential operators that accompanies the use of a degenerate metric, in what follows we will always assume that the metric defined implicitly by (2.5) is nondegenerate; we will briefly discuss the significance of this assumption at the end of the paper.

III. LINEARIZED EQUATIONS

We now study the equations governing perturbations of solutions to (2.5). If we denote the perturbation by C_a and make the replacement

$$A_a{}^{AB}\rightarrow A_a{}^{AB}+C_a{}^{AB}\ ,$$

then to first order in C_a (2.5) becomes

$$F_{[ab}{}^{(AB}D_cC_{d]}{}^{CD)}=0\ , \tag{3.1}$$

where we have assumed that the unperturbed background connection satisfies (2.5). Equation (3.1) represents a set of five first-order, linear partial differential equations for the perturbation and is *a priori* independent of any spacetime metric, although we are always free to replace F with Σ as in (2.6).

The linearized instanton equation (3.1) must necessarily have several degenerate "directions" in the space of perturbations as a consequence of SU(2)-gauge and diffeomorphism covariance. Thus, given a pair of su(2)-valued functions N and M, and a real-valued function f, any perturbation of the form

$$C_a=D_aN+(\nabla^bf)F_{ba}+[D^bM,F_{ba}] \tag{3.2}$$

will automatically satisfy (3.1) when the instanton equation, (2.6) in particular, is satisfied. Notice that in (3.2), and also in what follows, we are using an su(2) matrix notation that suppresses spinor indices. The first term in (3.2) represents local SU(2) transformations while the latter two terms are associated with diffeomorphism covariance [the bracket in the last term is an su(2) commutator]. We will show later (in Sec. V) how these terms correspond to the action of an infinitesimal diffeomorphism; for now, observe that the transformation is parametrized by seven functions, which is also the number of functions parametrizing the (semidirect) product of the local SU(2) and spacetime diffeomorphism groups. For the analysis to follow, it is necessary to use the "on-shell" form of the transformation given in (3.2); notice that this expression makes explicit use of the metric.

To understand the degeneracy associated with (3.2) it is convenient to employ the notion of the *principal symbol* of the differential operator appearing in (3.1). The principal symbol is obtained by replacing the highest-order partial derivatives with covectors, which we shall denote k_a, and setting any lower-order terms to zero. The symbol of the operator in (3.1) is thus a linear map from the vector space of perturbations at a point to totally symmetric valence-four spinor-valued four-forms at the same point. If this map were injective for every choice of (nonvanishing) k_a, then the corresponding differential operator would be elliptic.[14] Denoting the differential operator as D_1,

$$D_1 C := F_{[ab}{}^{(AB} D_c C_{d]}{}^{CD)} , \tag{3.3}$$

the symbol $\xi(D_1)$ is given by

$$\xi(D_1)\cdot C = F_{[ab}{}^{(AB} k_c C_{d]}{}^{CD)} . \tag{3.4}$$

In terms of the symbol, the degenerate directions correspond to perturbations which set $\xi(D_1)\cdot C$ to zero, i.e., are in the kernel of $\xi(D_1)$. Using (2.6), we have

$$C_a \in \text{kernel}\,\xi(D_1) \Longleftrightarrow \Sigma^{ab(AB} k_a C_b{}^{CD)} = 0 . \tag{3.5}$$

At a given point, k^a defines an orthogonal three-dimensional vector space; if we denote by $\sigma_a{}^{AB}$ the associated SU(2) soldering form, then (3.5) is equivalent to

$$\sigma^{a(AB} C_a{}^{CD)} = 0 . \tag{3.6}$$

From (3.6) the degenerate directions for the symbol are easily deduced. First, because $\sigma_a{}^{AB}$ is the soldering form for the three-space orthogonal to k^a, we can satisfy (3.5) by choosing

$$C_a = N k_a , \tag{3.7}$$

where N is an arbitrary su(2)-valued function. The remaining independent solutions of (3.5) will be orthogonal to k^a; to uncover their explicit form, observe that if we assume $C^a \perp k^a$, and remove the symmetrization of the spinor indices in (3.6), then the resulting map is just the usual isomorphism from [su(2)-valued] three-dimensional covectors into [su(2)-valued] SU(2) spinors. Thus the remaining degeneracies arise from the information associated with various traces on spinor indices that is lost when the indices are symmetrized; in particular, when $C^a \perp k^a$, the perturbation has nine independent components (at a given point) while the symmetrized product in (3.6) contains only five independent components. It is thus straightforward to verify that the remaining four components which satisfy (3.5) are of the form

$$C_a = f k^b F_{ba} \tag{3.8a}$$

and

$$C_a = k^b [M, F_{ba}] , \tag{3.8b}$$

where, as before, f and M are real and su(2)-valued functions, respectively. Comparison of (3.7) and (3.8) with (3.2) reveals that the degenerate directions for $\xi(D_1)$ correspond to the symbols of the differential operators contained in (3.2). We conclude that the existence of the gauge transformation (3.2) spoils the injectivity of $\xi(D_1)$ thereby preventing the operator D_1 from being elliptic.

We can summarize the structure of the linearized theory as follows. The solutions to the linearized instanton equation are elements of the kernel of a linear map D_1, which transforms su(2)-valued one-forms into valence-four (totally symmetric) spinor-valued four-forms,

$$D_1 : \Lambda_1{}^g \rightarrow \Lambda_4{}^{g \otimes g} , \tag{3.9}$$

with $D_1 C$ given in (3.3). Physically trivial perturbations (ignoring for now the question of linearization stability) are generated by another linear map D_0; the domain of D_0 is a product space consisting of real-valued and su(2)-valued functions, while the range of D_0 consists of su(2)-valued one-forms:

$$D_0 : \Lambda_0 \otimes \Lambda_0{}^g \otimes \Lambda_0{}^g \rightarrow \Lambda_1{}^g , \tag{3.10}$$

with $D_0(f,M,N)$ given in (3.2). Physically relevant perturbations are therefore equivalence classes $[C]$ corresponding to the kernel of D_1 modulo the image of D_0:

$$[C] = \text{kernel}D_1 / \text{image}D_0 . \tag{3.11}$$

From the discussion given above, we expect that elements of $[C]$ will be determined by linear elliptic operators and, hence, for a given compact spacetime, $[C]$ will be a finite-dimensional subspace (possibly with singularities) of all possible perturbations. To analyze the structure of $[C]$ in detail, it is useful to embed the problem of determining the equivalence classes (3.11) into the mathematical framework of elliptic complexes as we shall now discuss.

IV. ELLIPTIC COMPLEX

We saw in the last section that instanton perturbations are controlled by linear maps between sections of various vector bundles on the spacetime manifold M,

$$0 \rightarrow \Lambda_0 \otimes \Lambda_0{}^g \otimes \Lambda_0{}^g \xrightarrow{D_0} \Lambda_1{}^g \xrightarrow{D_1} \Lambda_4{}^{g\otimes g} \rightarrow 0 , \tag{4.1}$$

satisfying

$$D_1 D_0 = 0 . \tag{4.2}$$

Equations (4.1) and (4.2) define a complex. We have already pointed out that

$$\text{kernel}\,\xi(D_1) \equiv \text{image}\,\xi(D_0) , \tag{4.3}$$

which means that the complex is elliptic[14] [to verify this it is useful to observe that the sum of the dimensions of the vector spaces (fibers) associated with the first and third bundles in (4.1) is equal to the dimension of the vector space associated with the second bundle].

To proceed further we will need to define adjoints of the operators D_0, D_1 and for this purpose we require an inner product between sections of a given bundle. We define the inner products using the metric associated with the given solution of (2.5) to contract tensor indices and provide a volume element, while any free spinor indices are contracted (traced) using ϵ spinors in the usual way. For example, two sections $C, \bar{C} \in \Lambda_1{}^g$ have an inner product given by

$$(C,\bar{C}) = \int_M \sqrt{g}\, g^{ab} C_a{}^{AB} \bar{C}_{bAB} \equiv -\mathrm{tr} \int_M \sqrt{g}\, g^{ab} C_a \bar{C}_b \;, \tag{4.4}$$

where the integral is over the spacetime manifold M; we use analogous definitions for sections of the other two bundles. The adjoint operators are defined (schematically) via

$$(w, Dv) = (D^* w, v) \;, \tag{4.5}$$

and are easily computed to be

$$D_0{}^*: \Lambda_1{}^g \to \Lambda_0 \otimes \Lambda_0{}^g \otimes \Lambda_0{}^g \;, \quad D_0{}^* C = (\mathrm{tr} F^{ab} D_a C_b ; [D_a C_b, F^{ab}]; -D^a C_a) \;, \tag{4.6}$$

$$D_1{}^*: \Lambda_4{}^{g\otimes g} \to \Lambda_1{}^g \;, \quad D_1{}^* \omega = F^{cd}{}_{CD} D^b \omega^{ABCD}_{abcd} \;, \tag{4.7}$$

where the action of the derivative operator D_a is extended (when necessary) to include tensor indices via the unique torsion-free connection that is compatible with the metric obtained from the solution of (2.5). Notice that we have used the instanton equation to simplify (4.7).

In terms of the basic differential operators and their adjoints we can construct (elliptic, self-adjoint) "Laplacians" on sections of each bundle. They are defined as

$$\Delta_0: \Lambda_0 \otimes \Lambda_0{}^g \otimes \Lambda_0{}^g \to \Lambda_0 \otimes \Lambda_0{}^g \otimes \Lambda_0{}^g \;, \quad \Delta_0 := D_0{}^* D_0 \;, \tag{4.8}$$

$$\Delta_1: \Lambda_1{}^g \to \Lambda_1{}^g \;, \quad \Delta_1 := D_1{}^* D_1 + D_0 D_0^* \;, \tag{4.9}$$

$$\Delta_2: \Lambda_4{}^{g\otimes g} \to \Lambda_4{}^{g\otimes g} \;, \quad \Delta_2 := D_1 D_1{}^* \;. \tag{4.10}$$

Using the Fredholm alternative,[14,15] these Laplacians provide an orthogonal decomposition of sections of each vector bundle; thus,

$$\Lambda_0 \otimes \Lambda_0{}^g \otimes \Lambda_0{}^g = \mathrm{range} D_0{}^* \oplus \mathrm{kernel} \Delta_0 \;, \tag{4.11}$$

$$\Lambda_1{}^g = \mathrm{range} D_0 \oplus \mathrm{range} D_1{}^* \oplus \mathrm{kernel} \Delta_1 \;, \tag{4.12}$$

$$\Lambda_4{}^{g\otimes g} = \mathrm{range} D_1 \oplus \mathrm{kernel} \Delta_2 \;, \tag{4.13}$$

where the orthogonality of the summands is with respect to the inner products described above. Exactly as in the case of de Rham cohomology, one can show that the equivalence class $[C]$, defined in (3.11), can be identified with the kernel of the Laplacian (4.9) on Lie-algebra-valued one-forms:

$$\mathrm{kernel} D_1 / \mathrm{image} D_0 = \mathrm{kernel} \Delta_1 \;. \tag{4.14}$$

Using the definition (4.9), it is straightforward to verify that perturbations are elements of kernel Δ_1 if and only if

$$D_1 C = 0 = D_0{}^* C \;, \tag{4.15}$$

which can be viewed as a combination of the linearized instanton equation and "gauge fixing" conditions. Alternatively, we show in Appendix B that solutions to (4.15) must satisfy

$$(-D^a D_a + \lambda) C_b = 0 \;. \tag{4.16}$$

Equation (4.16) is clearly an elliptic second-order differential equation and, as mentioned above, because the kernel of an elliptic operator on a compact manifold is always finite dimensional, the physical instanton perturbations form a finite-dimensional subspace of all possible gravitational perturbations. But we can say even more: (4.16) has no solutions if $\lambda > 0$. The proof is standard: contract both sides of (4.16) with C_a and integrate over M to obtain

$$-\mathrm{tr} \int_M \sqrt{g}\, D^a C^b D_a C_b = \lambda\, \mathrm{tr} \int_M \sqrt{g}\, C^a C_a \;, \tag{4.17}$$

which can only be satisfied if $C_a = 0$. If $\lambda < 0$, there is no such obstruction to solutions of (4.16).

The simplicity of (4.16) and the resulting obstruction to solutions when the cosmological constant is positive are results that differ substantially from the corresponding results in non-Abelian gauge theory.[5,6] Indeed, (4.16) implies that the space of solutions to the full, nonlinear instanton equation consists of a discrete set of points when $\lambda > 0$ irrespective of the spacetime topology, while in gauge theory the space of instantons is a finite-dimensional manifold the dimension of which is controlled by the topology of the base manifold and the second Chern number of the principal fibration. Technically, the difference stems from the fact that in gauge theory one demands that the unperturbed curvature as well as its perturbation are self-dual, while in the gravitational case the unperturbed curvature is self-dual but the curvature perturbation is required to be *anti*-self-dual (modulo gauge transformations) as can be deduced from (4.15). In broad terms this difference between the gauge and gravitational instanton perturbation theories can be attributed to the way that the metric is treated in the gravitational case: the spacetime metric, needed to define self-duality, is not fixed *a priori* as it is in gauge theory, but is instead determined by solving the instanton equation.

V. DIFFEOMORPHISMS

As promised, we now return to the relationship between the transformation (3.2) and the action of infinitesimal diffeomorphisms on the spin connection. As we pointed out, this relationship holds only "on shell"; i.e., we must make explicit use of the spacetime metric which is obtained by solving the instanton equation.

An infinitesimal diffeomorphism can be represented by a complete vector field V^a on M, which acts on SU(2) connections as

$$A_a \to A_a + C_a \ , \qquad C_a = V^b F_{ba} \ . \tag{5.1}$$

This action of infinitesimal diffeomorphisms, because it is manifestly covariant with respect to local SU(2) transformations, is somewhat more convenient than the ordinary Lie derivative [which ignores su(2) indices]. Equation (5.1) differs from the Lie derivative by what is effectively an infinitesimal SU(2) transformation:

$$\begin{aligned} V^b F_{ba} &= V^b \partial_b A_a + A_b \partial_a V^b - \partial_a (V^b A_b) - [A_a, V^b A_b] \\ &= L_V A_a - D_a (V^b A_b) \ . \end{aligned} \tag{5.2}$$

Given a vector field representing an infinitesimal diffeomorphism and the metric obtained by solving the instanton equation, we can lower the vector index and treat the vector as a one-form; then, using the Hodge decomposition on one-forms,[14] we split the one-form into the sum of the gradient of a function, a divergence of a two-form, and a harmonic one-form:

$$V_a = g_{ab} V^b = \nabla_a f + \nabla^b \omega_{ba} + h_a \ . \tag{5.3}$$

In (5.3) ∇_a is the unique torsion-free, metric compatible derivative operator on tensors. Application of the Hodge decomposition to the two-form ω itself reveals that only its exact part contributes to (5.3), so for the purpose of our discussion ω can be chosen to satisfy

$$\nabla_{[a} \omega_{bc]} = 0 \ . \tag{5.4}$$

Finally, the one-form h is harmonic and therefore satisfies

$$\nabla_{[a} h_{b]} = 0 = \nabla^a h_a \ . \tag{5.5}$$

It is easy to see that we can replace ω with (twice) its self-dual part in (5.3). First, decompose ω into its anti-self-dual (L) and self-dual parts (M):

$$\omega_{ab} = (L_{ab} + M_{ab}) \ , \tag{5.6}$$

which satisfy, from (5.4),

$$\nabla_{[a} M_{bc]} = -\nabla_{[a} L_{bc]} \ . \tag{5.7}$$

If we take the dual of both sides of (5.7) we obtain

$$\nabla^a M_{ab} = \nabla^a L_{ab} \ . \tag{5.8}$$

Therefore we can replace the two-form ω with twice its self-dual part giving a contribution to (5.1) of the form

$$\begin{aligned} (\nabla^c \omega_c{}^b) F_{ba} &= 2(\nabla^c M_c{}^b) F_{ba} \\ &= [D^b M, F_{ba}] + \tfrac{1}{3} \lambda D_a M \ , \end{aligned} \tag{5.9}$$

where the su(2)-valued function M is defined via

$$M_{ab} := \tfrac{1}{2} \Sigma_{ab}{}^{AB} M_{AB} \ , \tag{5.10}$$

which is permissible because M_{ab} is self-dual.[16] To obtain the final equality in (5.9) one needs to use the instanton equation and various Σ identities (see Appendix A).

The last term in (5.9) is an SU(2) transformation and can be absorbed into a redefinition of N in (3.2); thus infinitesimal diffeomorphisms correspond to perturbations of the form

$$C_a = (\nabla^b f) F_{ba} + [D^b M, F_{ba}] + h^b F_{ba} \ . \tag{5.11}$$

Equation (5.11) agrees with the putative diffeomorphism part of (3.2) except for the term involving the harmonic one-form, which on an Einstein space satisfies

$$\nabla^b \nabla_b h_a = \lambda h_a \ . \tag{5.12}$$

The differential operator on the left-hand side of (5.12) is negative semidefinite; therefore, if $\lambda > 0$ there are no nontrivial solutions to (5.12) (the Bochner vanishing theorem). To see this explicitly, contract (5.12) with the harmonic form and integrate over M to obtain

$$-\int_M \sqrt{g}\, (\nabla^b h^a)(\nabla_b h_a) = \lambda \int_M \sqrt{g}\, h^a h_a \ . \tag{5.13}$$

If the cosmological constant is positive, then the left and right sides of the equality in (5.13) are negative and positive semidefinite, respectively; thus, the only solution is the trivial one. Incidentally, this implies that the manifold must be simply connected. If λ is negative then there is no such obstruction to the existence of harmonic one-forms, and we conclude that the transformation in (3.2) can fail to capture all infinitesimal diffeomorphisms when M is not simply connected and $\lambda < 0$. This fact did not spoil the ellipticity of the complex we described in Sec. IV because the space of harmonic forms is always finite dimensional. Furthermore, the additional finite number of diffeomorphism-orbit identifications that are not captured by the cohomology of the complex in the $\lambda < 0$ case are easily handled if one understands the topology of the spacetime manifold M.

VI. LINEARIZATION STABILITY

The issue of linearization stability deals with the question as to whether every solution of the linearized field equations is an approximation to an exact solution of the full, nonlinear equations (modulo gauge transformations). A well-known strategy for answering this question[15] is to use the fact that (true) perturbations are, in geometric terms, tangent vectors to the space of gauge-inequivalent solution **S** of the full, nonlinear equations. If the space of gauge-inequivalent solutions is a differentiable submanifold of all possible field configurations, then every tangent vector is necessarily tangent to a curve in that manifold; i.e., it is a first approximation to an exact solution. If there are singular points in **S**, then when the instanton equation is linearized around such points one can expect to obtain spurious solutions. Thus we aim to identify the conditions under which the space of solutions to the instanton equation represents a smooth submanifold of the space of SU(2) connections on M.[17]

To analyze the issue of linearization stability in the geometric framework described above, it is convenient to rearrange the three-term elliptic complex constructed in Sec. IV into an equivalent two-term elliptic complex as follows.[18] Consider the elliptic operator

$$D:\Lambda_1{}^g \to \Lambda_4{}^{g\otimes g} \otimes \Lambda_0 \otimes \Lambda_0{}^g \otimes \Lambda_0{}^g \ ,$$
$$D:=(D_1, D_0{}^*) \ , \tag{6.1}$$

and its adjoint

$$D^*:\Lambda_4{}^{g\otimes g} \otimes \Lambda_0 \otimes \Lambda_0{}^g \otimes \Lambda_0{}^g \to \Lambda_1{}^g \ ,$$
$$D^*:=(D_1{}^*, D_0) \ , \tag{6.2}$$

along with the associated Laplacians

$$\Delta:\Lambda_1{}^g \to \Lambda_1{}^g \ ,$$
$$\Delta:=D^*D \ , \tag{6.3}$$

$$\Delta':\Lambda_4{}^{g\otimes g} \otimes \Lambda_0 \otimes \Lambda_0{}^g \otimes \Lambda_0{}^g \to \Lambda_4{}^{g\otimes g} \otimes \Lambda_0 \otimes \Lambda_0{}^g \otimes \Lambda_0{}^g \ ,$$
$$\Delta':=DD^* \ . \tag{6.4}$$

Physical perturbations, i.e., elements of $[C]$, are elements of the kernel of D as can be verified by comparison with (4.15). Furthermore, the orthogonal decompositions obtained in Sec. IV are equivalent to the orthogonal decomposition provided by the Fredholm alternative used in conjunction with the elliptic operators D and D^*:

$$\Lambda_1{}^g = \mathrm{range} D^* \oplus \mathrm{kernel} D \ , \tag{6.5a}$$

$$\Lambda_4{}^{g\otimes g} \otimes \Lambda_0 \otimes \Lambda_0{}^g \otimes \Lambda_0{}^g = \mathrm{range} D \oplus \mathrm{kernel} D^* \ . \tag{6.5b}$$

If we think of the operator D as the differential of the map that defines the submanifold **S** then, given (6.5a), we can use the infinite-dimensional version of the implicit function theorem to conclude that **S** is a smooth submanifold, i.e., without singularities, wherever D is a surjective map.[14] From (6.5b), this requirement is equivalent to the statement that the map D^* is injective; thus we can guarantee linearization stability at points of **S** where

$$\mathrm{kernel} D^* = \mathrm{kernel} \Delta' = 0 \ . \tag{6.6}$$

We have not yet been able to definitively characterize the conditions under which (6.6) holds, so the issue of linearization stability, which is relevant for the moduli space of instantons with $\lambda < 0$, must be settled in future work.

VII. THE INDEX OF INSTANTON PERTURBATIONS

The analysis of Sec. IV has shown that when $\lambda > 0$, the space of instantons is discrete; in order to uncover the possibilities which exist for $\lambda < 0$, we introduce the notion of the analytical index[14] of instanton perturbations. The analytical index of the complex described in Sec. IV or VI is a topological invariant defined to be the alternating sum of the dimension of the kernels of the various Laplacians:

$$I = \dim(\mathrm{kernel}\Delta_0) - \dim(\mathrm{kernel}\Delta_1) + \dim(\mathrm{kernel}\Delta_2)$$
$$= \dim(\mathrm{kernel}\Delta') - \dim(\mathrm{kernel}\Delta)$$
$$= \dim(\mathrm{kernel}D^*) - \dim(\mathrm{kernel}D) \ . \tag{7.1}$$

From (7.1) we see that at points of linearization stability the analytical index is equal to (minus) the dimension of the space of physical perturbations, which in turn is just the dimension of the space of solutions to the instanton equation (2.5) modulo gauge transformations, i.e., $-I$ is the number of gauge-invariant free parameters entering into an SU(2)-spin connection on a given conformally anti-self-dual Einstein space.[17] Actually, to be completely precise, the above statement is valid provided that (in the $\lambda < 0$ case) there are no harmonic one-forms. If there are solutions to (5.12) for $\lambda < 0$, then the dimension of **S** is $-I$ less the number of harmonic one-forms, so this correction can be implemented provided one knows the first Betti number of the manifold M.

The Atiyah-Singer index theorem[19] can be applied here to equate I to the topological index associated with the various bundles we are using; we therefore have a concrete way of computing the dimension of **S** when $\lambda < 0$. Because this method of obtaining the dimension of **S** requires linearization stability—an unresolved issue—we will present these results elsewhere.

VIII. DISCUSSION

One of the many intriguing features of the Ashtekar variables is the degree of similarity they permit between gravitation and gauge theory. While this similarity has been used extensively in the canonical formalism,[7] we have seen that it is also useful in the covariant approach to gravitational instantons. Indeed, it is hard to imagine a simpler linearized instanton equation than (4.16), which in fact is considerably simpler than its gauge theory counterpart.

Perhaps the most striking consequence of the approach to gravitational instantons in terms of the Ashtekar variables is the proof that the gravitational moduli space is discrete when $\lambda > 0$. This implies that the 3+1 self-duality ansatz of Ashtekar and Renteln (see, for example, Refs. 4 and 9), which allows for an infinite dimensional family of solutions to the instanton equation, can only be locally valid when the spacetime is compact. Of course, they realized this; their approach relied on the existence of a foliation of the manifold, which is topologically quite restrictive. The moduli space for self-dual instantons with a negative cosmological constant is evidently going to be more interesting than that which occurs when $\lambda > 0$. The interplay between topology and geometry in the $\lambda < 0$ case can be analyzed at the linearized level via (7.1), especially if one can establish linearization stability.

The outstanding question that remains then is whether (or under what conditions) the instanton equation is linearization stable when the cosmological constant is negative. This is a nontrivial question in general relativity: Moncrief has shown that on Lorentzian spacetimes with a compact Cauchy surface the Einstein equations are linearization stable only if the unperturbed spacetime does not possess symmetries.[15] It seems likely that a similar conclusion could be reached in the compact Euclidean case, but this does not really help us here: gravitational instantons, obtained via solutions to (2.5), correspond to taking a (finite-dimensional) "slice" in the space of solutions to the Euclidean Einstein equations. It remains to be seen whether this slice possesses singularities.

If the linearization stability issue can be brought under control, then the approach of Sec. VII allows us to compute the dimension of the gravitational moduli space (for $\lambda<0$), which, in the semiclassical approximation to the Euclidean functional integral, is the space of zero modes of the small fluctuation operator. An interesting extension of this work would be to compute the measure on the moduli space; clearly one needs to consider the ghost contributions here. In addition, our approach to studying the space of gravitational instantons could be useful in the semiclassical evaluation of the Hartle-Hawking wave functional in the Ashtekar ("self-dual") representation. Here, new issues arise because one must use an elliptic complex for manifolds with boundary. For example, are there global obstructions to the use of the "self-dual" representation, i.e., will some form of the Atiyah-Patodi-Singer boundary conditions[20] be needed here?

The ability to treat gravitation in 2+1 dimensions as a (Chern-Simons) gauge theory has allowed for a substantial increase in our understanding of the quantum mechanics of this system.[21] One of the key ingredients in the analysis of the 2+1 theory has been the possibility of allowing for a degenerate spacetime metric. It is tantalizing to suppose that similar advances could be made in the 3+1 theory, and this expectation is given support by the results of canonical quantum gravity in terms of the Ashtekar variables.[7] As far as instantons are concerned, however, while the instanton equation and its linearized counterpart are indeed well defined when the metric (or soldering form) is allowed to be degenerate, it seems very little can be said about the solutions to such equations in the degenerate case (mainly because the relevant operators are no longer elliptic), and for this reason we have always assumed the metric is nondegenerate. It is clearly of great interest to find out what, if anything, can be said in the degenerate case; it seems that new techniques will be required to analyze this issue.

Finally, we should point out that while our results are designed to be valid globally on M, they do require the existence of a spin structure. It is well known that there are manifolds upon which spinors simply cannot be defined (without the introduction of additional structure); therefore, the approach we have used to analyze the space of gravitational instantons cannot be applied in such cases.

Note added in proof. Results pertinent to the linearization stability of (2.5) are now available; see Ref. 22.

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APPENDIX A: Σ IDENTITIES

We list below a pair of identities, involving products of the two-forms Σ, that were used in the paper. They can be derived from the definition (2.3) and the definition of the inverse soldering form:

$$\gamma_{AA'}{}^{a}\gamma_{a}{}^{BB'}=\delta_{A}{}^{B}\delta_{A'}{}^{B'} , \tag{A1}$$

$$g^{bc}\Sigma_{ab}{}^{AB}\Sigma_{cd}{}^{CD}=\epsilon^{AC}\gamma_{a}{}^{BR'}\gamma_{dR'}{}^{D}+\epsilon^{AD}\gamma_{a}{}^{BR'}\gamma_{dR'}{}^{C}+\epsilon^{BD}\gamma_{a}{}^{AR'}\gamma_{dR'}{}^{C}+\epsilon^{BC}\gamma_{a}{}^{AR'}\gamma_{dR'}{}^{D} , \tag{A2}$$

$$\Sigma_{ab}{}^{AB}\Sigma_{AB}{}^{cd}=4(\delta_{a}{}^{[c}\delta_{b}{}^{d]}+\tfrac{1}{2}\epsilon_{ab}{}^{cd}) . \tag{A3}$$

APPENDIX B: DERIVATION OF (4.16)

The linearized instanton equation (3.1) is equivalent to

$$D_1{}^{*}D_1C=0 , \tag{B1}$$

which implies

$$D^{b}(F_{abAB}F^{cd(AB}D_cC_d{}^{CD)})=0 . \tag{B2}$$

From (4.15) we also have

$$\mathrm{tr}F^{ab}D_aC_b=0 , \tag{B3}$$

$$[F^{ab},D_aC_b]=0 , \tag{B4}$$

$$D^aC_a=0 . \tag{B5}$$

Equations (B3) and (B4) imply

$$F^{abAB}D_aC_b{}^{CD}=F^{ab(AB}D_aC_b{}^{CD)} , \tag{B6}$$

so we can remove the symmetrization in (B2). Now, using (A3), (B2) can be replaced by

$$D^{b}[(\delta_{a}{}^{[c}\delta_{b}{}^{d]}+\tfrac{1}{2}\epsilon_{ab}{}^{cd})D_cC_d]=0 . \tag{B7}$$

After expanding out (B7) one encounters terms involving a gauge-covariant Laplacian, the Ricci tensor, the Riemann tensor, the SU(2) curvature, and the gradient of a divergence; the latter three of the these can be eliminated by using the cyclic identity for the Riemann tensor, the self-duality of the SU(2) curvature, and (B5), respectively. One thus finds (4.15) to imply

$$-D^aD_aC_b+R_b{}^{a}C_a=0 , \tag{B8}$$

where $R_b{}^{a}$ is the Ricci tensor of the unperturbed metric. For an Einstein space,

$$R_a{}^{b}=\lambda\delta_a{}^{b} , \tag{B9}$$

and (B8) becomes (4.16).

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