

*Research Article*

## **Advanced Discrete Halanay-Type Inequalities: Stability of Difference Equations**

**Ravi P. Agarwal,<sup>1</sup> Young-Ho Kim,<sup>2</sup> and S. K. Sen<sup>1</sup>**

<sup>1</sup> *Department of Mathematical Sciences, Florida Institute of Technology,  
150 West University Boulevard, Melbourne, FL 32901-6975, USA*

<sup>2</sup> *Department of Applied Mathematics, Changwon National University,  
Changwon, Kyeong-Nam 641-773, South Korea*

Correspondence should be addressed to Ravi P. Agarwal, agarwal@fit.edu

Received 7 December 2008; Accepted 21 January 2009

Recommended by Martin J. Bohner

We derive new nonlinear discrete analogue of the continuous Halanay-type inequality. These inequalities can be used as basic tools in the study of the global asymptotic stability of the equilibrium of certain generalized difference equations.

Copyright © 2009 Ravi P. Agarwal et al. This is an open access article distributed under the Creative Commons Attribution License, which permits unrestricted use, distribution, and reproduction in any medium, provided the original work is properly cited.

### **1. Introduction**

The investigation of stability of nonlinear difference equations with delays has attracted a lot of attention from many researchers such as Agarwal et al. [1–3], Bařnov and Simeonov [4], Bay and Phat [5], Cooke and Ivanov [6], Gopalsamy [7], Liz et al. [8–10], Niamsup et al. [11, 12], Mohamad and Gopalsamy [13], Pinto and Trofimchuk [14], and references cited therein. In [15], Halanay proved an asymptotic formula for the solutions of a differential inequality involving the “maximum” functional and applied it in the stability theory of linear systems with delay. Such an inequality was called *Halanay inequality* in several works. Some generalizations as well as new applications can be found, for instance, in Agarwal et al. [2], Gopalsamy [7], Liz et al. [8–10], Niamsup et al. [11, 12], Mohamad and Gopalsamy [13], and Pinto and Trofimchuk [14]. In particular, in [2, 6, 10, 12, 13], the authors considered discrete Halanay-type inequalities to study some discrete version of functional differential equations.

In the following results of Liz et al. [10], authors showed that some discrete versions of these (maximum) inequalities can be applied to study the global asymptotic stability of a family of difference equations.

**Theorem A.** Assume that  $(u, v)$  satisfies the system of inequalities

$$\begin{aligned}\Delta u_n &\leq -Au_n + B\tilde{u}_n + Cv_n + D\hat{v}_n, & n \geq 0, \\ v_n &\leq Eu_n + F\tilde{u}_n, & n \geq 0,\end{aligned}\tag{1.1}$$

where  $\Delta u_n = u_{n+1} - u_n$ ,  $\tilde{u}_n = \max\{u_n, \dots, u_{n-r}\}$ ,  $\hat{v}_n = \max\{v_{n-1}, \dots, v_{n-r}\}$ , and  $r > 0$  is a natural number. If  $B, C, D, E, F \geq 0$ ,  $FD + B > 0$ ,  $E + F > 0$  and

$$B + (E + F)(C + D) < A \leq 1,\tag{1.2}$$

then there exist constants  $K_1 \geq 0$ ,  $K_2 \geq 0$ , and  $\lambda_0 \in (0, 1)$  such that

$$u_n \leq K_1 \lambda_0^n, \quad v_n \leq K_2 \lambda_0^n, \quad n \geq 0.\tag{1.3}$$

Moreover,  $\lambda_0$  can be chosen as the smallest root in the interval  $(0, 1)$  of equation  $h(\lambda) = 0$ , where

$$h(\lambda) = \lambda^{2r+1} - (1 - A + CE)\lambda^{2r} - (B + FC + ED)\lambda^r - FD.\tag{1.4}$$

By a simple use of Theorem A, authors also demonstrated the validity of the following statement, namely, Theorem B.

**Theorem B.** Assume that  $f$  satisfies the following inequalities:

$$\begin{aligned}|f(n, x_n, \dots, x_{n-r})| &\leq \|(x_n, \dots, x_{n-r})\|_\infty, & \forall (x_n, \dots, x_{n-r}) \in \mathbb{R}^{r+1}, \\ |f(n, x_n, \dots, x_{n-r}) - x_n| &\leq r \|\Delta x_{n-1}, \dots, \Delta x_{n-r}\|_\infty, & \forall (x_n, \dots, x_{n-r}) \in \mathbb{R}^{r+1}.\end{aligned}\tag{1.5}$$

If either

- (a)  $0 \leq a \leq 1 - b$ , and  $0 < br < 1$ , or
- (b)  $a < 0$ , and  $0 < br < (a + b)(-a + b)^{-1}$

holds, then there exist  $K > 0$  and  $\lambda_0 \in (0, 1)$  such that for every solution  $\{x_n\}$  of

$$\Delta x_n = -ax_n - bf(n, x_n, x_{n-1}, \dots, x_{n-r}), \quad a > 0,\tag{1.6}$$

one has

$$|x_n| \leq (\max\{|x_i|\}) \lambda_0^n, \quad n \geq 0,\tag{1.7}$$

where  $\lambda_0$  can be calculated in the form established in Theorem A. As a consequence, the trivial solution of (1.6) is globally asymptotically stable.

The main aim of the present paper is to establish some new nonlinear retarded Halanay-type inequalities, which extend Theorem A, along with the derivation of new global stability conditions for nonlinear difference equations.

## 2. Halanay-Type Discrete Inequalities

Let  $\mathbb{R}$  denote the set of all real numbers,  $\mathbb{R}^+$  the set of positive real numbers,  $\mathbb{R}^0$  the set of nonnegative real numbers,  $\mathbb{Z}$  the set of integers,  $\mathbb{Z}^+$  the set of positive integers, and  $\mathbb{Z}^{-r} = \{z \in \mathbb{Z} : z \geq -r\}$ . Consider the following nonlinear difference equation:

$$\Delta x_n = f(n, x_n, x_{n-1}, \dots, x_{n-r}), \quad n \in \mathbb{Z}^+, \quad (2.1)$$

where  $\Delta x_n = x_{n+1} - x_n$ , and  $f : \mathbb{N} \times \mathbb{R}^{r+1} \rightarrow \mathbb{R}$ . Equation (2.1) is a generalized difference equation (see [3, Section 21] and [11]). The initial value problem for this equation requires the knowledge of the initial data  $\{x_{-r}, x_{-r+1}, \dots, x_0\}$ . This vector is called the initial string in [6]. For every initial string, there exists a unique solution  $\{x_n\}_{n \geq \mathbb{Z}^{-r}}$  of (2.1) that can be calculated using the explicit recurrence formula

$$x_{n+1} = x_n + f(n, x_n, x_{n-1}, \dots, x_{n-r}), \quad n \in \mathbb{Z}^0. \quad (2.2)$$

In this section, we introduce new discrete inequalities which will be used to derive global stability conditions in the next section.

**Theorem 2.1.** *Let  $a_i, b_i, c_i, d_i, e_i, f_i \in \mathbb{R}_0^+$ ,  $\sum_{i=0}^r d_i e_i > 0$ ,  $h_i \in \mathbb{Z}^0$  ( $i = 0, \dots, r$ ),  $0 = h_0 < h_1 < \dots < h_r$ ;  $h_r \in \mathbb{Z}^+$ , and*

$$b + (c + d)(e + f) < a \leq 1, \quad (2.3)$$

where  $a = \sum_{i=0}^r a_i$ ,  $b = \sum_{i=0}^r b_i$ ,  $c = \sum_{i=0}^r c_i$ ,  $d = \sum_{i=0}^r d_i$ ,  $e = \sum_{i=0}^r e_i$ , and  $f = \sum_{i=0}^r f_i$ . Also, let  $\{u_n, v_n\}_{n \in \mathbb{Z}^{-h_r}}$  be a sequence of nonnegative real numbers satisfying the system of inequalities

$$\begin{aligned} \Delta u_n &\leq \sum_{i=0}^r (-a_i u_n + b_i u_{n-h_i}^p + c_i v_n + d_i v_{n-h_i}), \quad n \in \mathbb{Z}^0, \\ v_n &\leq \sum_{i=0}^r (e_i u_n + f_i u_{n-h_i}^p), \quad n \in \mathbb{Z}^0, \end{aligned} \quad (2.4)$$

where  $p \geq 0$  is a constant. Then there exist constants  $K_1 \geq 0$ ,  $K_2 \geq 0$ , and  $\lambda_0 \in (0, 1)$  such that

$$u_n \leq K_1 \lambda_0^n, \quad v_n \leq K_2 \lambda_0^n, \quad n \in \mathbb{Z}^0, \quad (2.5)$$

where  $K_1 = \max_{0 \leq i \leq r} \{u_{-h_i}, \alpha^{-1} v_{-h_i}\}$ , and  $K_2 = \alpha K_1$  with  $\alpha = e + \sum_{i=0}^r f_i \lambda_0^{-n+(n-h_i)p}$ . Moreover,  $\lambda_0$  can be chosen as the smallest root in the interval  $(0, 1)$  of equation  $g(\lambda) = 0$ , where

$$\begin{aligned} g(\lambda) &= \lambda - (1 - a + ce) - \sum_{i=0}^r (b_i + c f_i) \lambda^{(n-h_i)p-n} \\ &\quad - \sum_{i=0}^r d_i e \lambda^{-h_i} - \sum_{i=0}^r \left( d_i \sum_{j=0}^r f_j \lambda^{(n-h_j-h_i)p-n} \right) \end{aligned} \quad (2.6)$$

with  $n \in \mathbb{Z}^0$ .

*Proof.* Let  $\{x_n, y_n\}_{n \in \mathbb{Z}^{-hr}}$  be a sequence of nonnegative real numbers satisfying the system of inequalities

$$\begin{aligned} x_n &= (1-a)^n x_0 + \sum_{j=0}^{n-1} (1-a)^{n-j-1} \\ &\quad \times \sum_{i=0}^r (-a_i x_j + b_i x_{j-h_i}^p + c_i y_j + d_i y_{j-h_i}), \\ y_n &= \sum_{i=0}^r (e_i x_n + f_i x_{n-h_i}^p), \end{aligned} \quad (2.7)$$

where  $n \in \mathbb{Z}^0$ . Since  $(1-a) \geq 0$ , it is easy to prove by induction that if  $u_n \leq x_n$  and  $v_n \leq y_n$  for  $n = -h_r, \dots, 0$ , then  $u_n \leq x_n$  and  $v_n \leq y_n$  for all  $n \in \mathbb{Z}^0$ .

On the other hand, the system (2.7) is equivalent to

$$\begin{aligned} \Delta x_n &= \sum_{i=0}^r (-a_i x_n + b_i x_{n-h_i}^p + c_i y_n + d_i y_{n-h_i}), \\ y_n &= \sum_{i=0}^r (e_i x_n + f_i x_{n-h_i}^p), \end{aligned} \quad (2.8)$$

where  $n \in \mathbb{Z}^0$ . Next we prove, under the assumptions of the theorem, that there exists a solution  $\{x_n, y_n\}_{n \in \mathbb{Z}^{-hr}}$  to system (2.8) in the form  $x_n = \lambda_0^n$ ,  $y_n = \alpha \lambda_0^n$  with  $\alpha > 0$ ,  $\lambda_0 \in (0, 1)$ . Indeed, such  $\{x_n, y_n\}_{n \in \mathbb{Z}^{-hr}}$  is a solution of (2.8) if and only if

$$\begin{aligned} \lambda_0^{n+1} &= (1-a)\lambda_0^n + \sum_{i=0}^r (b_i \lambda_0^{(n-h_i)p} + c_i \alpha \lambda_0^n + d_i (\alpha \lambda_0^{n-h_i})), \quad n \in \mathbb{Z}^0, \\ \alpha \lambda_0^n &= \sum_{i=0}^r (e_i \lambda_0^n + f_i \lambda_0^{(n-h_i)p}), \quad n \in \mathbb{Z}^0. \end{aligned} \quad (2.9)$$

This is equivalent to the existence of a solution  $\lambda_0 \in (0, 1)$  of equation  $g(\lambda) = 0$ , where  $g$  is the polynomial defined by (2.6).

Now,  $g(0) = \lim_{\lambda \rightarrow 0^+} g(\lambda) = -\infty < 0$  in view of  $\sum_{i=0}^r d_i e > 0$ . On the other hand,  $g(1) = a - b - (c + d)(e + f) > 0$  in view of (2.3). As a consequence, there exists  $\lambda_0 \in (0, 1)$  such that  $g(\lambda_0) = 0$ . Hence,  $(\lambda_0, \alpha)$  is a solution of (2.9) with  $\alpha = e + \sum_{i=0}^r f_i \lambda_0^{-n+(n-h_i)p} > 0$ .

For this value of  $\lambda_0$ , the pair  $\{K\lambda_0^n, K\alpha\lambda_0^n\}$  is a solution of (2.8) for every  $K \geq 0$ . Thus, choosing  $K = \max_{0 \leq i \leq r} \{u_{-h_i}, \alpha^{-1}v_{-h_i}\}$ , we have that  $u_n \leq K\lambda_0^n = x_n$ , and  $v_n \leq K\alpha\lambda_0^n = y_n$  for all  $n = -h_r, \dots, 0$ .

Hence, using the first part of the proof, we can conclude that  $u_n \leq x_n$ , and  $v_n \leq y_n$  for all  $n \in \mathbb{Z}_0$ .  $\square$

By the similar argument used in Theorem 2.1, we obtain the following result.

**Theorem 2.2.** Let  $a, b, c, d, e, f \in \mathbb{R}_0^+$ ,  $h_i \in \mathbb{Z}^0$ ,  $i = 0, \dots, r$ ,  $0 = h_0 < h_1 < \dots < h_r$ ;  $r \geq 1$ , and

$$b + c(e + f) + d(e + f)^{r+1} < a \leq 1 \quad (2.10)$$

with  $ce > 0$ . Also, let  $\{u_n, v_n\}_{n \in \mathbb{Z}^{-hr}}$  be a sequence of nonnegative real numbers satisfying the system of inequalities

$$\begin{aligned} \Delta u_n &\leq -au_n + \prod_{i=0}^r bu_{n-h_i} + cv_n + \prod_{i=0}^r dv_{n-h_i}, \quad n \in \mathbb{Z}^0, \\ v_n &\leq eu_n + \prod_{i=0}^r fu_{n-h_i}, \quad n \in \mathbb{Z}^0. \end{aligned} \quad (2.11)$$

Then there exist constants  $K_1 \geq 0$ ,  $K_2 \geq 0$ , and  $\lambda_0 \in (0, 1)$  such that

$$u_n \leq K_1 \lambda_0^n, \quad v_n \leq K_2 \lambda_0^n, \quad n \in \mathbb{Z}^0, \quad (2.12)$$

where  $K_1 = \max_{0 \leq i \leq r} \{u_{-h_i}, \rho^{-1}v_{-h_i}\}$ , and  $K_2 = \rho K_1$  with  $\rho = e + f \prod_{i=0}^r \lambda_0^{-h_i}$ . Moreover,  $\lambda_0$  can be chosen as the smallest root in the interval  $(0, 1)$  of equation  $F(\lambda) = 0$ , where

$$F(\lambda) = \lambda - (1 - a + ce) - [b + cf + d(e + f\lambda^{r-n-h})^{r+1}] \lambda^{r-n-h} \quad (2.13)$$

with  $n \in \mathbb{Z}^0$ ,  $h = \sum_{i=0}^r h_i$ .

*Proof.* Let  $\{x_n, y_n\}_{n \in \mathbb{Z}^{-hr}}$  be a sequence of nonnegative real numbers satisfying the system of inequalities

$$\begin{aligned} \Delta x_n &= -ax_n + b \prod_{i=0}^r x_{n-h_i} + cy_n + d \prod_{i=0}^r y_{n-h_i}, \quad n \in \mathbb{Z}^0, \\ y_n &= ex_n + f \prod_{i=0}^r x_{n-h_i}, \quad n \in \mathbb{Z}^0. \end{aligned} \quad (2.14)$$

Since  $(1 - a) \geq 0$ , it is easy to prove by induction that if  $u_n \leq x_n$  and  $v_n \leq y_n$  for  $n = -h_r, \dots, 0$ , then  $u_n \leq x_n$  and  $v_n \leq y_n$  for all  $n \in \mathbb{Z}^0$ .

Next we prove that, under the assumptions of the theorem, there exists a solution  $\{x_n, y_n\}_{n \in \mathbb{Z}^{-hr}}$  to system (2.14) in the form  $x_n = \lambda^n$ ,  $y_n = \rho \lambda^n$  with  $\rho > 0$ ,  $\lambda \in (0, 1)$ . Indeed, such  $\{x_n, y_n\}_{n \in \mathbb{Z}^{-hr}}$  is a solution of (2.14) if and only if

$$\begin{aligned} \lambda^{n+1} &= (1 - a)\lambda^n + b \prod_{i=0}^r \lambda^{n-h_i} + c\rho \lambda^n + d \prod_{i=0}^r \rho \lambda^{n-h_i}, \quad n \in \mathbb{Z}^0, \\ \rho \lambda^n &= e\lambda^n + f \prod_{i=0}^r \lambda^{n-h_i}, \quad n \in \mathbb{Z}^0. \end{aligned} \quad (2.15)$$

This is equivalent to the existence of a solution  $\lambda \in (0, 1)$  of equation  $F(\lambda) = 0$ , where  $F$  is the polynomial defined in (2.13).

Now, in view of  $ce > 0$ , we have  $F(0) = -1 + a - ac < 0$  in case  $rn > h$ ,  $F(0) = -1 + a - ac - [b + cf + d(e + f)^{r+1}] < 0$  in case  $rn = h$ , and  $F(0) = \lim_{\lambda \rightarrow 0^+} F(\lambda) = -\infty < 0$  in case  $rn < h$ .

On the other hand,  $F(1) = a - b - c(e + f) - d(e + f)^{r+1} > 0$  in view of (2.10). As a consequence, there exists  $\lambda_0 \in (0, 1)$  such that  $F(\lambda_0) = 0$ . Hence,  $(\lambda_0, \rho)$  is a solution of (2.15) with  $\rho = e + f \prod_{i=0}^r \lambda_0^{-h_i} > 0$ .

For this value of  $\lambda_0$ , the pair  $\{K\lambda_0^n, K\rho\lambda_0^n\}$  is a solution of (2.14) for every  $K \geq 0$ . Thus, choosing  $K = \max_{0 \leq i \leq r} \{u_{-h_i}, \rho^{-1}v_{-h_i}\}$ , we have  $u_n \leq K\lambda_0^n$ , and  $v_n \leq K\rho\lambda_0^n$  for all  $n = -h_r, \dots, 0$ . These imply  $u_n \leq x_n$ , and  $v_n \leq y_n$  for all  $n = -h_r, \dots, 0$ . Hence, using the first part of the proof, we can conclude that  $u_n \leq x_n$ , and  $v_n \leq y_n$  for all  $n \in \mathbb{Z}_0$ .  $\square$

*Remark 2.3.* In [10], a discrete Halanay-type inequality was given as in Theorem A, where the inequalities were replaced by

$$\begin{aligned} \Delta u_n &\leq -au_n + b\tilde{u}_n + cv_n + d\hat{v}_n, \quad n \geq 0, \\ v_n &\leq eu_n + f\tilde{u}_n, \quad n \geq 0, \end{aligned} \quad (2.16)$$

where  $\Delta u_n = u_{n+1} - u_n$ ,  $\tilde{u}_n = \max\{u_n, \dots, u_{n-r}\}$ ,  $\hat{v}_n = \max\{v_{n-1}, \dots, v_{n-r}\}$ , and  $r \geq 1$  is a natural number. Note that if a sequence  $\{u_n\}_{n \in \mathbb{Z}^-}$  of positive real numbers satisfies (2.16), then it also satisfies (2.4). On the other hand, let  $p = r = 1$ ,  $h_i = i$ ;  $a = \sum_{i=0}^1 a_i = 1$ ,  $b = b_0 = b_1 = 1/7$ ,  $c = c_0 = c_1 = 0$ ,  $d = d_0 = d_1 = 1/7$ ,  $e = e_0 = e_1 = 1/7$ , and  $f = f_0 = f_1 = 1/7$ . Then we might easily show that the sequence  $\{1/2^n\}_{n \in \mathbb{Z}^-}$  satisfies (2.4) but not (2.16). Indeed,

$$\begin{aligned} \Delta u_n &= \frac{1}{2^{n+1}} - \frac{1}{2^n} \\ &= -\frac{1}{2^{n+1}}, \\ &< -\frac{1}{2^n} + \frac{1}{7} \left( \frac{1}{2^n} + \frac{1}{2^{n-1}} \right) + \frac{1}{7} \left( \frac{1}{2^n} \frac{5}{7} + \frac{1}{2^{n-1}} \frac{5}{7} \right) \\ &= -\frac{13}{49} \frac{1}{2^n}, \end{aligned} \quad (2.17)$$

with  $\sum_{i=0}^1 b_i + (\sum_{i=0}^1 c_i + \sum_{i=0}^1 d_i)(\sum_{i=0}^1 e_i + \sum_{i=0}^1 f_i) < \sum_{i=0}^1 a_i = 1$ . On the other hand,

$$\begin{aligned} \Delta u_n &= -\frac{1}{2^{n+1}} \\ &> -\frac{1}{2^n} + \frac{1}{7} \max \left\{ \frac{1}{2^n}, \frac{1}{2^{n-1}} \right\} + \frac{1}{7} \left( \frac{1}{7} \frac{1}{2^{n-1}} + \frac{1}{7} \max \left\{ \frac{1}{2^{n-1}} \right\} \right) \\ &= -\frac{31}{49} \frac{1}{2^n}. \end{aligned} \quad (2.18)$$

Therefore, in the case of positive sequences, the discrete inequality (2.4) is less conservative than the discrete Halanay-type inequality given by (2.16).

### 3. Global Stability of Difference Equations

In order to show the applicability of the previous result, in this section we consider the generalized difference equation

$$\Delta x_n = -ax_n - bf(n, x_n, x_{n-h_1}, \dots, x_{n-h_r}), \quad (3.1)$$

where  $n, h_i \in \mathbb{Z}^+$ ,  $i = 1, \dots, r$ , and  $b > 0$ .

Although, for every initial string  $\{x_{-h_r}, x_{-h_r+1}, \dots, x_0\}$ , the solution  $\{x_n\}$  of (3.1) can be explicitly calculated by a recurrence formula similar to (2.2), it is in general difficult to investigate the asymptotic behavior of the solutions using that formula. The next result gives an asymptotic estimate by a simple use of the discrete Halanay inequality.

**Theorem 3.1.** For all  $(n, x_n, x_{n-h_1}, \dots, x_{n-h_r}) \in \mathbb{Z}^0 \times \mathbb{R}^{r+1}$ , assume that  $f$  satisfies the following inequalities:

$$|f(n, x_n, x_{n-h_1}, \dots, x_{n-h_r})| \leq \sum_{j=0}^r \beta_j |x_{n-h_j}|^p, \quad (3.2)$$

$$|f(n, x_n, x_{n-h_1}, \dots, x_{n-h_r}) - x_n| \leq \sum_{j=0}^r \gamma_j |\Delta x_{n-h_j}|, \quad (3.3)$$

where  $\beta_j, \gamma_j, p \in \mathbb{R}_0^+$ ,  $\sum_{i=0}^r \gamma_i |a| > 0$ ,  $h_j \in \mathbb{Z}^0$  ( $j = 0, \dots, r-1$ ), and  $h_r \in \mathbb{Z}^+$  with  $0 = h_0 < h_1 < \dots < h_r$ . If either

$$(a) \ 0 \leq a \leq 1 - b, \ 0 < b\gamma < 1, \text{ and } 0 < \beta \leq 1, \text{ or}$$

$$(b) \ a < 0 \text{ and } 0 < b\gamma < (a + b)(-a + b\beta)^{-1}$$

hold, then there exists a constant  $\lambda_0 \in (0, 1)$  for every solution  $\{x_n\}$  of (3.1) such that

$$|x_n| \leq \left( \max_{-h_r \leq i \leq 0} \{|x_i|, \alpha_1^{-1} |\Delta x_i|\} \right) \lambda_0^n, \quad n \in \mathbb{Z}^0, \quad (3.4)$$

where  $\alpha_1 = |a| + b \sum_{i=0}^r \beta_i \lambda_0^{-n+(n-h_i)p}$ ,  $\beta = \sum_{i=0}^r \beta_i$ ,  $\gamma = \sum_{i=0}^r \gamma_i$ , and  $\lambda_0$  can be chosen as the smallest root in the interval  $(0, 1)$  of equation  $g_1(\lambda) = 0$ , where

$$g_1(\lambda) = \lambda - (1 - a - b) - \sum_{i=0}^r b|a|\gamma_i \lambda^{-h_i} - \sum_{i=0}^r b\gamma_i \left( \sum_{j=0}^r b\beta_j \lambda^{(n-h_j-h_i)p-n} \right) \quad (3.5)$$

with  $n \in \mathbb{Z}^0$ .

As a consequence, the trivial solution of (3.1) is globally asymptotically stable.

*Proof.* Let  $\{x_n\}$  be a solution of (3.1). Equation (3.1) can be written in the form

$$\Delta x_n = -(a + b)x_n - b[f(n, x_n, x_{n-h_1}, \dots, x_{n-h_r}) - x_n]. \quad (3.6)$$

Hence, we know that

$$x_n = [1 - (a + b)]^n x_0 + \sum_{i=0}^{n-1} [1 - (a + b)]^{n-i-1} (-b) [f(i, x_i, x_{i-h_1}, \dots, x_{i-h_r}) - x_i], \quad (3.7)$$

where  $n \in \mathbb{Z}^0$ . Thus, using inequality (3.3), we obtain

$$|x_n| \leq [1 - (a + b)]^n |x_0| + \sum_{i=0}^{n-1} \sum_{j=0}^r [1 - (a + b)]^{n-i-1} b \gamma_j |\Delta x_{i-h_j}|. \quad (3.8)$$

Denote  $u_n = |x_n|$  for  $n = -h_r, \dots, 0$ , and

$$u_n = [1 - (a + b)]^n |x_0| + \sum_{i=0}^{n-1} \sum_{j=0}^r [1 - (a + b)]^{n-i-1} b \gamma_j |\Delta x_{i-h_j}| \quad (3.9)$$

for  $n \in \mathbb{Z}^+$ . Then we have  $|x_n| \leq u_n$  and, from inequality (3.9), we obtain

$$\Delta u_n = -(a + b)u_n + \sum_{j=0}^r b \gamma_j |\Delta x_{n-h_j}| \quad (3.10)$$

for  $n \in \mathbb{Z}^+$ . On the other hand, using hypothesis (3.2) in (3.1), we have

$$\begin{aligned} |\Delta x_n| &\leq | -a | |x_n| + b \sum_{j=0}^r \beta_j |x_{n-h_j}|^p \\ &\leq |a| u_n + b \sum_{j=0}^r \beta_j u_{n-h_j}^p. \end{aligned} \quad (3.11)$$

Denote  $v_n = |\Delta x_n|$ . We can apply Theorem 2.1 to the system of inequalities (3.10) and (3.11) with  $\sum_{i=0}^r a_i = a + b$ ,  $b_i = 0$ ,  $c_i = 0$ ,  $d_i = b \gamma_j$ ,  $\sum_{i=0}^r e_i = |a|$ , and  $f_i = b \beta_j$ . Consequently, Theorem 2.1 ensures the validity of the following inequality:

$$|x_n| \leq \left( \max_{-h_r \leq i \leq 0} \{ |x_i|, \alpha_1^{-1} |\Delta x_i| \} \right) \lambda_0^n, \quad n \in \mathbb{Z}^0, \quad (3.12)$$

where  $\lambda_0$  and  $\alpha_1$  are chosen as in Theorem 3.1. This completes the proof of the theorem.  $\square$

Next, we obtain new conditions for the asymptotic stability of (3.1) using inequality (3.13) instead of (3.3).

**Corollary 3.2.** For all  $(n, x_n, x_{n-h_1}, \dots, x_{n-h_r}) \in \mathbb{Z}^0 \times \mathbb{R}^{r+1}$ , assume that  $f$  satisfies inequality (3.2) and the following condition:

$$|f(n, x_n, x_{n-h_1}, \dots, x_{n-h_r}) - x_n| \leq \sum_{j=0}^r (\gamma_j |x_{n-h_j}|^p + \delta_j |\Delta x_n| + \eta_j |\Delta x_{n-h_j}|), \tag{3.13}$$

where  $\gamma_j, \delta_j, \eta_j, p \in \mathbb{R}_0^+$ ,  $\sum_{j=0}^r \eta_j |a| > 0$ ,  $h_j \in \mathbb{Z}^0$  ( $j = 0, \dots, r-1$ ), and  $h_r \in \mathbb{Z}^+$  with  $0 = h_0 < h_1 < \dots < h_r$ . If

$$b\gamma + b(\delta + \eta)(|a| + b\beta) < a + b \leq 1 \tag{3.14}$$

holds, where  $\beta = \sum_{i=0}^r \beta_i$ ,  $\gamma = \sum_{i=0}^r \gamma_i$ ,  $\delta = \sum_{i=0}^r \delta_i$ , and  $\eta = \sum_{i=0}^r \eta_i$ , then there exists a constant  $\lambda_0 \in (0, 1)$  for every solution  $\{x_n\}$  of (3.1) such that

$$|x_n| \leq \left( \max_{-h_r \leq i \leq 0} \{|x_i|, \alpha_1^{-1} |\Delta x_i|\} \right) \lambda_0^n, \quad n \in \mathbb{Z}^0, \tag{3.15}$$

where  $\alpha_1 = |a| + b \sum_{i=0}^r \beta_i \lambda_0^{-n+(n-h_i)p}$ , and  $\lambda_0$  can be chosen as the smallest root in the interval  $(0, 1)$  of equation  $g_2(\lambda) = 0$ , where

$$g_2(\lambda) = \lambda - (1 - a - b + b|a|\delta) - \sum_{i=0}^r b(\gamma_i + b\delta\beta_i)\lambda^{(n-h_i)p-n} - \sum_{i=0}^r b|a|\eta_i\lambda^{-h_i} - \sum_{i=0}^r b\eta_i \left( \sum_{j=0}^r b\beta_j \lambda^{(n-h_j-h_i)p-n} \right) \tag{3.16}$$

with  $n \in \mathbb{Z}^0$ .

As a consequence, the trivial solution of (3.1) is globally asymptotically stable.

Similarly, using Theorem 2.2 instead of Theorem 2.1, we obtain the following result.

**Theorem 3.3.** For all  $(n, x_n, x_{n-h_1}, \dots, x_{n-h_r}) \in \mathbb{Z}^0 \times \mathbb{R}^{r+1}$ , assume that  $f$  satisfies the following inequalities:

$$|f(n, x_n, x_{n-h_1}, \dots, x_{n-h_r})| \leq \beta \prod_{j=0}^r |x_{n-h_j}|, \tag{3.17}$$

$$|f(n, x_n, x_{n-h_1}, \dots, x_{n-h_r}) - x_n| \leq \gamma \prod_{j=0}^r |x_{n-h_j}| + \delta |\Delta x_n| + \eta \prod_{j=0}^r |\Delta x_{n-h_j}|,$$

where  $\beta, \gamma, \delta, \eta \in \mathbb{R}_0^+$ ,  $h_j \in \mathbb{Z}^0$ ,  $j = 0, \dots, r-1$ , and  $h_r \in \mathbb{Z}^+$  with  $0 = h_0 < h_1 < \dots < h_r$ . If  $|a|\delta > 0$  and

$$b\gamma + b\delta(|a| + b\beta) + b\eta(|a| + b\beta)^{r+1} < a + b \leq 1, \tag{3.18}$$

then there exists a constant  $\lambda_0 \in (0, 1)$  for every solution  $\{x_n\}$  of (3.1) such that

$$|x_n| \leq \left( \max_{-h_r \leq i \leq 0} \{|x_i|, \rho_1^{-1} |\Delta x_i|\} \right) \lambda_0^n, \quad n \in \mathbb{Z}^0, \quad (3.19)$$

where  $\rho_1 = |a| + b\beta \prod_{i=0}^r \lambda_0^{h_i}$ , and  $\lambda_0$  can be chosen as the smallest root in the interval  $(0, 1)$  of equation  $F_1(\lambda) = 0$ , where

$$F_1(\lambda) = \lambda - (1 - (a + b) + |a|b\delta) - b \left[ \gamma + b\beta\delta + \eta(|a| + b\beta\lambda^{r-n-h})^{r+1} \right] \lambda^{r-n-h} \quad (3.20)$$

with  $n \in \mathbb{Z}^0$ ,  $h = \sum_{i=0}^r h_i$ .

As a consequence, the trivial solution of (3.1) is globally asymptotically stable.

**Remark 3.4.** Equation (3.1) covers a variety of difference equations. For instance, we can consider the following difference equation:

$$\Delta x_n = -ax_n - bf(x_{n-k}), \quad b > 0. \quad (3.21)$$

Next, we study the asymptotic behavior of the solutions of (3.21). We can apply Theorem 3.1, Corollary 3.2, or Theorem 3.3 to obtain some relations between coefficients  $a$  and  $b$  that ensure the global asymptotic stability of the zero solution. Moreover, from Theorem 3.1 we know that if there exists  $\beta, \gamma \in \mathbb{R}^+$  such that  $|f(x)| \leq \beta|x|^p$ ,  $|f(x) - x| \leq \gamma|\Delta x|$  for all  $x$ , and if either

- (a)  $0 < a \leq 1 - b$ ,  $0 < b\gamma < 1$ , and  $0 < \beta \leq 1$ , or
- (b)  $a < 0$  and  $0 < b\gamma < (a + b)(-a + b\beta)^{-1}$

hold, then all solutions of (3.21) converge to zero.

## Acknowledgment

The authors thank the referees of this paper for their careful and insightful critique.

## References

- [1] R. P. Agarwal, S. Deng, and W. Zhang, "Generalization of a retarded Gronwall-like inequality and its applications," *Applied Mathematics and Computation*, vol. 165, no. 3, pp. 599–612, 2005.
- [2] R. P. Agarwal, Y.-H. Kim, and S. K. Sen, "New discrete Halanay inequalities: stability of difference equations," *Communications in Applied Analysis*, vol. 12, no. 1, pp. 83–90, 2008.
- [3] R. P. Agarwal and P. J. Y. Wong, *Advanced Topics in Difference Equations*, vol. 404 of *Mathematics and Its Applications*, Kluwer Academic Publishers, Dordrecht, The Netherlands, 1997.
- [4] D. Bařnov and P. Simeonov, *Integral Inequalities and Applications*, vol. 57 of *Mathematics and Its Applications*, Kluwer Academic Publishers, Dordrecht, The Netherlands, 1992.
- [5] N. S. Bay and V. N. Phat, "Stability analysis of nonlinear retarded difference equations in Banach spaces," *Computers & Mathematics with Applications*, vol. 45, no. 6–9, pp. 951–960, 2003.
- [6] K. L. Cooke and A. F. Ivanov, "On the discretization of a delay differential equation," *Journal of Difference Equations and Applications*, vol. 6, no. 1, pp. 105–119, 2000.
- [7] K. Gopalsamy, *Stability and Oscillations in Delay Differential Equations of Population Dynamics*, vol. 74 of *Mathematics and Its Applications*, Kluwer Academic Publishers, Dordrecht, The Netherlands, 1992.

- [8] E. Liz and S. Trofimchuk, "Existence and stability of almost periodic solutions for quasilinear delay systems and the Halanay inequality," *Journal of Mathematical Analysis and Applications*, vol. 248, no. 2, pp. 625–644, 2000.
- [9] E. Liz and J. B. Ferreira, "A note on the global stability of generalized difference equations," *Applied Mathematics Letters*, vol. 15, no. 6, pp. 655–659, 2002.
- [10] E. Liz, A. F. Ivanov, and J. B. Ferreira, "Discrete Halanay-type inequalities and applications," *Nonlinear Analysis: Theory, Methods & Applications*, vol. 55, no. 6, pp. 669–678, 2003.
- [11] P. Niamsup and V. N. Phat, "Asymptotic stability of nonlinear control systems described by difference equations with multiple delays," *Electronic Journal of Differential Equations*, vol. 2000, no. 11, pp. 1–17, 2000.
- [12] S. Udpin and P. Niamsup, "New discrete type inequalities and global stability of nonlinear difference equations," to appear in *Applied Mathematics Letters*.
- [13] S. Mohamad and K. Gopalsamy, "Continuous and discrete Halanay-type inequalities," *Bulletin of the Australian Mathematical Society*, vol. 61, no. 3, pp. 371–385, 2000.
- [14] M. Pinto and S. Trofimchuk, "Stability and existence of multiple periodic solutions for a quasilinear differential equation with maxima," *Proceedings of the Royal Society of Edinburgh. Section A*, vol. 130, no. 5, pp. 1103–1118, 2000.
- [15] A. Halanay, *Differential Equations: Stability, Oscillations, Time Lags*, Academic Press, New York, NY, USA, 1966.

Copyright of Journal of Inequalities & Applications is the property of Hindawi Publishing Corporation and its content may not be copied or emailed to multiple sites or posted to a listserv without the copyright holder's express written permission. However, users may print, download, or email articles for individual use.