# Investigations on the development of a mixed displacement-pressure formulation for an anelastic displacement-field finite element

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## ABSTRACT

Space and weapon delivery systems contain guidance components and payload that need to be protected from the extremely harsh acoustic excitation present during launch operations. The above example represents just one application where high-damping viscoelastic materials are used in the design of shock and vibration isolation components. The shock transients generally encountered are characterized by a broad frequency spectrum. Widely available finite element codes do not offer the proper tools to model the frequency-dependent mechanical properties of viscoelastic materials over the frequency domain of interest. An added difficulty is the large Poissson's ratio exhibited by some of these materials, which indicates that previously developed displacement–based finite element formulations should be complemented with mixed pressure-displacement finite element formulations. A pure displacement-based finite element generally predicts the displacements well, if the mesh used is fine enough, but the same thing may not be said about the values of the predicted stresses. The Anelastic Displacement Fields (ADF) method is employed herein to model frequency-dependence of material properties within a time-domain finite element framework and using a mixed displacement-pressure finite element formulation. Finite elements based on this new formulation are developed.

**Keywords:** Damping modeling, ADF, finite elements, mixed formulation

## 1. INTRODUCTION

The finite element formulation is pursued here for a single ADF-, and for a two-dimensional, four-node, plane-strain fnite element. The rationale for this theoretical development is that pure displacement-based finite elements do not predict stresses well for structures made of nearly incompressible materials. The formulation used in this paper is for an almost incompressible case, when the bulk modulus $\beta_K$ is large yet finite. The Poisson's ratio is very close to 0.5, yet it does not reach that value. The pressure formulation considers that the pressure is a variable internal to the

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element only and no continuous pressure distribution is enforced. This is a first approximation of the problem and in subsequent models; a mixed u/p c element will be developed. The extra variables considered in this u/p formulation will be the pressure developed as a result of the total deformation, p, and the pressure caused by the anelastic field $p^A$.

## 2. FINITE ELEMENT DEVELOPMENT

In the plane-strain problem, the following relations are true:

$$\begin{aligned} &\sigma_1, \sigma_2, \sigma_3, \tau_{12}\left(=\sigma_6\right) \neq 0 \\ &\varepsilon_3 = 0 \end{aligned} \tag{1}$$

### 2.1 ADF background

The basis for the ADF method is the assumption that, for a linear viscoelastic material, the total displacement is the sum of the elastic displacement $u^E$ and of the anelastic displacement $u^A$.[1,2,3].

$$u(x,t) = u^E(x,t) + u^A(x,t) \tag{2}$$

To these elastic and anelastic quantities, we'll associate the corresponding strains ε and $\varepsilon^A$. From thermodynamic considerations and for small deformations, the strain energy density function is a positive definite function of the strains. The quadratic form of the free energy function is

$$F = \frac{1}{2}\left(E_{ijkl}\varepsilon_{ij}\varepsilon_{kl} - 2E_{ijkl}\varepsilon_{ij}\varepsilon_{kl}^A + E_{ijkl}^A\varepsilon_{kl}^A\varepsilon_{ij}^A\right) \tag{3}$$

The equation above, in matrix form is:

$$F = \frac{1}{2}\begin{Bmatrix} \{\varepsilon\} \\ \{\varepsilon^A\} \end{Bmatrix}^T \begin{bmatrix} [E] & [-E] \\ [-E] & [E^A] \end{bmatrix} \begin{Bmatrix} \{\varepsilon\} \\ \{\varepsilon^A\} \end{Bmatrix} \tag{4}$$

$[E^A]$ is a matrix made of anelastic material constants.
For isothermal reversible processes, the strain energy density function equals the Helmholtz function. Both anelastic and total stress must satisfy the following conditions respective to the Helmholtz function [1,2,3]:

$$\begin{aligned} &\sigma = \frac{\partial F}{\partial \varepsilon}; \\ &\sigma^A = -\frac{\partial F}{\partial \varepsilon^A} \end{aligned} \tag{5}$$

where $[E^A]$ is a matrix made of anelastic material constants. The elastic stresses are, in tensor form:

$$\sigma_{ij} = E_{ijkl}\varepsilon_{kl}^E = E_{ijkl}\varepsilon_{kl} - E_{ijkl}\varepsilon_{kl}^A \tag{6}$$

In general, the anelastic stress components, in tensor form and, respectively, matrix form are expressed as a function of the total and anelastic strains $\varepsilon$, $\varepsilon^A$:

$$\sigma_{ij}^A = E_{ijkl}\varepsilon_{kl} - E_{ijkl}^A \varepsilon_{kl}^A \tag{7}$$

$$\{\sigma^A\} = [E]\{\varepsilon\} - [E^A]\{\varepsilon^A\} \tag{8}$$

[E], $[E^A]$ are the elasticity matrices for the elastic and, respectively, anelastic fields.

For the plane-strain case, the following are also true:

$$\begin{aligned} &\sigma_1^A, \sigma_2^A, \sigma_3^A, \tau_{12}^A \left(= \sigma_6^A\right) \neq 0 \\ &\varepsilon_3^A = 0 \end{aligned} \tag{9}$$

## 2.2 Mixed u-p finite element development considerations

In general, if the material approaches incompressibility (Poisson's ratio approaches 0.5), it is more advantageous to write the elastic stress in terms of deviatoric strains and pressure, and eliminate the dependence on the bulk modulus, $\beta_K$ , since this quantity becomes very large for almost incompressible materials [4]:

$$\sigma_{ij} = \beta_K \varepsilon_V \delta_{ij} + 2\beta_G \varepsilon'_{ij} = -p\delta_{ij} + 2\beta_G \varepsilon'_{ij} \tag{10}$$

The quantity $\beta_G$ is the shear modulus, $\varepsilon_V$ is the volumetric strain, $\delta_{ij}$ is the Kronecker delta and $\varepsilon_{ij}$' are the deviatoric strains, p is the pressure.

The deviatoric strain vector components are functions of the volumetric strain [4]:

$$\varepsilon_{ij}^{'} = \varepsilon_{ij} - \frac{1}{3}\varepsilon_V \delta_{ij} \tag{11}$$

The principle of virtual work as a function of independent variables, which are displacements and pressure vectors is [4]:

$$\int_V \bar{\varepsilon}'^T \sigma' dV - \int_V p\bar{\varepsilon}_V = R \tag{12}$$

The bared terms in the Equation above are virtual quantities.

The Hellinger-Reissner (HR) functional for the mixed u-p formulation is [4]:

$$\Pi_{HR}^*(u,p) = \int_V \frac{1}{2}\varepsilon'^T C' \varepsilon' dV - \int_V \frac{1}{2}\frac{p^2}{\beta_K} - \int_V p\varepsilon_V dV - (work\ \ due\ \ to\ \ body\ forces\ and\ surface\ tractions) \tag{13}$$

The condition that the HR functional is stationary with respect to the displacements and pressure leads to a system of two equations [4]:

$$\Pi^*_{HR}(u,p) = \int_V \delta\varepsilon'^T C'\varepsilon' dV - \int_V p\delta\varepsilon_V dV = R$$
$$\int_V \left(\frac{p}{\beta_K} + \varepsilon_V\right)\delta p dV = 0 \tag{14}$$

R is the virtual work of all externally applied loading.

### 2.3 Plane-strain finite element development

The total deviatoric and, respectively, anelastic deviatoric strains for the plane-strain case are, in matrix form [4]:

$$\begin{bmatrix} \varepsilon' \\ \varepsilon'_A \end{bmatrix} = \begin{bmatrix} \varepsilon_{xx} - \frac{1}{3}(\varepsilon_{xx}+\varepsilon_{yy}) \\ \varepsilon_{yy} - \frac{1}{3}(\varepsilon_{xx}+\varepsilon_{yy}) \\ \gamma_{xy} \\ -\frac{1}{3}(\varepsilon_{xx}+\varepsilon_{yy}) \\ \varepsilon^A{}_{xx} - \frac{1}{3}(\varepsilon^A{}_{xx}+\varepsilon^A{}_{yy}) \\ \varepsilon^A{}_{yy} - \frac{1}{3}(\varepsilon^A{}_{xx}+\varepsilon^A{}_{yy}) \\ \gamma^A{}_{xy} \\ -\frac{1}{3}(\varepsilon^A{}_{xx}+\varepsilon^A{}_{yy}) \end{bmatrix} = \begin{bmatrix} \left(\frac{2}{3}\frac{\partial u}{\partial x} - \frac{1}{3}\frac{\partial v}{\partial y}\right) \\ \left(\frac{2}{3}\frac{\partial v}{\partial y} - \frac{1}{3}\frac{\partial u}{\partial x}\right) \\ \left(\frac{\partial v}{\partial x} + \frac{\partial u}{\partial y}\right) \\ -\frac{1}{3}\left(\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y}\right) \\ \left(\frac{2}{3}\frac{\partial u^A}{\partial x} - \frac{1}{3}\frac{\partial v^A}{\partial y}\right) \\ \left(\frac{2}{3}\frac{\partial v^A}{\partial y} - \frac{1}{3}\frac{\partial u^A}{\partial x}\right) \\ \left(\frac{\partial v^A}{\partial x} + \frac{\partial y^A}{\partial y}\right) \\ -\frac{1}{3}\left(\frac{\partial u^A}{\partial x} + \frac{\partial v^A}{\partial y}\right) \end{bmatrix} \tag{15}$$

The total and, respectively, anelastic volumetric strains are:

$$\varepsilon_v = (\varepsilon_{xx} + \varepsilon_{yy})$$
$$\varepsilon^A{}_v = (\varepsilon^A{}_{xx} + \varepsilon^A{}_{yy}) \tag{16}$$

These strains will be related to the pressure caused by the total displacements and, respectively, to the pressure caused by the anelastic displacements, $p^A$.

The deviatoric stresses are a function of deviatoric strain, in the almost incompressible case:

$$\{\sigma'\} = C'(\{\varepsilon'\} - \{\varepsilon'^A\}) \tag{17}$$

$$\{\sigma'\} = [C'][\varepsilon' - \varepsilon'^A] = [[C'] \quad -[C']]\begin{bmatrix} \varepsilon' \\ \varepsilon'^A \end{bmatrix} \tag{18}$$

$$\left\{\sigma'^{A}\right\}=\left[\left[C'\right] \quad -\left[C'^{A}\right]\right]\begin{bmatrix}\varepsilon' \\ \varepsilon'^{A}\end{bmatrix} \tag{19}$$

$$\left[C'\right]=\begin{bmatrix}2\beta_G & 0 & 0 & 0 \\ 0 & 2\beta_G & 0 & 0 \\ 0 & 0 & \beta_G & 0 \\ 0 & 0 & 0 & 2\beta_G\end{bmatrix} \tag{20}$$

$$\left[C'^{A}\right]=C_G\begin{bmatrix}2\beta_G & 0 & 0 & 0 \\ 0 & 2\beta_G & 0 & 0 \\ 0 & 0 & \beta_G & 0 \\ 0 & 0 & 0 & 2\beta_G\end{bmatrix} \tag{21}$$

where $C_G = \frac{1+\Delta_G}{\Delta_G}$ [1,2,3] and describes the coupling between the physical relaxation process (characterized by the relaxation strength Δ) and the total displacement field (G refers to the shear modulus). The total pressure in the body, p, is related to the total volumetric strain ($\beta_K$ is the bulk modulus):

$$p = -\beta_K \varepsilon_V \tag{22}$$

The pressure in the body due to the anelastic displacements ONLY is $p^A$, and is related to the total volumetric strain:

$$p^A = -\beta_K \varepsilon_V^A \tag{23}$$

The displacement interpolation matrices [H] are (i , j, m, n identify the nodes that make up the element)[4]:

$$\begin{Bmatrix}\{u\} \\ \{v\} \\ \{u^A\} \\ \{v^A\}\end{Bmatrix}=\left[H\right]\left[u_i \quad u_j \quad u_m \quad u_n \quad v_i \quad v_j \quad v_m \quad v_n \quad u_i^A \quad u_j^A \quad u_m^A \quad u_n^A \quad v_i^A \quad v_j^A \quad v_m^A \quad v_n^A\right]^T =$$

$$\ldots = \left[H\right]\left\{\hat{u}\right\} \tag{24}$$

$$\left[H\right]=\begin{bmatrix}[h] & [0] & [0] & [0] \\ [0] & [h] & [0] & [0] \\ [0] & [0] & [h] & [0] \\ [0] & [0] & [0] & [h]\end{bmatrix} \tag{24b}$$

$$
\begin{aligned}
[h] &= \lfloor h_i \quad h_j \quad h_m \quad h_n \rfloor \\
h_i &= \frac{1}{4}(1+x)(1+y) \\
h_j &= \frac{1}{4}(1-x)(1-y) \\
h_m &= \frac{1}{4}(1-x)(1+y) \\
h_n &= \frac{1}{4}(1+x)(1-y)
\end{aligned} \tag{25}
$$

The strain-displacement interpolation matrices $B_D$ are [4]:

$$
B_D = \begin{bmatrix}
\frac{2}{3}h_{i,x} & \frac{2}{3}h_{j,x} & \frac{2}{3}h_{m,x} & \frac{2}{3}h_{n,x} & -\frac{1}{3}h_{i,y} & -\frac{1}{3}h_{j,y} & -\frac{1}{3}h_{m,y} & -\frac{1}{3}h_{n,y} \\
-\frac{1}{3}h_{i,x} & -\frac{1}{3}h_{j,x} & .. & .. & \frac{2}{3}h_{i,y} & \frac{2}{3}h_{j,y} & .. & .. \\
h_{i,y} & h_{j,y} & .. & .. & h_{i,x} & h_{j,x} & .. & .. \\
-\frac{1}{3}h_{i,x} & -\frac{1}{3}h_{j,x} & .. & .. & -\frac{1}{3}h_{i,y} & -\frac{1}{3}h_{j,y} & .. & ..
\end{bmatrix} \tag{26}
$$

The deviatoric strains are, in matrix form, a function of the displacement vector multiplied by the strain-displacement interpolation matrix:

$$
[\varepsilon'] = [B_D]\begin{bmatrix} \{u\} \\ \{v\} \\ \{u^A\} \\ \{v^A\} \end{bmatrix} \tag{27}
$$

The volumetric strain-displacement interpolation matrices $B_V$ are [4]:

$$
B_V = \lfloor h_{1,x} \quad h_{2,x} \quad h_{3,x} \quad h_{4,x} \quad h_{1,y} \quad h_{2,y} \quad h_{3,y} \quad h_{4,y} \quad h_{1,x} \quad h_{2,x} \quad h_{3,x} \quad h_{4,x} \quad h_{1,y} \quad h_{2,y} \quad h_{3,y} \quad h_{4,y} \rfloor \tag{28}
$$

The volumetric total and anelastic strains are:

$$
\begin{aligned}
[\varepsilon_V] &= [B_V]\begin{Bmatrix} \{u\} \\ \{v\} \end{Bmatrix} \\
[\varepsilon_V^A] &= [B_V]\begin{Bmatrix} \{u^A\} \\ \{v^A\} \end{Bmatrix}
\end{aligned} \tag{28}
$$

The constant pressure inside the element means that the interpolation functions $[H_p]$ are unity, thus [4]:

$$p = \lfloor H_p \rfloor \{\hat{p}\}$$
$$p^A = \lfloor H_p \rfloor \{\hat{p}^A\} \tag{29}$$
$$\lfloor H_p \rfloor = [1]$$

The unknown vector contains the total and anelastic displacements, as well as the total and anelastic pressures. The stiffness matrix for the mixed u-p, single ADF element will be:

$$\begin{Bmatrix} \{u\} \\ \{v\} \\ \{u^A\} \\ \{v^A\} \\ \hat{p} \\ \hat{p}^A \end{Bmatrix} \tag{30}$$

By substituting the interpolation equations into the virtual work equation, we'll get the stiffness matrix K:

$$[k] = \begin{bmatrix} K_{uu} & K_{uu^A} & K_{up} & K_{up^A} \\ K_{u^A u} & K_{u^A u^A} & K_{u^A p} & K_{u^A p^A} \\ K_{pu} & K_{pu^A} & K_{pp} & K_{pp^A} \\ K_{p^A u} & K_{p^A u^A} & K_{p^A p} & K_{p^A p^A} \end{bmatrix} \tag{31}$$

Each one of the terms that this stiffness matric incorporates is listed below [4]

$$\begin{aligned}
K_{uu} &= \int_V B_D^T C' B_D dV = \iint B_D^T C' B_D t dx dy \\
K_{uu^A} &= -\int_V B_D^T C'^A B_D dV = -\iint B_D^T C'^A B_D t dx dy \\
K_{u^A u^A} &= \int_V B_D^T C'^A B_D dV = \iint B_D^T C'^A B_D t dx dy \\
K_{up} &= -\int_V B_V^T H_p dV = K_{pu}^T \\
K_{u^A p} &= -\int_V B_V^T H_p dV = K_{pu^A}^T \\
K_{up^A} &= -\int_V B_V^T H_p dV = K_{p^A u}^T \\
K_{p^A p^A} &= -\int_V H_{p^A}^T \frac{1}{\beta_k} H_{p^A} dV \\
K_{pp} &= -\int_V H_p^T \frac{1}{\beta_k} H_p dV
\end{aligned} \tag{32}$$

## 2.4 Motion equations

The tensor form of the equation of motion for a material of density ρ, acted upon by a body force $f_i$, in Cartesian direction i (i=1,2,3) is [1,2]

$$\rho u_{i,tt} - \sigma_{ij,j} = f_i \tag{33}$$

The stress may now be expressed in terms of the total and anelastic strains; Equation above now becomes [1,2]

$$\rho u_{i,tt} - \left(E_{ijkl}\left(\varepsilon_{kl} - \varepsilon_{kl}^{A}\right)\right)_{,j} = f_i \tag{34}$$

Note that both total pressure and the pressure due to the anelastic displacement are constant and do not vary with the coordinates. Only the deviatoric strains will be contributing to the motion equation.

$$\rho u_{i,tt} - \left(E_{ijkl}\left(\varepsilon'_{kl} - \varepsilon'^{A}_{kl}\right)\right)_{,j} = f_i \tag{35}$$

The mass matrix that will ensue will be similar to that of the pure-displacement, single-ADF mass matrix obtained in previous work[3]:

$$[M] = \begin{bmatrix} [M_{uu}] & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \tag{36}$$

That mass matrix was obtained from the kinetic energy:

$$T(x,y,t) = \frac{1}{2}\rho\int_V \left[ \begin{Bmatrix} \{\dot{u}\} \\ \{\dot{v}\} \end{Bmatrix}^T \begin{Bmatrix} \{\dot{u}\} \\ \{\dot{v}\} \end{Bmatrix} \right] dV = \frac{1}{2}\rho\int_V \left[ \begin{Bmatrix} \{\dot{\hat{u}}\} \\ \{\dot{\hat{v}}\} \end{Bmatrix}^T [H]^T [H] \begin{Bmatrix} \{\dot{\hat{u}}\} \\ \{\dot{\hat{v}}\} \end{Bmatrix} \right] dV \tag{36b}$$

$$[M_{uu}] = \begin{bmatrix} [M_d]_{4x4} & 0 \\ 0 & [M_d]_{4x4} \end{bmatrix}$$

$$[M_d] = \begin{bmatrix} h_i^2 & h_i h_j & h_i h_m & h_i h_n \\ & h_j^2 & h_j h_m & h_j h_n \\ & symm. & h_m^2 & h_m h_n \\ & & & h_n^2 \end{bmatrix} \tag{37}$$

## 2.5 Relaxation equations

The relaxation equations may be obtained from a non-equilibrium thermodynamics assumption, which states that the time rate of change of the state variable is proportional to the conjugate quantity. That means there that the rate of change of the anelastic strain is proportional to the anelastic stress [1,2]

$$\varepsilon_{kl,t}^{A} = L_{klij}\sigma_{ij}^{A} \tag{38}$$

If a single anelastic displacement field (ADF) is considered and if Ω is the inverse of the relaxation time at constant strain, the rate of change of the anelastic strain is proportional to the difference between its equilibrium and actual value ( $\bar{\varepsilon}^A$ ) [1,2]

$$\varepsilon_{kl,t}^A = -\Omega\left(\varepsilon_{kl}^A - \bar{\varepsilon}_{kl}^A\right) \tag{39}$$

The equilibrium value of the anelastic strain $\bar{\varepsilon}^A$ corresponds to a null anelastic stress [1,2]

$$\bar{\varepsilon}^A = \left(E_{ijkl}^A\right)^{-1} E_{ijkl}\varepsilon_{kl} \tag{40}$$

The time rate of change of the anelastic strain becomes

$$E_{ijkl}^A \varepsilon_{kl,t}^A = -\Omega\left(E_{ijkl}^A \varepsilon_{kl}^A - E_{ijkl}\varepsilon_{kl}\right) \tag{41}$$

If one extracts the anelastic stress,

$$\frac{1}{\Omega} E_{ijkl}^A \varepsilon_{kl,t}^A = E_{ijkl}\varepsilon_{kl} - E_{ijkl}^A \varepsilon_{kl}^A = \sigma_{ij}^A \tag{42}$$

Calculate the divergence of the above:

$$\left(\frac{1}{\Omega} E_{ijkl}^A \varepsilon_{kl,t}^A\right)_{,j} - \sigma_{ij,j}^A = 0 \tag{43}$$

Equation (44) may be developed in terms of the strains, as [1,2]

$$\left(\frac{1}{\Omega} E_{ijkl}^A \varepsilon_{kl,t}^A - E_{ijkl}\varepsilon_{kl} + E_{ijkl}^A \varepsilon_{kl}^A\right)_{,j} = 0 \tag{44}$$

Once again, it may be seen that both total and anelastic pressures are constant throughout the element (their derivatives are null with respect to position) and because of the Equation (45) below, only the deviatoric part of the strains will remain in the relationship from Equation 44 as seen in Equation 46.

$$\varepsilon_{ij}' = \varepsilon_{ij} - \frac{\varepsilon_V}{3}\delta_{ij} = \varepsilon_{ij} + \frac{p}{3\beta_K}\delta_{ij} \tag{45}$$

$$\left(\frac{1}{\Omega} C'^A_{ijkl}\ \varepsilon'^A_{kl,t} - C'_{ijkl}\ \varepsilon'_{kl} + C'^A_{ijkl}\ \varepsilon'^A_{kl}\right)_{,j} = 0 \tag{46}$$

The damping matrix will resemble the damping matrix obtained in the pure displacement finite element formulation of the ADF, see Equation 47. Equation 47 also assumes that we take into consideration a rectangular system.

$$[C_{damping}] = \frac{1}{\Omega}\int_V {B_D}^T \begin{bmatrix} 0 & 0 \\ 0 & \left[C'^A\right] \end{bmatrix} B_D dV = \frac{1}{\Omega}\iint {B_D}^T \begin{bmatrix} 0 & 0 \\ 0 & \left[C'^A\right] \end{bmatrix} B_D t dx dy \tag{47}$$

The overall damping matrix will be:

$$[C_{overall}] = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & \left[C_{damping}\right] & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \tag{48}$$

## 3. CONCLUSIONS

A new, four-noded, plane-strain, ADF finite element has been developed using a mixed displacement-pressure formulation. Displacements, total pressure and anelastic pressure (produced by anelastic displacements) were introduced as variables. The mass, stiffness and damping matrix formulations were presented. The main challenge of this development will be the time-integration of the dynamic equations, due to the presence of pressure as one of the variables.

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