New Bounds for the k-out-of-n Type Probabilities and Their Applications

by

Ahmed M. Binmahfoudh

Master of Science
in Engineering Management
Florida Institute of Technology
2008

Bachelor of Science
Computer Engineering
King Fahad University of Petroleum and Mineral
2005

A dissertation
submitted to Florida Institute of Technology
in partial fulfillment of the requirements
for the degree of

Doctorate of Philosophy
in
Operations Research

Melbourne, Florida
April, 2017

We the undersigned committee
hereby approve the attached dissertation

NEW BOUNDS FOR THE k-OUT-OF-n TYPE PROBABILITIES AND THEIR APPLICATIONS by Ahmed M. Binmahfoudh

---

Munevver Mine Subasi, Ph.D.
Associate Professor
Department of Mathematical Sciences
Committee Chair

Luis Daniel Otero, Ph.D.
Associate Professor
Department of Engineering Systems
Outside Committee Member

Jewgeni Dshalalow, Ph.D.
Professor
Department of Mathematical Sciences
Committee Member

Muzaffar Shaikh, Ph.D.
Distinguished Professor & Department Head
Department of Engineering Systems
Committee Member

Ugur Abdulla, Ph.D., Dr. Sci.
Professor & Department Head
Department of Mathematical Sciences

ABSTRACT

Title:

New Bounds for the k-out-of-n Type Probabilities and Their Applications

Author:

Ahmed M. Binmahfoudh

Major Advisor:

Munevver Mine Subasi, Ph.D.

The contribution of the shape information of the underlying distribution in probability bounding problem is investigated and an efficient linear programming based bounding methodology, which takes advantage of the advanced optimization techniques, probability theory, and the state-of-the-art tools, to obtain robust and efficiently computable bounds for the probabilities that at least k and exactly k-out-of-n events occur is developed. The k-out-of-n type probability bounding problem is formulated as linear programs under the assumption that the probability distribution is unimodal. The dual feasible bases structures of the relaxed versions of linear programs involved are fully described. The bounds for the probability that at least k and exactly k-out-of-n events occur are obtained in the form of formulas. A dual based linear programming algorithm is proposed to obtain bounds as the customized algorithmic solutions of the LP's formulated. Numerical examples are presented to show that the use of shape constraint significantly improves on the bounds for the probabilities that at least k and exactly k-out-of-n events occur when only first a few binomial moments are known. An application in PERT, where the shape of the underlying probability distribution can be used to obtain bounds for the distribution of the critical path length, is presented.

Table of Contents

List of Figures

List of Tables

Acknowledgments

I would like to express the deepest appreciation to my committee chair, Dr. Munevver Mine Subasi, who has continually and persuasively conveyed a spirit of adventure in regard to research and an excitement in regard to teaching. She has been a tremendous mentor for me. I would like to express thanks to her for encouraging my research and for allowing me to grow as a research scientist. Without her supervision and constant help this dissertation would not have been possible.

I would also like to thank my committee members, Dr. Jewgeni Dshalalow, Dr. Muzaffar Shaikh, and Dr. Luis Daniel Otero for serving as my committee members even at hardship. I also want to thank you for letting my defense be an enjoyable moment, and for your brilliant comments and valuable suggestions.

I owe special note of thank you to Saudi Arabia Cultural Mission (SACM) for their financial support granted through pre-doctoral fellowship.

Special thanks to my family. Words cannot express how grateful I am to my mother, Salwa, and my father, Mohammed, for all of the sacrifices they have made on my behalf. Your prayer for me was what sustained me thus far.

Last but not least I would like express appreciation to my beloved wife Samah and my children, Jurri, Lamar, Mayar, and Mohammed, who spent sleepless nights with and was always my support in the moments when there was no one to answer my queries.

Dedication

I dedicate this dissertation to my mom Salwa, my father Mohammed, my wife Samah, my daugthers Jurri, Lamar, Mayar, and my son Mohammed.

Chapter 1

Introduction

The problem to find or approximate the probability of the union, intersection and other Boolean functions of random events arises in a wide range of applications in probability theory, statistics, reliability theory, and stochastic programming: (i) stochastic transportation networks [41, 28]; (ii) communication network reliability [42]; (iii) telecommunication networks [20]; (iv) PERT [44, 56]; (v) maximum satisfiability [7]; (vi) estimation of multivariate normal distribution [10]; (vii) bit error rate performance [2]; $(viii)$ reliability [29, 47, 16]; and (ix) financial risk [12, 23]. Further applications can be found in [1] and [52].

In this dissertation we shall present a linear programming based probability bounding methodology to obtain improved bounds for the probability that at least k-out-of-n events occur. The typical application is to estimate the reliability evaluations of k-out-of-n systems such as multistate networks (oil and gas supply systems, communication networks, power generation and transmission systems, etc.) and fault tolerant systems (multidisplay system in a cockpit, multiengine system in an airplane, and multipump system in a hydraulic control system, etc.).

Let $A_1, \cdots, A_n$ be arbitrary events in an arbitrary probability space Ω and let ξ denote the number of those events that occur. The well-known Jordan's formulas (Jordan, 1927) [31] are available to compute the probabilities that at least k $(1 \leq k \leq n)$ and exactly k $(0 \leq k \leq n)$ out of n events occur:

$$P(\xi \geq k) = \sum_{j=k}^{n} (-1)^{j-k} \binom{k-1}{j-1} S_j \quad \text{and} \quad P(\xi = k) = \sum_{j=k}^{n} (-1)^{j-k} \binom{k}{j} S_j \tag{1.1}$$

where S_j is the jth binomial moment of the random variable ξ defined by

$$S_j = \sum_{1 \leq i_1 < \cdots < i_j \leq n} P(A_{i_1} \cap \cdots \cap A_{i_j}), \quad j = 1, ..., n \tag{1.2}$$

and $S_0 = 1$, by definition. However, if n is large, it may not be possible to find the exact values of the probabilities in (1.1) because the total number of terms in (1.2) becomes an exponential function of n. On the other hand, it may be possible to compute a few of the binomial moments and give lower and upper bounds for the probabilities in (1.1). Indeed, it is frequently observed that the probability distribution of the random variable is unknown, but a few moments of it are known or can be computed from historical data. The goal of the probability bounding approaches is to obtain the best possible approximation of the probabilities of type $P(\xi \geq k)$ and $P(\xi = k)$ based on the knowledge of the first m moments, $S_0, \cdots, S_m, (m < n)$, or any finite collection of binomial moments.

The probability bounding problem has a very long tradition throughout the history of probability theory. It presents many interesting challenges in concept and computation. Boole (1854) [4], paying much attention to explore the connection between logic and probability theory, discovered a basic inequality and a general approximation scheme for the probability that at least one out of n events occurs (probability of the union of events). He was the first to suggest algebraic methods to find the bounds for the probability of the union of events. Hailperin (1965) [25] showed that Boole's method is equivalent to Fourier-Motzkin elimination. Other well-known bounds for the probability of the union of events are due to Bonferroni

(1937) [5] . However, Boole and Bonferroni bounds are weak in general. Bounds based on the knowledge of the first binomial moment were obtained by Fréchet (1940) [17]. Dawson and Sankoff (1967) [13] presented the sharp Bonferroni bounds based on the first two binomial moments. Kwerel (1975) [32] reproduced and extended Dawson-Sankoff results to present lower and upper bounds for $P(\xi = k)$, probability that exactly k-out-of-n occurs, using the first three binomial moments.

Prékopa (1988-1990) [37, 38, 39, 40] discovered that the sharp $S_1, ..., S_m$ lower and upper bounds can be obtained as optimum values of linear programs (LP's) which were called "binomial moment problems" by Prékopa. Prékopa [37, 38, 39, 40] also obtained fundamental results in connection with binomial moment problems and presented dual feasible basis structure theorems to obtain closed form bounds for probabilities $P(\xi \leq k)$ and $P(\xi = k)$ for the case of small m values. Boros and Prékopa (1989) [6] used a linear programming approach to produce the closed form Bonferroni bounds and bounds for the probabilities that at least k and exactly k-out-of-n events occur, based on the knowledge of first four binomial moments. Prékopa (1990) [40] generalized the results for other moments which started the area of "discrete moment problems". Samuels and Studden (1989) [53] independently discovered the sharp Bonferroni inequalities and moment problems, however, they used the classical approach for the general moment problem, and determined the solutions in closed form whenever possible; their method is applicable only to small size problems. Other closed form probability bounds were presented by Sathe et al. (1980) [54] and Galambos et al. (1980, 1996) [18, 19]. Móri and Székely (1985) [35] presented a method that can be used to prove Bonferroni-Galambos type inequalities. Bound formulas for the probability of the union of events as well as the k-out-of-n type probabilities, using aggregation and disaggregation in linear programs, were presented in [21, 46].

Probability bounds based on the probabilities of the individual events and their intersections, and graph structures also exist in literature [30, 9, 10, 59, 11]. The reader is referred to papers by Veneziani (2009) [60], Boros, Scozzari, Tardella, and Veneziani (2014) [8], Prékopa, Ninh, and Alexe (2016) [48], and Prékopa and Yoda (2016) [49] for recent linear programming based probability bounds. Other studies on probability bounding can be found in [14, 27, 36, 50].

Despite the numerous procedures developed to estimate the probability of the Boolean functions of events; little has been done to propose an efficient methodology that takes into account the shape of the underlying probability distribution. While some attractive probability bounds were presented in the above mentioned papers, none of them takes into account the shape of the probability distribution under study. The papers by Prékopa, Subasi, and Subasi (2008) [47] and Subasi, Subasi, and Prékopa (2009) [56] are the first, where shape information is used to obtain robust bounds for the probability of the union of events based on the knowledge of first two binomial moments and for the expectations of higher order convex functions based on the knowledge of first two power moments.

The objective of this dissertation is to use the shape information of the distribution of the random variable ξ to obtain sharp bounds for the probability that at least k-out-of-n events occur, $P(\xi \geq k)$, and the probability that exactly k-out-of-n events occur, $P(\xi = k)$, based on the knowledge of m (not necessarily consecutive) binomial moments of ξ.

Let $p_i = P(\xi = i)$, $i = 0, 1, \cdots, n$. It is well-known that the jth binomial moment S_j can be written in the following form (Takács, 1995) [58]:

$$S_j = E\left[\binom{\xi}{j}\right] = \sum_{i=0}^{n} \binom{i}{j} p_i \ , \quad j = 0, 1, \cdots, n. \tag{1.3}$$

Note that the probabilities $p_i, i = 0, 1, \cdots, n$, can be uniquely determined from the above system (see, e.g., Prékopa, 1995).

Our investigation starts with the following two linear programming problems:

$$\begin{aligned} &\min(\max) \sum_{i=k}^{n} p_i \\ &\text{subject to} \\ &\sum_{i=0}^{n} \binom{i}{j} p_i = S_j, \quad j = 0, 1, \cdots, m \\ &p_i \geq 0, \quad i = 0, 1, \cdots, n \end{aligned} \tag{1.4}$$

where $1 \leq k \leq n$ and

$$\begin{aligned} &\min(\max) p_k \\ &\text{subject to} \\ &\sum_{i=0}^{n} \binom{i}{j} p_i = S_j, \quad j = 0, 1, \cdots, m \\ &p_i \geq 0, \quad i = 0, 1, \cdots, n \end{aligned} \tag{1.5}$$

where $0 \leq k \leq n$, $S_0 = 1$ (by definition), and $m < n$. Note that there are infinitely many probability distributions satisfying the constraints of problems (1.4) and (1.5). One of them is the true distribution of the random variable ξ.

Problems (1.4) and (1.5) are called the discrete binomial moment problems and have been extensively studied by Prékopa (1988-1990) [37, 38, 39, 40]. Note that, given $S_1, \cdots, S_m$, the two optimum values of problem (1.4) and problem (1.5) provide us with the best possible lower and upper bounds for the probability that at least k and exactly k out of n events occur, respectively.

In what follows we assume that binomial moments $S_1, \cdots, S_m$ are known for some $m < n$. We do not assume the knowledge of the probability distribution $\{p_i\}$, but assume that it is unimodal with a known/given mode, i.e., there exists an integer M $(0 \leq M \leq n)$ such that

$$p_0 \leq \cdots \leq p_{M-1} \leq p_M \quad \text{and} \quad p_M \geq p_{M+1} \geq \cdots \geq p_n. \tag{1.6}$$

The organization of this dissertation is as follows. In Chapter 2 we formulate linear programming problems for bounding the probability that at least k out of n events occur, where the probability distribution of the occurrences is unimodal with a known mode and some of the binomial moments of the events are also known. Dual feasible basis structures of the LP's are fully described. In Chapter 3 binomial moment problems with finite, preassigned support are formulated to obtain lower and upper bounds based on the knowledge of first m binomial moments to obtain bounds for the probability that exactly k out of n events occur, where the probability distribution of the occurrences is supposed to be unimodal with mode M. We also characterize the dual feasible basis structures of the LP's involved. In Chapter 4 we use the bases structure theorems obtained in Chapters 2-3 to obtain lower and upper bounds for the probability that exactly k out of n events occur, where the probability distribution of the occurrences is supposed to be unimodal with known mode, based on the knowledge of first two binomial moments. In Chapter 4.5 a dual type algorithm is presented to obtain customized algorithmic solutions of the LP's involved. Numerical examples are presented in Chapter 5 to compare the bounds obtained by the binomial moment problem with and without the unimodality constraint (shape of the underlying probability distribution). The results show that the use of the shape constraint significantly improves on bounds. In Chapter 6 we present applications of our bounding methodology in PERT, where shape information about the unknown probability distribution can be used to approximate the distribution of the critical path length.

Chapter 2

Bounding the Probability that at Least k-out-of-n Events Occur

We assume that the probability distribution of the occurrences is unimodal with known mode M, i.e., condition (1.6) is satisfied, and some of the binomial moments of the events are also known. We consider the following problem

$$\min(\max) \sum_{i=k}^{n} p_i$$

subject to

$$\begin{aligned} \sum_{i=0}^{n} \binom{i}{j} p_i = S_j, \quad j = 0, 1, \cdots, m \\ p_0 \le \cdots \le p_{M-1} \le p_M \\ p_M \ge p_{M+1} \ge \cdots \ge p_n \\ p_i \ge 0, \quad i = 0, 1, \cdots, n \end{aligned} \tag{2.1}$$

where $1 \le k \le n$, $S_0 = 1$, and $m < n$.

Problem (2.1) can be used in connection with arbitrary events $A_1, \cdots, A_n$, to obtain best possible lower and upper bounds for the probability that at least k-out-of-n events occur, where $1 \le k \le n$. If $m < n$, then there are infinitely many probability distributions satisfying the constraints of problem (2.1). One of them is the true distribution of ξ. This implies that the optimum value of the min (max) problem (2.1) is a lower (upper) bound for the probability $P(\xi \geq k)$. These bounds have the property that, given $S_1, \cdots, S_m$, no better bounds can be obtained for $P(\xi \geq k)$ under the assumption that the probability distribution of ξ is unimodal with mode M. In view of this fact, we call them sharp bounds.

As it is known in linear programming theory, the objective function value corresponding to any dual feasible basis in the minimization (maximization) problem provides us with a lower (upper) bound for the optimum value of the problem. The purpose of this section is to derive a general theorem in connection with problem (2.1) that characterizes the dual feasible bases of its relaxed version, further, to present closed form and algorithmic bounds for the probability that at least k-out-of-n events occur.

First, we reformulate problem (2.1) by introducing new variables $v_0, ..., v_n$. This can be done in two different ways:

$$\begin{gathered} v_0 = p_0, v_1 = p_1 - p_0, \cdots, v_M = p_M - p_{M-1}, \\ v_{M+1} = p_M - p_{M+1}, v_{M+2} = p_{M+1} - p_{M+2}, \cdots, v_n = p_n \end{gathered} \tag{2.2}$$

and

$$\begin{gathered} v_0 = p_0, v_1 = p_1 - p_0, \cdots, v_{M-1} = p_{M-1} - p_{M-2}, \\ v_M = p_M - p_{M+1}, v_{M+1} = p_{M+1} - p_{M+2}, \cdots, v_n = p_n \end{gathered} \tag{2.3}$$

If the distribution of ξ is increasing, we have $M = n$ and this case is included in representation (2.2). Similarly, if the distribution of ξ is known to be decreasing, then $M = 0$ and this is included in representation (2.3).

In order to obtain bounds for $P(\xi \geq k)$ we shall consider the cases where k is smaller than or equal to the mode M and k is greater than the mode M.

2.1 Characterization of dual feasible bases for $P(\xi \geq k), 1 \leq k \leq M$

Let $1 \leq k \leq M$. If we use representation (2.2) in problem (2.1), we obtain the following problem:

$$\min(\max)\ \{(M-k+1)\sum_{i=0}^{k} v_i + \sum_{i=k+1}^{M}(M-i+1)v_i + \sum_{i=M+1}^{n}(i-M)v_i\}$$

subject to

$$\sum_{i=0}^{M}(M-i+1)v_i + \sum_{i=M+1}^{n}(i-M)v_i = 1$$

$$\sum_{i=0}^{M}\left[\binom{i}{j} + \cdots + \binom{M}{j}\right] v_i + \sum_{i=M+1}^{n}\left[\binom{M+1}{j} + \cdots + \binom{i}{j}\right] v_i = S_j, j = 1, ..., m$$

$$v_0 + ... + v_M - v_{M+1} - ... - v_n \geq 0$$

$$v_i \geq 0\ , \quad i = 0, \cdots, n\ . \tag{2.4}$$

In case of representation (2.3) the problem can be formulated as follows:

$$\min(\max)\ \{(M-k)\sum_{i=0}^{k} v_i + \sum_{i=k+1}^{M-1}(M-i)v_i + \sum_{i=M}^{n}(i-M+1)v_i\}$$

subject to

$$\sum_{i=0}^{M-1}(M-i)v_i + \sum_{i=M}^{n}(i-M+1)v_i = 1$$

$$\sum_{i=0}^{M-1}\left[\binom{i}{j}+\cdots+\binom{M-1}{j}\right]v_i + \sum_{i=M}^{n}\left[\binom{M}{j}+\cdots+\binom{i}{j}\right]v_i = S_j, j=1,...,m$$

$$v_M + ... + v_n - v_0 - ... - v_{M-1} \geq 0$$

$$v_i \geq 0\ , \quad i = 0, \cdots, n\ . \tag{2.5}$$

In what follows we shall refer to problem (2.4) without constraint "$v_0+...+v_M-v_{M+1}-...-v_n \geq 0$" and problem (2.5) without "$v_M+...+v_n-v_0-...-v_{M-1} \geq 0$" the *relaxed problems*. For both relaxed problems let $A = (a_0, ..., a_n)$ designate the matrix of the equality constraints, b the right hand side vector and c the vector of coefficients of the objective function.

Consider Table 2.1 which gives in detailed form the matrix $\binom{c^T}{A}$, where A is the coefficient matrix of the equality constraints, and c^T is the vector of objective function coefficients in problem (2.4).

Table 2.1: Matrix of equality constraints in (2.4)

	0	1	...	k	$k+1$	...	M	$M+1$	$M+2$	...	n
c^T	$M-k+1$	$M-k+1$	...	$M-k+1$	$M-k$	...	1	1	2	...	$n-M$
0	$M+1$	M	...	$M-k+1$	$M-k$	...	1	1	2	...	$n-M$
1	$0+...+M$	$1+...+M$	...	$k+...+M$	$k+1+...+M$	...	M	$M+1$	$M+1+M+2$	...	$M+1+...+n$
2	$\binom{2}{2}+...+\binom{M}{2}$	$\binom{2}{2}+...+\binom{M}{2}$	...	$\binom{k}{2}+...+\binom{M}{2}$	$\binom{k+1}{2}+...+\binom{M}{2}$	...	$\binom{M}{2}$	$\binom{M+1}{2}$	$\binom{M+1}{2}+\binom{M+2}{2}$	...	$\binom{M+1}{2}+...+\binom{n}{2}$
$\vdots$	$\vdots$	$\vdots$	...	$\vdots$	$\vdots$	...	$\vdots$	$\vdots$	$\vdots$	...	$\vdots$
m	$\binom{m}{m}+...+\binom{M}{m}$	$\binom{m}{m}+...+\binom{M}{m}$	...	$\binom{k}{m}+...+\binom{M}{m}$	$\binom{k+1}{m}+...+\binom{M}{m}$	...	$\binom{M}{m}$	$\binom{M+1}{m}$	$\binom{M+1}{m}+\binom{M+2}{m}$	...	$\binom{M+1}{m}+...+\binom{n}{m}$

Let a_i, $i = 0, 1, ..., n$ denote the columns of matrix A in Table 2.1, where the top two rows and leftmost column are ignored. We apply the following column subtraction procedures on A:

- Starting from the column of 0, $i = 0$, subtract a_{i+1} from a_i and write it to a_i, that is,

$$a_i - a_{i+1} \longrightarrow a_i, \; i = 0, ..., M-1.$$

- Starting from the column of $n-1$, $i = n-1$, subtract a_i from a_{i+1} and write it to a_{i+1}, that is,

$$a_{i+1} - a_i \longrightarrow a_{i+1}, \; i = n-1, ..., M+1.$$

Note that the only columns which are not affected by the above matrix operations are the columns of M and $M+1$. One can easily see that the resulting matrix is the Pascal matrix P consisting of binomial coefficients:

$$P = \begin{pmatrix} 1 & 1 & 1 & \dots & 1 & \dots & 1 & 1 & \dots & 1 \\ 0 & 1 & 2 & \dots & k & \dots & M & M+1 & \dots & n \\ 0 & 0 & \binom{2}{2} & \dots & \binom{k}{2} & \dots & \binom{M}{2} & \binom{M+1}{2} & \dots & \binom{n}{2} \\ \vdots & \vdots & \vdots & \dots & \vdots & \dots & \vdots & \vdots & \dots & \vdots \\ 0 & 0 & 0 & \dots & \binom{k}{m} & \dots & \binom{M}{m} & \binom{M+1}{m} & \dots & \binom{n}{m} \end{pmatrix}. \qquad (2.6)$$

A similar column subtraction technique could be applied to obtain the matrix of equality constraints in problem (2.5) as the Pascal matrix P. We know that any minor of P that has all positive entries in its main diagonal, is positive (see, e.g., [22, 37]). In what follows we make use of this assertion.

Below we characterize the dual feasible bases of the relaxed version of problems (2.4) and (2.5) whose optimum values provide us with lower and upper bounds for

$P(\xi \geq k)$, the probability that at least k-out-of-n events occur.

Theorem 1. *Any dual feasible basis of any of the relaxed problems* (2.4) *and* (2.5) *has one of the following structures, presented in terms of the subscripts, where* I_B *is the set of subscripts of the vectors that are in the basis* B*. In addition, all dual feasible bases are dual nondegenerate, except for those with* $I_B \subset \{k, ..., n\}$ *which are dual degenerate.*

Minimization problem, $m+1$ even

- $\{i, i+1, ..., j, j+1, k-1, k, k+1, r, r+1, ..., t, t+1, n\}$ *if* $1 \leq k \leq n-2$
- $\{0, i, i+1, ..., j, j+1, k-1, k, k+1, r, r+1, ..., t, t+1\}$ *if* $2 \leq k \leq n-1$
- $\{i, i+1, ..., j, j+1, r, r+1, ..., t, t+1\}$ *if* $j+1 < k \leq r$

Minimization problem, $m+1$ odd

- $\{i, i+1, ..., j, j+1, k-1, k, k+1, r, r+1, ..., t, t+1\}$ *if* $1 \leq k \leq n-1$
- $\{0, i, i+1, ..., j, j+1, k-1, k, k+1, r, r+1, ..., t, t+1, n\}$ *if* $2 \leq k \leq n-2$
- $\{0, i, i+1, ..., j, j+1, r, r+1, ..., t, t+1\}$ *if* $j+1 < k \leq r$
- $\{i, i+1, ..., j, j+1, r, r+1, ..., t, t+1, n\}$ *if* $j+1 < k \leq r$

Maximization problem, $m+1$ even

- $I_B \subset \{k, ..., n\}$ *if* $n-k \geq m$
- $\{0, i, i+1, ..., j, j+1, k, r, r+1, ..., t, t+1\}$ *if* $1 \leq k \leq n$
- $\{i, i+1, ..., j, j+1, k, r, r+1, ..., t, t+1, n\}$ *if* $1 \leq k \leq n-1$

Maximization problem, $m+1$ odd

- $I_B \subset \{k, ..., n\}$ *if* $n-k \geq m$
- $\{i, i+1, ..., j, j+1, k, r, r+1, ..., t, t+1\}$ *if* $1 \leq k \leq n$
- $\{0, i, i+1, ..., j, j+1, k, r, r+1, ..., t, t+1, n\}$ *if* $1 \leq k \leq n-1$.

Proof. We carry out the proof for the relaxed problem (2.4). The proof of the assertion for problem (2.5) is similar.

Table 2.1 gives in detailed form the matrix $\binom{c^T}{A}$, where A is the coefficient matrix of the equality constraints and c^T is the vector of objective function coefficients in problem (2.4). Note that if $M = 0$ or $M = n$, then the underlying probability distribution is increasing or decreasing, respectively.

If we subtract the row of 0 from the row of c^T in Table 2.1, then the row corresponding to the objective function coefficients becomes

	0	1	$\cdots$	$k-1$	k	$k+1$	$\cdots$	M	$M+1$	$\cdots$	n
c^T	$-k$	$-(k-1)$	$\cdots$	-1	0	0	$\cdots$	0	0	$\cdots$	0

A basis B in the minimization problem (2.4) is dual feasible if the following inequality holds:

$$c_B^T B^{-1} a_p \le c_p \quad \text{for any nonbasic } p\ . \tag{2.7}$$

For the maximization problem the dual feasibility of a basis is defined by the reversed inequalities. A basis B is dual degenerate if there is at least one nonbasic p such that $c_p - c_B^T B^{-1} a_p = 0$. Since we have

$$\begin{pmatrix} 1 & c_B^T \\ 0 & B \end{pmatrix} \begin{pmatrix} c_p - c_B^T B^{-1} a_p \\ B^{-1} a_p \end{pmatrix} = \begin{pmatrix} c_p \\ a_p \end{pmatrix},$$

the first component of the solution of this equation can be expressed as

$$c_p - c_B^T B^{-1} a_p = \frac{1}{|B|} \begin{vmatrix} c_p & c_B^T \\ a_p & B \end{vmatrix} \ge 0 \text{ for any nonbasic } p\ . \tag{2.8}$$

Hence, we are interested in the sign of determinants $|B|$ and $\begin{vmatrix} c_p & c_B^T \\ a_p & B \end{vmatrix}$. Prékopa, Subasi, and Subasi (2008) [47] proved that for any basis B we have $|B| > 0$: Since B is a basis, it follows that $|B| \neq 0$. The entries in the first row can be written up as sum of 1's so that the number of terms in any position in that row is equal to the number of terms in any entry in its column. Then we apply a column subtraction procedure, further, split the obtained determinant into a sum of determinants.

Any determinant in the obtained sum is either zero, or positive because they are the minors crossed out of the matrix Pascal matrix P in (2.6). At least one term must be positive because $|B| \neq 0$. It follows that $|B| > 0$.

In connection with the term $\begin{vmatrix} c_p & c_B^T \\ a_p & B \end{vmatrix}$ we prove the following lemma.

Lemma 1. *For* $0 \leq i_1 < ... < i_t < ... < i_{m+2} \leq n$ *let us define*

$$\Delta_t = \begin{vmatrix} -(k-i_1) & -(k-i_2) & \cdots & -(k-i_t) & 0 & 0 & \cdots & 0 \\ a_{i_1} & a_{i_2} & \cdots & a_{i_t} & a_{i_{t+1}} & a_{i_{t+2}} & \cdots & a_{i_{m+2}} \end{vmatrix}, \quad (2.9)$$

where $0 \leq i_t \leq k \leq M$, $1 \leq t \leq m+1$, *and* $a_{i_t} \in \mathbb{R}^m$ *are the columns of* A. *Then*

$$\Delta_t = \sum_{j=1}^{t-1} (-1)^j (i_{j+1} - i_j) M_{1j} + (-1)^t (k - i_t) M_{1t} \ , \quad (2.10)$$

and $M_{1j}, j = 1, ..., t$ *are positive, where* $M_{1j}, j = 1, ..., t$ *are the minors of (2.9) formed by eliminating row 1 and column* j.

Proof. We apply the following column subtraction procedures on the determinant Δ_t in the order they are written:

- $col_j - col_{j+1} \longrightarrow col_j, \quad j = 1, ..., t-1$
- $col_{j+1} - col_j \longrightarrow col_{j+1}, \quad j = m+1, ..., t+1$

where col_j denotes the jth column of Δ_t. The only columns which remain unaffected by these operations are the t^{th} and $(t+1)^{st}$ columns and hence, the determinant Δ_t in (2.9) is obtained as

$$\begin{vmatrix} i_1 - i_2 & \cdots & i_{t-1} - i_t & -(k-i_t) & 0 & 0 & \cdots & 0 \\ a_{i_1} - a_{i_2} & \cdots & a_{i_{t-1}} - a_{i_t} & a_{i_t} & a_{i_{t+1}} & a_{i_{t+2}} - a_{i_{t+1}} & \cdots & a_{i_{m+2}} - a_{i_{m+1}} \end{vmatrix},$$

$$(2.11)$$

where $0 \le i_t \le k \le M$ and $1 \le t \le m+1$. We note that

$$a_r - a_{r+1} = \begin{pmatrix} 1 \\ r \\ \binom{r}{2} \\ \vdots \\ \binom{r}{m} \end{pmatrix}, \ 0 \le r \le M \quad \text{and} \quad a_r - a_{r-1} = \begin{pmatrix} 1 \\ r \\ \binom{r}{2} \\ \vdots \\ \binom{r}{m} \end{pmatrix}, \ M+1 \le r \le n$$

where a_r is the rth column of the coefficient matrix A of problem (2.4).

Hence, we have

$$a_l - a_r = a_l - a_{l+1} + ... + a_{r-1} - a_r = \begin{pmatrix} 1 \\ l \\ \binom{l}{2} \\ \vdots \\ \binom{l}{m} \end{pmatrix} + ... + \begin{pmatrix} 1 \\ r-1 \\ \binom{r-1}{2} \\ \vdots \\ \binom{r}{m} \end{pmatrix}, \ 0 \le l < r \le M \tag{2.12}$$

and

$$a_r - a_l = a_r - a_{r-1} + ... + a_{l+1} - a_l = \begin{pmatrix} 1 \\ r \\ \binom{r}{2} \\ \vdots \\ \binom{r}{m} \end{pmatrix} + ... + \begin{pmatrix} 1 \\ l \\ \binom{l}{2} \\ \vdots \\ \binom{l}{m} \end{pmatrix}, \ M+1 \le l < r \le n \ . \tag{2.13}$$

Developing determinant (2.11) according to the first row, we get

$$\Delta_t = \sum_{j=1}^{t-1} (-1)^j (i_{j+1} - i_j) M_{1j} + (-1)^t (k - i_t) M_{1t} \ , \tag{2.14}$$

where $M_{1j}, j = 1, ..., t$ are the minors of (2.9) formed by eliminating row 1 and column j.

We remark that if we use a similar column subtraction procedure as we applied above, then, by the use of (2.12) and (2.13), the minors M_{1j} , $j = 1, ..., t$ can be further split into a sum of determinants. Any determinant in the obtained sum is either zero, or positive because they are the minors crossed out of the matrix Pascal matrix P in (2.6). This implies that minors M_{1j}, $j = 1, ..., t$, are positive. □

Returning to the proof of the Theorem 1, let us first consider the case $I_B \subset \{k, ..., n\}$ in the maximization problem. Then (2.10) implies that $c_p - c_B^T B^{-1} a_p \leq 0$ for every nonbasic a_p and $c_p - c_B^T B^{-1} a_p = 0$ when $c_p = 0$. This proves that $I_B \subset \{k, ..., n\}$ is a dual degenerate basis in the maximization problem.

Let $h(k)$ designate the number of subscripts that are smaller than k in matrix B. Consider the minimization problem and assume that $m + 1$ is even. Let $1 \leq k \leq n - 2$. If $h(k)$ is odd, then the vectors $a_{k-1}, a_k, a_{k+1}, a_n$ must be in the basis. Otherwise, for $p = k - 1, k, k + 1, n$ the determinant (2.10) is negative. This implies that $c_p - c_B^T B^{-1} a_p$ is negative and hence, B cannot be dual feasible. If there are further vectors in the basis, the subscripts of which are smaller than $k - 1$ or greater than $k + 1$, then those must form consecutive pairs to ensure that, by putting $\binom{c_p}{a_p}$ in its right place (the column subscripts are in increasing order) in the determinant $\begin{vmatrix} c_p & c_B^T \\ a_p & B \end{vmatrix}$, the determinant in (2.10) is positive, which implies that $c_p - c_B^T B^{-1} a_p > 0$.

If we assume that $2 \leq k \leq n - 1$ and $h(k)$ is even, the second structure is obtained by including a_0 and deleting a_n. Note that this does not mean that a_n

cannot be in the basis if it does not appear in the structure. The third structure (minimization problem, $m+1$ is even) is obtained by assuming $h(k)$ is even and the basis is formed by including the consecutive pairs less than k or consecutive pairs greater than or equal to k. We remark that in the third structure a_0 or a_n can be in the basis, but if a_0 is in the basis, then a_1 must be in the basis. Similarly, if a_n is in the basis, then a_{n-1} is in the basis as well.

In order to obtain the dual feasible basis structures in the minimization problem when $m+1$ is odd we assume that $1 \leq k \leq n-1$ and $h(k)$ is odd. In this case the fourth structure can be obtained from the first structure by deleting a_n. Similarly, if $2 \leq k \leq n-2$ and $h(k)$ is even, then the fifth structure is obtained from the second one by including a_n. The sixth structure is obtained from the third structure by assuming $h(k)$ is odd and including a_0 in the basis. Similarly, the seventh structure is obtained from the third structure by assuming $h(k)$ is even and including a_n in the basis.

The justification of the nondegenerate dual feasible basis structures in the maximization problem can be obtained using the ideas discussed above and therefore, is left to the reader. $\square$

Theorem 1 fully describes the dual feasible basis structures for the relaxed versions of problems (2.4) and (2.5) for the case of $1 \leq k \leq M$ to obtain bounds for the probability that at least k-out-of-n events occur, $P(\xi \geq k)$, where the underlying probability distribution is unimodal with mode M and we know the first $m+1$ binomial moments.

2.2 Characterization of dual feasible bases for $P(\xi \geq k), M+1 \leq k \leq n$

In this section we characterize the dual feasible basis structures for problem (2.1), where k is greater than the mode M, that is, $M+1 \leq k \leq n$. If we use representation (2.2) in problem (2.1), we obtain the next problem:

$$\min(\max)\ \{v_k + 2v_{k+1} + ... + (n-k+1)v_n\}$$

subject to

$$\sum_{i=0}^{M}(M-i+1)v_i + \sum_{i=M+1}^{n}(i-M)v_i = 1$$

$$\sum_{i=0}^{M}\left[\binom{i}{j} + \cdots + \binom{M}{j}\right]v_i + \sum_{i=M+1}^{n}\left[\binom{M+1}{j} + \cdots + \binom{i}{j}\right]v_i = S_j, j = 1, ..., m$$

$$v_0 + ... + v_M - v_{M+1} - ... - v_n \geq 0$$

$$v_i \geq 0\ , \quad i = 0, \cdots, n\ . \tag{2.15}$$

In case of representation (2.3) the problem can be formulated as follows:

$$\min(\max)\ \{v_k + 2v_{k+1} + ... + (n-k+1)v_n\}$$

subject to

$$\sum_{i=0}^{M-1}(M-i)v_i + \sum_{i=M}^{n}(i-M+1)v_i = 1$$

$$\sum_{i=0}^{M-1}\left[\binom{i}{j} + \cdots + \binom{M-1}{j}\right]v_i + \sum_{i=M}^{n}\left[\binom{M}{j} + \cdots + \binom{i}{j}\right]v_i = S_j, j = 1, ..., m$$

$$v_M + ... + v_n - v_0 - ... - v_{M-1} \geq 0$$

$$v_i \geq 0\ ,\quad i = 0, \cdots, n\ . \tag{2.16}$$

In what follows problem (2.15) without the constraint "$v_0 + ... + v_M - v_{M+1} - ... - v_n \geq 0$" and problem (2.16) without "$v_M + ... + v_n - v_0 - ... - v_{M-1} \geq 0$" will be called the relaxed problems.

In the following theorem we fully describe dual feasible bases structures of the relaxed versions of problems (2.15) and (2.16). Note that the only difference between problems (2.4)-(2.5) (when k is less than or equal to the mode M) and problems (2.15)-(2.16) (when k is greater than the mode M) is observed in the objective function. Thus, some of the assertions from Section 2.1 also apply here.

Theorem 2. *Any dual feasible basis of any of the relaxed problems* (2.15) *and* (2.16) *has one of the following structures, presented in terms of the subscripts, where* I_B *is the set of subscripts of the vectors that are in the basis* B. *In addition all dual feasible bases are dual nondegenerate, except for those with* $I_B \subset \{0, ..., k-1\}$ *which are dual degenerate.*

<u>Minimization problem, $m+1$ even</u>

- $I_B \subset \{0, ..., k-1\}$ *if* $k \geq m+1$
- $\{i, i+1, ..., j, j+1, k-1, r, r+1, ..., t, t+1, n\}$ *if* $0 \leq k-1 \leq n-1$
- $\{0, i, i+1, ..., j, j+1, k-1, r, r+1, ..., t, t+1\}$ *if* $1 \leq k-1 \leq n$

<u>Minimization problem, $m+1$ odd</u>

- $I_B \subset \{0, ..., k-1\}$ *if* $k \geq m+1$
- $\{i, i+1, ..., j, j+1, k-1, r, r+1, ..., t, t+1\}$ *if* $0 \leq k-1 \leq n-1$
- $\{0, i, i+1, ..., j, j+1, k-1, r, r+1, ..., t, t+1, n\}$ *if* $1 \leq k-1 \leq n-1$

<u>Maximization problem, $m+1$ even</u>

- $\{i, i+1, ..., j, j+1, k-2, k-1, k, r, r+1, ..., t, t+1, n\}$ *if* $1 \leq k-1 \leq n-2$
- $\{0, i, i+1, ..., j, j+1, k-2, k-1, k, r, r+1, ..., t, t+1\}$ *if* $2 \leq k-1 \leq n-1$
- $\{i, i+1, ..., j, j+1, r, r+1, ..., t, t+1\}$ *if* $j+1 < k-1 \leq r$

<u>Maximization problem, $m+1$ odd</u>

- $\{i, i+1, ..., j, j+1, k-2, k-1, k, r, r+1, ..., t, t+1\}$ *if* $1 \leq k-1 \leq n-1$
- $\{0, i, i+1, ..., j, j+1, k-2, k-1, k, r, r+1, ..., t, t+1, n\}$ *if* $2 \leq k-1 \leq n-2$
- $\{0, i, i+1, ..., j, j+1, r, r+1, ..., t, t+1\}$ *if* $j+1 < k-1 \leq r$
- $\{i, i+1, ..., j, j+1, r, r+1, ..., t, t+1, n\}$ *if* $j+1 < k-1 \leq r$.

Proof. We shall prove Theorem 2 for problem (2.15). The discussions can be easily carried over to the case of problem (2.16). Table 2.2 gives in detailed form the matrix $\binom{c^T}{A}$, where A is the coefficient matrix of the equality constraints and c^T is the vector of objective function coefficients in problem (2.15).

Table 2.2: Matrix of equality constraints in (2.15)

	0	1	...	M	$M+1$	...	k	$k+1$	...	n
c^T	0	0	...	0	0	...	1	2	...	$n-k+1$
0	$M+1$	M	...	1	1	...	$k-M$	$k-M+1$	...	$n-M$
1	$0+...+M$	$1+...+M$	...	M	$M+1$	...	$M+1+...+k$	$M+1+...+k+1$	...	$M+1+...+n$
2	$\binom{2}{2}+...+\binom{M}{2}$	$\binom{2}{2}+...+\binom{M}{2}$	...	$\binom{M}{2}$	$\binom{M+1}{2}$	...	$\binom{M+1}{2}+...+\binom{k}{2}$	$\binom{M+1}{2}+...+\binom{k+1}{2}$	...	$\binom{M+1}{2}+...+\binom{n}{2}$
$\vdots$	$\vdots$	$\vdots$	...	$\vdots$	$\vdots$	...	$\vdots$	$\vdots$	...	$\vdots$
m	$\binom{m}{m}+...+\binom{M}{m}$	$\binom{m}{m}+...+\binom{M}{m}$	...	$\binom{M}{m}$	$\binom{M+1}{m}$	...	$\binom{M+1}{m}+...+\binom{k}{m}$	$\binom{M+1}{m}+...+\binom{k+1}{m}$	...	$\binom{M+1}{m}+...+\binom{n}{m}$

Let B be an arbitrary basis in problem (2.15). Since problems (2.4) and (2.15) share the same constraints, we know $|B| > 0$ and hence, the sign of $c_p - c_B^T B^{-1} a_p$ is determined by the sign of the determinant $\begin{vmatrix} c_p & c_B^T \\ a_p & B \end{vmatrix}$ as discussed in the proof of Theorem 1. In this case we have the following assertion.

Lemma 2. *For* $0 \le i_1 < ... < i_s < ... < i_{m+2} \le n$ *let us define*

$$\Delta_s = \begin{vmatrix} 0 & 0 & \cdots & 0 & 0 & i_{s+1}-k+1 & i_{s+2}-k+1 & \cdots & i_{m+2}-k+1 \\ a_{i_1} & a_{i_2} & \cdots & a_{i_{s-1}} & a_{i_s} & a_{i_{s+1}} & a_{i_{s+2}} & \cdots & a_{i_{m+2}} \end{vmatrix}, \tag{2.17}$$

where $0 \le i_s \le M < M+1 \le k \le n$*,* $1 \le s \le m+1$*, and* $a_{i_s} \in \mathbb{R}^m$ *are the columns of* A*. Then*

$$\Delta_s = (-1)^{s+1}(i_{s+1}-k+1)M_{1,s+1} + \sum_{j=s+2}^{m+2} (-1)^j (i_j - i_{j-1}) M_{1j} \ , \tag{2.18}$$

and $M_{1j}, j = s+1, ..., m+2$ *are positive, where* $M_{1j}, j = s+1, ..., m+2$ *are the minors of (2.17) formed by eliminating row 1 and column* j *.*

Proof. One can show that, by the use of elementary column subtraction operations used in the proof of Lemma 1, the determinant Δ_s in (2.17) can be obtained as

$$\begin{vmatrix} 0 & \cdots & 0 & 0 & i_{s+1}-k+1 & i_{s+2}-i_{s+1} & \cdots & i_{m+2}-i_{m+1} \\ a_{i_1}-a_{i_2} & \cdots & a_{i_{s-1}}-a_{i_s} & a_{i_s} & a_{i_{s+1}} & a_{i_{s+2}}-a_{i_{s+1}} & \cdots & a_{i_{m+2}}-a_{i_{m+1}} \end{vmatrix}, \tag{2.19}$$

where $0 \le i_s \le M < M+1 \le k \le n$ and $1 \le s \le m+1$.

Note that the assertions (2.12) and (2.13) are also valid for the determinant in

(2.19). Thus, developing it according to the first row, we obtain

$$\Delta_s = (-1)^{s+1}(i_{s+1} - k + 1)M_{1,s+1} + \sum_{j=s+2}^{m+2} (-1)^j (i_j - i_{j-1}) M_{1j} \ ,$$

where $M_{1j}, j = s+1, ..., m+2$ are the minors of (2.17) formed by eliminating row 1 and column j. A similar column subtraction procedure as we applied in the proof of Lemma 1 together with (2.12) and (2.13) enables us to split the minors M_{1j} , $j = s+1, ..., m+2$ into a sum of determinants some of which are zero or positive. This completes the proof of Lemma 2. □

Now we return to the proof of Theorem 2. First consider the case $I_B \subset \{0, ..., k-1\}$ in the minimization problem. Then (2.18) implies that $c_p - c_B^T B^{-1} a_p \geq 0$ for every nonbasic a_p and $c_p - c_B^T B^{-1} a_p = 0$ when $c_p = 0$, that is, when $p \leq k-1$. This proves that $I_B \subset \{0, ..., k-1\}$ is a dual degenerate basis in the minimization problem when $m+1$ is even or odd.

Let $h(k)$ designate the number of subscripts that are smaller than $k-1$ in basis B. Consider the minimization problem and assume that $m+1$ is even. Let $0 \leq k \leq n-1$. If $h(k)$ is even, then the vectors a_{k-1}, a_n must be in the basis. Otherwise, for $p = k-1, n$ the determinant (2.18) is negative and hence, basis B cannot be dual feasible. If there are further vectors in the basis, then those must form consecutive pairs to ensure that, by putting $\binom{c_p}{a_p}$ in its right place (the column subscripts are in increasing order) in the determinant the determinant in (2.18) is positive, which implies that $c_p - c_B^T B^{-1} a_p > 0$ and B is dual feasible in the minimization problem. If $2 \leq k \leq n-1$ and $h(k)$ is odd, the third structure is obtained by including a_0 and deleting a_n. Note that this does not mean that a_0 or a_n cannot be in the basis if one does not appear in the structure.

Now, let us assume that $m+1$ is odd in the minimization problem. If $h(k)$, $0 \leq k-1 \leq n-1$, is even, then the fourth basis structure is obtained from the second one by deleting a_n. Similarly, if $h(k)$, $1 \leq k-1 \leq n-1$, is odd, then the fifth structure is obtained by including a_0, a_{k-1}, a_n and any number of consecutive pairs to form an $(m+1) \times (m+1)$ matrix for which the determinant Δ_s in (2.17) is positive, i.e., B is dual feasible.

The seventh dual feasible basis structure (maximization problem, $m+1$ is even) is obtained by assuming $h(k)$, $1 \leq k-1 \leq n-2$, is odd. In this case, a_{k-2}, a_{k-1}, a_k, and a_n must be in the basis, otherwise the determinant $\Delta_s > 0$, and thus, B cannot be dual feasible. If $2 \leq k-1 \leq n-1$ and $h(k)$ is even, then the eight structure is obtained from the seventh one by removing a_n and including a_0 in the basis. In the ninth structure we choose any number of consecutive pairs less than $k-1$ and any number of consecutive pairs greater than or equal to $k-1$ to form an $(m+1) \times (m+1)$ matrix for which Δ_s is negative and therefore the corresponding basis is dual feasible in the maximization problem. We remark that a_0, a_{k-1}, or a_n can appear in this structure, however, if a_0 is in the basis than a_1 must be in the basis. Similarly if a_{k-1} or a_n is in the basis, then a_k or a_{n-1} must be respectively included in the ninth structure. The justification of the nondegenerate dual feasible basis structures in the maximization problem for the case of $m+1$ being odd does not require new ideas and therefore, is left to the reader. □

Theorems 1 and 2 fully describes the dual feasible basis structures for the relaxed versions of problems (2.4) and (2.15) for the cases of $1 \leq k \leq M$ and $M+1 \leq k \leq n$, respectively. The characterization of the basis structures enables us to solve those problems algorithmically to obtain bounds for the probability that at least k-out-of-n events occur, $P(\xi \geq k)$, where the underlying probability distribution is unimodal with mode M and we know the first m binomial moments when m is large. For small m values bounds for $P(\xi \geq k)$ can be obtained in the form of formulas. Next, we present dual feasible basis structure theorems in connection with $P(\xi = k)$ under unimodality condition.

Chapter 3

Bounding the Probability that Exactly k-out-of-n Events Occur

Consider the binomial moment problem (1.5). If we assume that the probability distribution of the occurrences is unimodal with mode M $(0 \leq M \leq n)$, i.e., condition (1.6) is satisfied, we obtain the following linear programming problem:

$$\begin{aligned} & \min(\max) p_k \\ \text{subject to} & \\ & \sum_{i=0}^{n} \binom{i}{j} p_i = S_j, \quad j = 0, 1, \cdots, m \\ & p_0 \leq ... \leq p_M \\ & p_M \geq ... \geq p_n \\ & p_i \geq 0, \quad i = 0, 1, \cdots, n \end{aligned} \tag{3.1}$$

Then the two optimum values of problem (3.1) provide us with best possible lower and upper bounds for the probability that exactly k-out-of-n events occur, $P(\xi = k)$, where $0 \leq k \leq n$, under the assumption that the probability distribution of the random variable ξ is unimodal with mode M.

As in Chapter 2 we consider the two cases where k is less than or equal to or is greater than the mode M. The reformulation of problem (3.1) for those cases require special attention and shall be presented in separate sections.

3.1 Characterization of dual feasible bases for $P(\xi = k), 0 \leq k \leq M$

We assume that $0 \leq k \leq M$. In case of representation (2.2) problem (3.1) can be formulated as

$$\min(\max)\ \{v_0 + ... + v_k\}$$

subject to

$$\sum_{i=0}^{M}(M-i+1)v_i + \sum_{i=M+1}^{n}(i-M)v_i = 1$$

$$\sum_{i=0}^{M}\left[\binom{i}{j} + \cdots + \binom{M}{j}\right] v_i + \sum_{i=M+1}^{n}\left[\binom{M+1}{j} + \cdots + \binom{i}{j}\right] v_i = S_j, j = 1, ..., m$$

$$v_0 + ... + v_M - v_{M+1} - ... - v_n \geq 0 \tag{3.2}$$

$$v_i \geq 0\ , \quad i = 0, \cdots, n\ .$$

By the use of representation (2.3) in problem (3.1), we obtain the following problem

$$\min(\max)\ \{v_0 + ... + v_k\}$$

subject to

$$\sum_{i=0}^{M-1}(M-i)v_i + \sum_{i=M}^{n}(i-M+1)v_i = 1$$

$$\sum_{i=0}^{M-1}\left[\binom{i}{j} + \cdots + \binom{M-1}{j}\right]v_i + \sum_{i=M}^{n}\left[\binom{M}{j} + \cdots + \binom{i}{j}\right]v_i = S_j, j = 1, ..., m$$

$$v_M + ... + v_n - v_0 - ... - v_{M-1} \geq 0 \tag{3.3}$$

$$v_i \geq 0\ , \quad i = 0, \cdots, n\ .$$

We shall call problem (3.2) without the constraint "$v_0 + ... + v_M - v_{M+1} - ... - v_n \geq 0$" and problem (3.3) without "$v_M + ... + v_n - v_0 - ... - v_{M-1} \geq 0$" the relaxed problems.

As before for both relaxed problems $A = (a_0, ..., a_n)$ shall designate the coefficient matrix of the equality constraints, b the right hand side vector, and c the vector of objective function coefficients. Table 3.1 gives in detailed form the matrix $\binom{c^T}{A}$ in problem (3.2).

Table 3.1: Matrix of equality constraints in (3.2)

	0	1	...	k	$k+1$	...	M	$M+1$	$M+2$	...	n
c^T	1	1	...	1	0	...	0	0	0	...	0
0	$M+1$	M	...	$M-k+1$	$M-k$	...	1	1	2	...	$n-M$
1	$0+...+M$	$1+...+M$	...	$k+...+M$	$k+1+...+M$	...	M	$M+1$	$M+1+M+2$	...	$M+1+...+n$
2	$\binom{2}{2}+...+\binom{M}{2}$	$\binom{2}{2}+...+\binom{M}{2}$	...	$\binom{k}{2}+...+\binom{M}{2}$	$\binom{k+1}{2}+...+\binom{M}{2}$	...	$\binom{M}{2}$	$\binom{M+1}{2}$	$\binom{M+1}{2}+\binom{M+2}{2}$	...	$\binom{M+1}{2}+...+\binom{n}{2}$
$\vdots$	$\vdots$	$\vdots$	...	$\vdots$	$\vdots$	...	$\vdots$	$\vdots$	$\vdots$	...	$\vdots$
m	$\binom{m}{m}+...+\binom{M}{m}$	$\binom{m}{m}+...+\binom{M}{m}$	...	$\binom{k}{m}+...+\binom{M}{m}$	$\binom{k+1}{m}+...+\binom{M}{m}$	...	$\binom{M}{m}$	$\binom{M+1}{m}$	$\binom{M+1}{m}+\binom{M+2}{m}$	...	$\binom{M+1}{m}+...+\binom{n}{m}$

The following theorem characterizes the dual feasible bases in relaxed versions of problems (3.2) and (3.3).

Theorem 3. *A basis B in (3.2) or (3.3) is dual feasible if and only if it has one of the following structures in terms of the subscripts of the basic vectors, where in all parentheses the numbers are arranged in increasing order. Let I_B denote the basis subscript. Those bases for which $I_B \subset \{k+1,...,n\}, n-k \geq m+1$ and $I_B \subset \{0,...,k\}, k \geq m$ are dual degenerate in the minimization and maximization problems, respectively. The bases in all other cases are dual nondegenerate.*

Minimization problem, $m+1$ even

- $I_B \subset \{k+1,...,n\}$ *if* $n-k \geq m+1$
- $\{0,i,i+1,...,j,j+1,k+1,r,r+1,...,t,t+1\}$ *if* $0 \leq k \leq n-1$
- $\{i,i+1,...,j,j+1,k+1,r,r+1,...,t,t+1,n\}$ *if* $0 \leq k \leq n-2$

Minimization problem, $m+1$ odd

- $I_B \subset \{k+1,...,n\}$ *if* $n-k \geq m+1$
- $\{0,i,i+1,...,j,j+1,k+1,r,r+1,...,t,t+1,n\}$ *if* $0 \leq k \leq n-2$
- $\{i,i+1,...,j,j+1,k+1,r,r+1,...,t,t+1\}$ *if* $0 \leq k \leq n-1$

Maximization problem, $m+1$ even

- $I_B \subset \{0,...,k\}$ *if* $k \geq m$
- $\{i,i+1,...,j,j+1,k,r,r+1,...,t,t+1,n\}$ *if* $0 \leq k \leq n-1$
- $\{0,i,i+1,...,j,j+1,k,r,r+1,...,t,t+1\}$ *if* $1 \leq k \leq n$

Maximization problem, $m+1$ odd

- $I_B \subset \{0,...,k\}$ *if* $k \geq m$
- $\{i,i+1,...,j,j+1,k,r,r+1,...,t,t+1\}$ *if* $0 \leq k \leq n$
- $\{0,i,i+1,...,j,j+1,k,r,r+1,...,t,t+1,n\}$ *if* $1 \leq k \leq n-1$.

Proof. We carry out the proof for the relaxed problem (3.2). The proof of the assertion for the relaxed problem (3.3) goes along the same line. Let B be an arbitrary basis and a_p a nonbasic vector. From the proof of Theorem 1 we know that $|B| > 0$, hence the sign of $c_p - c_B^T B^{-1} a_p$ is determined by the sign of the determinant $\begin{vmatrix} c_p & c_B^T \\ a_p & B \end{vmatrix}$. In order to determine the sign of this determinant we first prove the following lemma.

Lemma 3. *Let* $0 \le i_1 < ... < i_t < ... < i_{m+2} \le n$ *and consider the determinant*

$$\nabla_t = \begin{vmatrix} 1 & 1 & \cdots & 1 & 0 & 0 & \cdots & 0 \\ a_{i_1} & a_{i_2} & \cdots & a_{i_t} & a_{i_{t+1}} & a_{i_{t+2}} & \cdots & a_{i_{m+2}} \end{vmatrix} \tag{3.4}$$

where $0 \le i_t \le k < i_{t+1}$, $1 \le t \le m+1$, *and* $a_{i_t} \in \mathbb{R}^m$ *are the columns of* A. *Then*

$$(-1)^{t+1} \nabla_t > 0 \ . \tag{3.5}$$

Proof. First note that the constraints of problem (2.4) are the same. Therefore, the columns $a_i, i = 0, ..., n$ of the coefficient matrix A in (3.2) satisfy (2.12) and (2.13). Applying the columns substraction procedure used in the proof of Lemma 1 and then developing the determinant (3.4) according to the first row, we get

$$\nabla_t = (-1)^{t+1} | \, a_{i_1} - a_{i_2}, a_{i_2} - a_{i_3}, ..., a_{i_{t-1}} - a_{i_t}, a_{i_{t+1}}, a_{i_{t+2}} - a_{i_{t+1}}, ..., a_{i_{m+2}} - a_{i_{m+1}} | \ .$$

If we substitute the vector differences by the sums in accordance with (2.12) and (2.13), then $(-1)^{t+1}\nabla_t$ can be split into the sum of

$$i_{t+1}(i_1 - i_2)...(i_{t-1} - i_t)(i_{t+2} - i_{t+1})...(i_{m+2} - i_{m+1})$$

determinants. Using a column subtraction procedure similar to the one we applied above the determinants in the sum can be obtained as the minors of Pascal matrix in which all positive entries stand in the upper triangle. This implies that $(-1)^{t+1}\nabla_t > 0$. Thus, Lemma 3 is proved. $\square$

Let us get back to the proof of Theorem 3. It is easy to see that if $I_B \subset \{k+1, ..., n\}$ in the minimization problem, then $c_p - c_B^T B^{-1} a_p \geq 0$ and if $I_B \subset \{0, ..., k\}$ in the maximization problem, then $c_p - c_B^T B^{-1} a_p \leq 0$ for every nonbasic a_p. This proves the dual feasibility of I_B. We also observe that if $c_p = 0$ in the minimization problem and $c_p = 1$ in the maximization problem, then $c_p - c_B^T B^{-1} a_p = 0$. Thus, the bases I_B in all four cases is dual degenerate.

Similar to the proofs of Lemmas 1 and 2, here too we work with the number of subscripts, designated by $h(k)$, in B which are less than or equal to k.

First consider the minimization problem where $m+1$ is even. Assume that $h(k), 0 \leq k \leq n-1$ is odd. First note that if $p = 0$ or $p = k+1$, then the determinant $\begin{vmatrix} c_p & c_B^T \\ a_p & B \end{vmatrix}$ is positive since B is dual feasible in the minimization problem, however, (3.5) implies that it is negative, which is a contradiction. Therefore a_0 and a_{k+1} must be in the basis.

If there are further vectors in the basis, the subscripts of which are smaller than or equal to k or greater than $k+1$, then those must form consecutive pairs to ensure that, by putting $\binom{c_p}{a_p}$ in its right place (the column subscripts are in increasing order) in the determinant $\begin{vmatrix} c_p & c_B^T \\ a_p & B \end{vmatrix}$, it passes an even number of columns. By Lemma 3, we have $c_p - c_B^T B^{-1} a_p > 0$. Hence, we obtain the second dual feasible basis structure.

If we assume that $0 \leq k \leq n-2$ and $h(k)$ is even, the third structure is obtained by including a_n and deleting a_0. We remark that a_n and a_n can be a basic vector in the second and third structures, respectively. However, if a_n (or a_0) is included in the second (or third) structure, then a_{n-1} (or a_1) must be in the basis as well.

Now let us consider the minimization problem where $m+1$ is odd and assume that $0 \leq k \leq n-2$ and $h(k)$ is odd, then the fifth structure in the minimization problem is obtained from the third one by including a_0. If $0 \leq k \leq n-1$ and $h(k)$

is even, then the sixth structure is obtained from the second structure by deleting a_0.

The dual feasibility of the bases in the maximization problem can similarly be verified where the number of basic subscripts, designated by $g(k)$, smaller than k are assumed to be even or odd. □

3.2 Characterization of dual feasible bases for $P(\xi = k), M+1 \leq k \leq n$

Now we assume that $M+1 \leq k \leq n$. In case of representation (2.2) problem (3.1) can be formulated as

$$\min(\max)\ \{v_k + ... + v_n\}$$

subject to

$$\sum_{i=0}^{M}(M-i+1)v_i + \sum_{i=M+1}^{n}(i-M)v_i = 1$$

$$\sum_{i=0}^{M}\left[\binom{i}{j} + \cdots + \binom{M}{j}\right]v_i + \sum_{i=M+1}^{n}\left[\binom{M+1}{j} + \cdots + \binom{i}{j}\right]v_i = S_j, j = 1, ..., m$$

$$v_0 + ... + v_M - v_{M+1} - ... - v_n \geq 0 \tag{3.6}$$

$$v_i \geq 0\ , \quad i = 0, \cdots, n\ .$$

By the use of representation (2.3) in problem (3.1), we obtain the following problem

$$\min(\max)\ \{v_k + ... + v_n\}$$

subject to

$$\sum_{i=0}^{M-1}(M-i)v_i + \sum_{i=M}^{n}(i-M+1)v_i = 1$$

$$\sum_{i=0}^{M-1}\left[\binom{i}{j} + \cdots + \binom{M-1}{j}\right]v_i + \sum_{i=M}^{n}\left[\binom{M}{j} + \cdots + \binom{i}{j}\right]v_i = S_j, j = 1, ..., m$$

$$v_M + ... + v_n - v_0 - ... - v_{M-1} \geq 0 \tag{3.7}$$

$$v_i \geq 0\ , \quad i = 0, \cdots, n\ .$$

Problem (3.6) without the constraint "$v_0 + ... + v_M - v_{M+1} - ... - v_n \geq 0$" and problem (3.7) without "$v_M + ... + v_n - v_0 - ... - v_{M-1} \geq 0$" shall be called relaxed problems.

Let $A = (a_0, ..., a_n)$ designate the coefficient matrix of the equality constraints, b the right hand side vector, and c the vector of objective function coefficients. Table 3.2 gives in detailed form the matrix $\binom{c^T}{A}$ in problem (3.6).

Table 3.2: Matrix of equality constraints in (3.2)

	0	1	...	M	$M+1$	...	k	$k+1$	...	n
c^T	0	0	...	0	0	...	1	2	...	$n-k+1$
0	0	0	...	0	0	...	1	1	...	1
1	$0+...+M$	$1+...+M$	...	M	$M+1$	...	$M+1+...+k$	$M+1+...+k+1$	...	$M+1+...+n$
2	$\binom{2}{2}+...+\binom{M}{2}$	$\binom{2}{2}+...+\binom{M}{2}$	...	$\binom{M}{2}$	$\binom{M+1}{2}$	...	$\binom{M+1}{2}+...+\binom{k}{2}$	$\binom{M+1}{2}+...+\binom{k+1}{2}$	...	$\binom{M+1}{2}+...+\binom{n}{2}$
$\vdots$	$\vdots$	$\vdots$	...	$\vdots$	$\vdots$	...	$\vdots$	$\vdots$	...	$\vdots$
m	$\binom{m}{m}+...+\binom{M}{m}$	$\binom{m}{m}+...+\binom{M}{m}$	...	$\binom{M}{m}$	$\binom{M+1}{m}$	...	$\binom{M+1}{m}+...+\binom{k}{m}$	$\binom{M+1}{m}+...+\binom{k+1}{m}$	...	$\binom{M+1}{m}+...+\binom{n}{m}$

The next theorem characterizes the dual feasible basis in the relaxed problems (3.6) and (3.7). Since the proof of Theorem 4 is similar to that of Theorem 3 we leave it to the reader.

Theorem 4. *Let I_B be the basis subscript set. A basis in (3.6) or (3.7) is dual feasible if and only if it has one of the following structures in terms of the subscripts of the basic vectors, where in all parentheses the numbers are arranged in increasing order. Those bases for which $I_B \subset \{0, ..., k-1\},\ k \geq m+1$ in the minimization problem and $I_B \subset \{k, ..., n\},\ n-k \geq m$ in the maximization problem are dual degenerate. The bases in all other cases are dual nondegenerate.*

<u>Minimization problem, $m+1$ even</u>

- $I_B \subset \{0, ..., k-1\}$ *if* $k \geq m+1$
- $\{0, i, i+1, ..., j, j+1, k-1, r, r+1, ..., t, t+1\}$ *if* $2 \leq k \leq n-1$
- $\{i, i+1, ..., j, j+1, k-1, r, r+1, ..., t, t+1, n\}$ *if* $1 \leq k \leq n$

<u>Minimization problem, $m+1$ odd</u>

- $I_B \subset \{0, ..., k-1\}$ *if* $k \geq m+1$
- $\{0, i, i+1, ..., j, j+1, k-1, r, r+1, ..., t, t+1, n\}$ *if* $2 \leq k \leq n$
- $\{i, i+1, ..., j, j+1, k-1, r, r+1, ..., t, t+1\}$ *if* $1 \leq k \leq n-1$

<u>Maximization problem, $m+1$ even</u>

- $I_B \subset \{k, ..., n\}$ *if* $n-k \geq m$
- $\{i, i+1, ..., j, j+1, k, r, r+1, ..., t, t+1, n\}$ *if* $1 \leq k \leq n-1$
- $\{0, i, i+1, ..., j, j+1, k, r, r+1, ..., t, t+1\}$ *if* $1 \leq r \leq n$

<u>Maximization problem, $m+1$ odd</u>

- $I_B \subset \{k, ..., n\}$ *if* $n-k \geq m$
- $\{i, i+1, ..., j, j+1, k, r, r+1, ..., t, t+1\}$ *if* $1 \leq k \leq n$
- $\{0, i, i+1, ..., j, j+1, k, r, r+1, ..., t, t+1, n\}$ *if* $1 \leq k \leq n-1$.

Proof. We prove the theorem for the case of the relaxed problem (3.6). The proof for the relaxed problem (3.7) can be carried out in a similar way. Let B be an arbitrary basis and a_p a nonbasic vector. First note that the constraints of problems (2.4) and (3.6) are the same. Therefore, from the proof of Theorem 1, we know that $|B| > 0$ and the sign of $c_p - c_B^T B^{-1} a_p$ is determined by the sign of the determinant $\begin{vmatrix} c_p & c_B^T \\ a_p & B \end{vmatrix}$. In connection with this determinant we have the following assertion.

Lemma 4. *Let* $0 \le i_1 < ... < i_s < ... < i_{m+2} \le n$ *and consider the determinant*

$$\nabla_s = \begin{vmatrix} 0 & 0 & \cdots & 0 & 1 & 1 & \cdots & 1 \\ a_{i_1} & a_{i_2} & \cdots & a_{i_s} & a_{i_{s+1}} & a_{i_{s+2}} & \cdots & a_{i_{m+2}} \end{vmatrix} \tag{3.8}$$

where $0 \le i_s < k \le i_{s+1}$, $1 \le s \le m+1$, *and* $a_{i_s} \in \mathbb{R}^m$ *are the columns of* A. *Then*

$$(-1)^{s+2} \nabla_s > 0 \ . \tag{3.9}$$

Proof. Since problems (2.4) and (3.6) share the same constraints (2.12) and (2.13) hold true for the columns $a_i, i = 0, ..., n$ of the coefficient matrix A in (3.6). Applying the columns substraction procedure used in the proof of Lemma 1 and then developing the determinant (3.8) according to the first row, we get

$$\nabla_s = (-1)^{s+1} |\ a_{i_1} - a_{i_2}, a_{i_2} - a_{i_3}, ..., a_{i_{s-1}} - a_{i_s}, a_{i_s}, a_{i_{s+2}} - a_{i_{s+1}}, ..., a_{i_{m+2}} - a_{i_{m+1}} |\ .$$

If we substitute the vector differences by the sums in accordance with (2.12) and (2.13), then $(-1)^{s+2}\nabla_s$ can be split into the sum of

$$i_s(i_1 - i_2)...(i_{s-1} - i_s)(i_{s+2} - i_{s+1})...(i_{m+2} - i_{m+1})$$

determinants. Using a series similar column subtraction operations the determinants in the sum can be obtained as the minors of Pascal matrix in which all positive entries stand in the upper triangle. This implies that $(-1)^{s+2}\nabla_s > 0$. □

The proof of Theorem 4 follows directly from the assertion in Lemma 4 and the ideas used in the proof of Theorems 1-3.

□

Theorems 3 and 4 fully describes the dual feasible basis structures for the relaxed versions of problems (3.2) and (3.6) for the cases of $0 \leq k \leq M$ and $M+1 \leq k \leq n$, respectively. The characterization of the basis structures enables us to solve those problems to obtain closed form bounds for the probability that exactly k-out-of-n events occur, $P(\xi = k)$, where the underlying probability distribution is unimodal with mode M and we know the first m binomial moments, when m is small. For larger values of m we can solve the LPs (3.2) and (3.6) to obtain sharp bounds for $P(\xi = k)$.

Chapter 4

Closed Form and Algorithmic Bounds

In this chapter we shall present closed form bounds for the probabilities that at least k and exactly k-out-of-n events occur, based on the first two binomial moments S_1 and S_2.

4.1 Closed form bounds for $P(\xi \geq k),\ 1 \leq k \leq M$, based on the binomial moments S_1, S_2

Let $1 \leq k \leq M$. We consider the relaxed problem (2.4) and present closed form bounds for $P(\xi \geq k)$, the probability that at least k out of n events occur when $m = 2$. If only the first two binomial moments S_1 and S_2 are known, the relaxed version of problem (2.4) can be written in the form (see [47]):

$$\min(\max)\ \{(M-k+1)\sum_{i=0}^{k} v_i + \sum_{i=k+1}^{M} (M-i+1)v_i + \sum_{i=M+1}^{n} (i-M)v_i\}$$

subject to

$$\begin{gathered}
\sum_{i=0}^{M}(M-i+1)v_i + \sum_{i=M+1}^{n} (i-M)v_i = 1 \\
\sum_{i=0}^{M}\left[\binom{M+1}{2} - \binom{i}{2}\right]v_i + \sum_{i=M+1}^{n}\left[\binom{i+1}{2} - \binom{M+1}{2}\right]v_i = S_1 \\
\sum_{i=0}^{M}\left[\binom{M+1}{3} - \binom{i}{3}\right]v_i + \sum_{i=M+1}^{n}\left[\binom{i+1}{3} - \binom{M+1}{3}\right]v_i = S_2 \\
v_i \geq 0\ , \quad i = 0, \cdots, n\ .
\end{gathered} \tag{4.1}$$

Since $m = 2$, and hence, $m + 1$ is odd, by the use of Theorem 1, a dual feasible basis for the minimization problem (4.1), presented in terms of the subscripts, is of the form

$$B_{min} = \begin{cases} \{i, i+1, n\}, & 0 \leq i \leq n-2 \\ \{0, i, i+1\}, & k+1 \leq i \leq n-1 \\ \{k-1, k, k+1\} & \end{cases}$$

and a dual feasible basis for the maximization problem (4.1) is of the form

$$B_{max} = \begin{cases} \{s, u, t\}, & k \leq s < u < t \leq n \\ \{k, j, j+1\}, & 1 \leq j \leq n-1 \\ \{0, k, n\} & \end{cases}$$

Below we present the optimality conditions for the dual feasible bases B_{min} and B_{max} in problem (4.1) as well as the lower and upper bounds on $P(\xi \geq k)$ obtained in connection with those bases.

4.1.1 Lower bounds for $P(\xi \geq k)$, $1 \leq k \leq M$

Optimality conditions for B_{min} and the corresponding lower bounds

• If $k \leq i \leq n-1$ and $B_{min} = \{i, i+1, n\}$ is optimal, then the lower bound for $P(\xi \geq k)$, that is the objective function value corresponding to the basis B_{min}, is 0.

• If the basis is $B_{min} = \{i, i+1, n\}$, $0 \leq i \leq k-1$, then the lower bound for $P(\xi \geq k)$ is given by

$$\begin{aligned} P(\xi \geq k) \;\geq\; 1 \;-\; & \frac{(k-i)[n(M+1)+i(M+1)+ni+M+1]}{(M-i+1)(n-i+1)} \\ + & \frac{(k-i-1)[n(M+1)+(M+1)i+ni-n]}{(M-i)(n-i)} \\ + & \frac{2(M-i)(n-i)(n+M+k-1)S_1}{(M-i)(M-i+1)(n-i)(n-i+1)} \\ - & \frac{2(k-i-1)(n+M-2i+1)(n+M+i-1)S_1}{(M-i)(M-i+1)(n-i)(n-i+1)} \\ - & \frac{6[(M-i)(n-i)-(k-i-1)(n+M-2i+1)]S_2}{(M-i)(M-i+1)(n-i)(n-i+1)}\,, \end{aligned} \tag{4.2}$$

where i satisfies the following conditions

$$\begin{aligned} 2(n+M+i-1)S_1 - 6S_2 &\geq n(M+i) + i(M+1) \\ 2(M+2i-1)S_1 - 6S_2 &\leq i(2M+i+1) \\ 2(n+M+i)S_1 - 6S_2 &\leq (M+1)(n+i+1) + ni \ . \end{aligned} \tag{4.3}$$

Note that conditions (4.3) ensures the primal feasibility of the basis $B_{min} = \{i, i+1, n\}$.

• Now, consider the basis $B_{min} = \{0, i, i+1\}$, where $k+1 \leq i \leq n-1$. Then the corresponding lower bound for $P(\xi \geq k)$ is given by

$$P(\xi \geq k) \ \geq \ 1 + \frac{2k(2i+M+1)S_1 - 6kS_2 - k(i+1)(i+2M+2)}{(M+1)(i+1)(i+2)} \ , \tag{4.4}$$

where i is determined by

$$\begin{aligned} 2(i+M)S_1 - 6S_2 &\geq M(i+1) \\ 2(i+M-1)S_2 - 6S_2 &\leq iM \\ 2(2i+M+1)S_1 - 6S_2 &\leq (j+1)(j+2M+2) \ . \end{aligned} \tag{4.5}$$

• If the basis is $B_{min} = \{k-1, k, k+1\}$, then the lower bound for $P(\xi \geq k)$ is

$$P(\xi \geq k) \leq \frac{2(M+2k)S_1 - 6S_2 + (k+1)(k-6) - 2M(k-1)}{3(M-k+2)} \tag{4.6}$$

if and only if

$$\begin{aligned} 2(M+2k)S_1 - 6S_2 &\leq M(2k+1) + k(k+2) \\ 2(M+2k-1)S_1 - 6S_2 &\geq k(k+1) + 2(Mk-1) \\ 2(M+2k-3)S_1 - 6S_2 &\geq (k-1)(2M+k) \ . \end{aligned} \tag{4.7}$$

4.1.2 Upper bounds for $P(\xi \geq k)$, $1 \leq k \leq M$

Optimality conditions for B_{max} and the corresponding upper bounds

Below we present upper bounds for $P(\xi \geq k)$ by the use of dual feasible basis for the maximization problem (4.1).

- If the basis $B_{max} = \{s, t, u\}$, where $k \leq s < t < u \leq n$, is optimal, then the upper bound for $P(\xi \geq k)$ is equal to 1.

- Let $B_{max} = \{k, j, j+1\}$ be the optimal basis. First, we note that if $j = M$, $k+1 \leq j \leq M-1$, or $M+1 \leq j \leq n-1$, then the upper bound for $P(\xi \geq k)$ is equal to 1.

Assume that $0 \leq j \leq k-2$. In this case the upper bound for $P(\xi \geq k)$ is given by

$$
\begin{aligned}
P(\xi \geq k) \ \leq \ & \frac{6S_2 - 2(M+2j-1)S_1 + 3j(j-1)}{3j^2 - k^2 + 2jM + 2jk - 3j + k} \\
& + \frac{(M-k+1)(k+j-1)6S_2 - 2[M(k-j-2) + k(k-2) - j^2 + 1]S_1}{(m-j+1)(3j^2 - k^2 + 2jM + 2jk - 3j + k)} \\
& + \frac{(M-k+1)(k(kM+k-M-1) - jM(j+M+2k+1) - jk(k+j-2))}{(m-j+1)(3j^2 - k^2 + 2jM + 2jk - 3j + k)} \\
& + \frac{(M-k+1)[k(2M+k-2) - j(j+M-2)]2S_1 - 6(j+k)S_2}{j^2M + jk^2 + 2jkm - jk + 2jM^2 - 3jM - k^2M + kM - 3j^2 - 3j^2 - 2j^2k + 3j^2} \\
& + \frac{(M-k+1)[kM(k-2j-1) - jM(2M+j-3) - jk(j+k-4)]}{j^2M + jk^2 + 2jkm - jk + 2jM^2 - 3jM - k^2M + kM - 3j^2 - 3j^2 - 2j^2k + 3j^2}
\end{aligned}
\tag{4.8}
$$

where j is determined by the inequalities

$$\begin{aligned}
2(M+2j-1)S_1 - 6S_2 &\le 3j(j-1) \\
2[M(k-j-2)+k(k-2)-j^2+1]S_1 - 6(k+j-1)S_2 &\le \\
k(kM+k-M-1) - jM(j+M+2k+1) - jk(k+j-2) & \qquad (4.9)\\
2[k(2M+k-2)-j(j+M-2)]S_1 - 6(j+k)S_2 &\ge \\
kM(k-2j-1) - jM(2M+j-3) - jk(j+k-4)\ . &
\end{aligned}$$

• The upper bound for $P(\xi \ge k)$ corresponding to the basis $B_{max} = \{0, k, n\}$ is expressed by

$$P(\xi \ge k) \ \le \ \frac{2(n+k+M-1)S_1 - 6S_2 - (n+M+1)(k-1)}{(n+1)(M+1)} \qquad (4.10)$$

where

$$\begin{aligned}
2(n+M-1)S_1 - 6S_2 &\ge nM \\
2(k+M)S_1 - 6S_2 &\le M(k-1) \qquad (4.11)\\
2(n+k+M-1)S_1 - 6S_2 &\le k(n+M+1) + nM\ .
\end{aligned}$$

We remark that if we use the relaxed version of problem (2.5), rather than that of problem (2.4), then the lower and upper bounds change in such a way that M is replaced by $M-1$ in the formulas presented above.

4.2 Closed form bounds for $P(\xi \geq k),\ M+1 \leq k \leq n$, based on the binomial moments S_1, S_2

Using problem (2.15) for the case of $m = 2$ and $M + 1 \leq k \leq n$ we obtain

$$\min(\max)\ \{v_k + 2v_{k+1} + ... + (n-k+1)v_n\}$$

subject to

$$\sum_{i=0}^{M}(M-i+1)v_i + \sum_{i=M+1}^{n}(i-M)v_i = 1$$

$$\sum_{i=0}^{M}\left[\binom{M+1}{2} - \binom{i}{2}\right] v_i + \sum_{i=M+1}^{n}\left[\binom{i+1}{2} - \binom{M+1}{2}\right] v_i = S_1 \tag{4.12}$$

$$\sum_{i=0}^{M}\left[\binom{M+1}{3} - \binom{i}{3}\right] v_i + \sum_{i=M+1}^{n}\left[\binom{i+1}{3} - \binom{M+1}{3}\right] v_i = S_2$$

$$v_i \geq 0\ , \quad i = 0, \cdots, n\ .$$

We present closed form bounds for $P(\xi \geq k)$ by the use of problem (4.12). Since $m+1$ is odd, Theorem 2 provides us with the following dual feasible bases for problem (2.15):

$$B_{min} = \begin{cases} \{s, u, t\}, & 0 \leq s < u < t \leq k-1 \\ \{k-1, i, i+1\}, & 0 \leq i \leq n-1 \\ \{0, k-1, n\} & \end{cases}$$

for the minimization problem and

$$B_{max} = \begin{cases} \{0, j, j+1\}, & k \le j \le n-1 \\ \{j, j+1, n\}, & 0 \le j \le k-2 \\ \{k-2, k-1, k\} & \end{cases}$$

for the maximization problem.

4.2.1 Lower bounds for $P(\xi \ge k)$, $M+1 \le k \le n$

Optimality conditions for B_{min} and the corresponding lower bounds

• First we remark that if $B_{min} = \{s, u, t\}$, $0 \le s < u < t \le k-1$, or $B_{min} = \{k-1, i, i+1\}$, $0 \le i \le k-3$, is optimal, then the lower bound for $P(\xi \ge k)$ (that is the objective function value of (2.15) corresponding to the basis B_{min}) is equal to 0.

Below we present the optimality conditions for the other dual feasible bases B_{min} for the minimization problem (2.15) and the corresponding lower bounds for $P(\xi \ge k)$.

• Let $B_{min} = \{k-1, i, i+1\}$. Note that, for the cases $0 \le i \le M-1$, $M+1 \le i \le k-3$, or $i = M$, the lower bound corresponding to this basis is 0.

• Now, consider the case $M+1 \le k-1 < i \le n-1$. Then the basis $B_{min} = \{k-1, i, i+1\}$ is primal feasible if and only if i is determined by the inequalities

$$\begin{aligned} 2(M+2i+1)S_1 - 6S_2 &\le M(2i+2) + 3i + 1 \\ 2(k+M-2i)S_1 - 6S_2 &\le M(i+1) + k(M+i+2) \\ 2(M+k+i-1)S_1 - 6S_2 &\ge k(i+1) + M(i+k)\ . \end{aligned} \tag{4.13}$$

In this case the corresponding lower bound for $P(\xi \geq k)$, $M+1 \leq k-1$, is given by

$$P(\xi \geq k) \geq \frac{(i-k+1)[6S_2 - 2(k+M-2i)S_1 + M(i+1) + k(M+i+2)]}{i(k-1) + M(i-k+1) + 1}$$
$$- \frac{(i-k+2)[6S_2 - 2(M+k+i-1)S_1 + k(i+1) + M(i+k)]}{k(i-M+1) + i(M-3) + 2M - 1} \ . \tag{4.14}$$

• The basis $B_{min} = \{k-1, i, i+1\}$, $k-1 = M < i \leq n-1$, is optimal if and only if i is determined by the following inequalities:

$$\begin{aligned} (3i + Mi + M - 1)S_1 - 3(i+1)S_2 &\leq 2i(M+1) - 1 \\ 2(i+2M)S_1 - 6S_2 &\leq M^2 + M(2i+3) \\ 2(2M+i-1)S_1 - 6S_2 &\geq M^2 + M(2i+1) \ . \end{aligned} \tag{4.15}$$

The corresponding lower bound for $P(\xi \geq k)$, $k = M+1$, is given by

$$P(\xi \geq k) \geq \frac{(i-k+1)[6S_2 - 2(i+2M)S_1 + M^2 + M(2i+3)]}{M(2i-M+1) - i + 1}$$
$$+ \frac{(i-k+2)[6S_2 - 2(2M+i-1)S_1 + k(i+1) + M^2 + M(2i+1)]}{M(2i-M+3) - 3i - 1} \ . \tag{4.16}$$

• The optimality of the basis $B_{min} = \{0, k-1, n\}$ is ensured if and only if the following conditions are satisfied:

$$\begin{aligned} 2(n+M-1)S_1 - 6S_2 &\geq nM \\ 2(k+M-2)S_1 - 6S_2 &\leq M(k-1) \\ 2(n+k+M-1)S_1 - 6S_2 &\leq k(n+1) + M(n+k) \ . \end{aligned} \tag{4.17}$$

Then the corresponding lower bound for $P(\xi \geq k)$ is given by

$$P(\xi \geq k) \geq \frac{M(k-1) - 2(k+M-2)S_1 + 6S_2}{(n+1)(n-M)} \ . \tag{4.18}$$

4.2.2 Upper bounds for $P(\xi \geq k)$, $M+1 \leq k \leq n$

Optimality conditions for B_{max} and the corresponding upper bounds

- The primal feasibility conditions for $B_{max} = \{0, j, j+1\}$, $k \leq j \leq n-1$, are

$$\begin{aligned} 2(j+M)S_1 - 6S_2 &\geq M(j+1) \\ 2(j+M-1)S_1 - 6S_2 &\leq Mj \\ 2(2j+M+1)S_1 - 6S_2 &\leq (j+2M+2)(j+1) \ . \end{aligned} \tag{4.19}$$

The corresponding upper bound for $P(\xi \geq k)$ is

$$\begin{aligned} P(\xi \geq k) \leq & \frac{2[(j-k+1)(j+M)(2j-M+2) - (j-M)(j+1)(k+M-2)]S_1}{(j-M)(j-M+1)(j+1)(j+2)} \\ & - \frac{6[(j-k+1)(2j-k+2) - (j-M)(j+1)]S_2}{(j-M)(j-M+1)(j+1)(j+2)} \\ & + \frac{M(j-k+2)j}{(j-M+1)(j+2)} - \frac{(j-k+1)M}{j-M} \ . \end{aligned} \tag{4.20}$$

- The basis $B_{max} = \{j, j+1, n\}$, $0 \leq j \leq k-2$, is primal feasible if and only if j is determined by the conditions

$$\begin{aligned} 2(n+j+M-1)S_1 - 6S_2 &\geq nM + j(n+M+1) \\ 2(n+j+M)S_1 - 6S_2 &\leq j(n+M+2) + M(n+1) \\ 2(M+2j-1)S_1 - 6S_2 &\leq j(2M+j+1) \ . \end{aligned} \tag{4.21}$$

The upper bound corresponding to the basis $B_{max} = \{j, j+1, n\}$, $0 \le j \le k-2$, is given by

$$P(\xi \ge k) \le \frac{j(n-k+1)(2M+j+1)}{(n-M)(n-j+1)(n-j)}$$
$$- \frac{2(n-k+1)(M+2j-1)S_1 - 6(n-k+1)S_2}{(n-M)(n-j+1)(n-j)} \ . \tag{4.22}$$

• The optimality of the basis $B_{max} = \{k-2, k-1, k\}$, where $k-2 \ge M+1$, is ensured by the inequalities

$$\begin{aligned} (2M+4k-2)S_1 - 6S_2 &\le k^2 + 2kM + k \\ (2M+4k-4)S_1 - 6S_2 &\ge k^2 + 2kM - M - 1 \\ (2M+4k-6)S_1 - 6S_2 &\le k^2 + 2kM - 2M - k \ . \end{aligned} \tag{4.23}$$

The corresponding upper bound for $P(\xi \ge k)$, $k-2 \ge M+1$, is

$$P(\xi \ge k) \le \frac{6S_2 + (6-2M-4k)S_1 + k^2 + 2kM - 2M - k}{2(k-M)} \ . \tag{4.24}$$

• The basis $B_{max} = \{k-2, k-1, k\}$, $k = M+1$, is optimal if the following conditions are satisfied:

$$\begin{aligned} 2MS_1 - 2S_2 &\le M(M+1) \\ (6M-2)S_1 - 6S_2 &\ge (3M-2)(M+1) \\ (2M-2)S_1 - 2S_2 &\le M(M-1) \ . \end{aligned} \tag{4.25}$$

In this case the corresponding upper bound for $P(\xi \geq k)$, $k = M + 1$, is

$$P(\xi \geq k) \leq (1 - M)S_1 + S_2 + M(M - 1)/2 \ . \tag{4.26}$$

• The optimality of the basis $B_{max} = \{k - 2, k - 1, k\}$, $k = M + 2$, is ensured by the inequalities

$$\begin{aligned} (2M + 2)S_1 - 2S_2 &\leq (M + 1)(M + 2) \\ (2 + 6M)S_1 - 6S_2 &\geq M(3M + 5) \\ 2MS_1 - 2S_2 &\leq M(M + 1) \ . \end{aligned} \tag{4.27}$$

The corresponding upper bound for $P(\xi \geq k)$, $k = M + 2$, is

$$P(\xi \geq k) \leq S_2 - MS_1 + M(M + 1)/2 \ . \tag{4.28}$$

4.3 Closed form bounds for $P(\xi = k)$, $0 \le k \le M$, based on the binomial moments S_1, S_2

We consider the relaxed problem (3.2) and present closed form bounds for the probability that exactly k out of n events occur when $m = 2$. As shown in Prékopa-Subasi-Subasi (2008) if only the first two binomial moments S_1 and S_2 are known, the relaxed version of problem (3.2), when $0 \le k \le M$, can be written in the form:

$$\min(\max)\ \{v_0 + ... + v_k\}$$

$$\text{subject to}$$

$$\sum_{i=0}^{M}(M-i+1)v_i + \sum_{i=M+1}^{n}(i-M)v_i = 1$$

$$\sum_{i=0}^{M}\left[\binom{M+1}{2} - \binom{i}{2}\right]v_i + \sum_{i=M+1}^{n}\left[\binom{i+1}{2} - \binom{M+1}{2}\right]v_i = S_1 \tag{4.29}$$

$$\sum_{i=0}^{M}\left[\binom{M+1}{3} - \binom{i}{3}\right]v_i + \sum_{i=M+1}^{n}\left[\binom{i+1}{3} - \binom{M+1}{3}\right]v_i = S_2$$

$$v_i \ge 0\ , \quad i = 0, \cdots, n\ .$$

By the use of Theorem 3, a dual feasible basis B_{min} for the minimization problem (4.29), presented in terms of the subscripts, has the form

$$B_{min} = \begin{cases} \{s,u,t\}, & k+1 \le s < u < t \le n \\ \{0, k+1, n\} & \\ \{k+1, i, i+1\}, & 0 \le i \le n-1 \end{cases}$$

and a dual feasible basis B_{max} for the maximization problem (4.29) has the form

$$B_{max} = \begin{cases} \{s,u,t\}, & 0 \le s < u < t \le k \\ \{0,k,n\} & \\ \{k,j,j+1\} & 0 \le j \le n-1 \end{cases},$$

where $0 \le i, j \le n-1$.

4.3.1 Lower bounds for $P(\xi = k)$, $0 \le k \le M$

Optimality conditions for B_{min} and the corresponding lower bounds

- First, we remark that the objective function value corresponding to the basis $B_{min} = \{s,u,t\}$, where $k+1 \le s < u < t \le n$, is equal to 0. Hence, the corresponding lower bound for $P(\xi = k)$ is 0.

- The basis $B_{min} = \{0, k+1, n\}$, $k+1 \le M$, is primal feasible if

$$\begin{aligned} 2(n+M-1)S_1 - 6S_2 &\ge nM \ , \\ 2(k+M-1)S_1 - 6S_2 &\le kM \ , \\ 2(n+k+M)S_1 - 6S_2 &\le (k+1)(n+1) + M(n+k+1) \ . \end{aligned} \tag{4.30}$$

The lower bound can then be obtained as

$$P(\xi = k) \ge \frac{6S_2 - 2(n+k+M)S_1 + (k+1)(n+1) + M(n+k+1)}{(k+1)(M+1)(n+1)} \ . \tag{4.31}$$

• The basis $B_{min} = \{0, k+1, n\}$, where $k = M$, is primal feasible if

$$\begin{aligned} 2(n+M-1)S_1 - 6S_2 &\geq nM \ , \\ 2(2M)S_1 - 6S_2 &\leq M(M+1) \ , \\ 2(n+2M+1)S_1 - 6S_2 &\leq (M+2)(n+1) + M(n+M+2) \ . \end{aligned} \tag{4.32}$$

The corresponding lower bound is given by

$$P(\xi = k) \geq \frac{6S_2 - 2(n+2M+1)S_1 + (M+2)(n+1) + M(n+M+2)}{(M+1)(M+2)(n+1)} \ . \tag{4.33}$$

• Next we present the conditions that ensure the primal feasibility of the basis $B_{min} = \{k+1, i, i+1\}$.

• If $k+2 \leq i \leq n-1$, then the lower bound corresponding to $B_{min} = \{k+1, i, i+1\}$ is equal to 0.

• If $0 \leq i \leq k-1$, then the primal feasibility of $B_{min} = \{k+1, i, i+1\}$ is ensured if and only if i is determined by

$$\begin{aligned} 2(k+M+i-1)S_1 - 6S_2 &\geq i(k+M+1) + kM \ , \\ 2(2i+M-1)S_1 - 6S_2 &\leq i(i+2M+1) \ , \\ 2(k+i+M)S_1 - 6S_2 &\leq (k+i+1)(M+1) + ki \ . \end{aligned} \tag{4.34}$$

Then the lower bound for $P(\xi = k)$ is obtained as

$$P(\xi = k) \geq \frac{(k+i+1)(M+1)+kM}{(M-i+1)(k-i+1)} - \frac{i(k+M+1)+ki}{(M-i)(k-i)}$$
$$+ \frac{2[k(k+i+M)+(M-i+1)(M+2i-1)]S_1 - 6(M+k-i+1)S_2}{(M-i)(M-i+1)(k-i)(k-i+1)} \tag{4.35}$$

• If $k = M$, then $B_{min} = \{M+1, i, i+1\}$ is primal feasible if and only if

$$\begin{aligned} 2(2M+i)S_1 - 6S_2 &\geq M(M+3i+1)+2i \ , \\ 2(2i+M-1)S_1 - 6S_2 &\leq i(i+2M+1) \ , \\ 2(2M+i+1)S_1 - 6S_2 &\leq M(M+3i+3)+2(i+1) \ . \end{aligned} \tag{4.36}$$

The corresponding lower bound is

$$P(\xi = k) \geq \frac{M(M+3i+3)+2(i+1)}{(M-i+1)(M-i+2)} - \frac{M(M+3i+1)+2i}{(M-i)(M-i+1)}$$
$$\frac{6(M+i)S_1 - 12S_2}{(M-i)(M-i+1)(M-i+2)} \tag{4.37}$$

4.3.2 Upper bounds for $P(\xi = k)$, $0 \le k \le M$

Optimality conditions for B_{max} and the corresponding upper bounds

• If $B_{max} = \{0, k, n\}$ is also primal feasible, i.e., (4.11) is satisfied, then the upper bound for $P(\xi = k)$ is given by the following formula:

$$
\begin{aligned}
P(\xi = k) \le \frac{nk + nM + kM + k}{k(M+1)(n+1)} &- \frac{nM}{k(M-k+1)(n-k+1)} \\
+ \frac{2(n^2 + M^2 - k^2 + nM + 3k - 3)S_1 - 6(n + M - k + 2)S_2}{(M+1)(n+1)(M-k+1)(n-k+1)} &.
\end{aligned} \tag{4.38}
$$

• If $B_{max} = \{k, j, j+1\}, \quad 0 \le j \le k-2$ is primal feasible if and only if

$$
\begin{aligned}
2(2j + M - 1)S_1 - 6S_2 &\le j(j + 2M + 1) \\
2(j + k + M - 1)S_1 - 6S_2 &\le k(j + M + 1) + jM \\
2(j + k + M - 2)S_1 - 6S_2 &\ge M(j + k - 1) + kj
\end{aligned} \tag{4.39}
$$

Then the upper bound for $P(\xi = k)$ is given by the following formula:

$$
P(\xi = k) \le \frac{j(j + 2k - 1) - 2(k + 2j - 2)S_1 + 6S_2}{(M-j)(M-j+1)(M-k+1)} . \tag{4.40}
$$

• $B_{max} = \{k, j, j+1\}, \quad k+1 \le j \le M-2$ is primal feasible if and only if j is determined by

$$
\begin{aligned}
2(M + j + k - 1)S_1 - 6S_2 &\ge k(M + j + 1) + Mj \\
2(M + j + k - 2)S_1 - 6S_2 &\le M(k + j - 1) + kj \\
2(M + 2j - 1)S_1 - 6S_2 &\le j(2M + j + 1) .
\end{aligned} \tag{4.41}
$$

In this case the upper bound for $P(\xi = k)$ is given by the formula:

$$P(\xi = k) \leq \frac{j(2M + j + 1) - 2(M + 2j - 1)S_1 + 6S_2}{(j - k)(j - k + 1)(M - k + 1)} \ . \tag{4.42}$$

• The basis $B_{max} = \{k, j, j + 1\}$, where $M + 1 \leq j \leq n - 1$, is primal feasible if and only if

$$\begin{aligned} 2(kj + M + 1)S_1 - 6S_2 &\leq (j + 1)(j + 2M + 2) \\ 2(j + M + k)S_1 - 6S_2 &\geq k(j + M + 2) + M(j + 1) \\ 2(j + k + M - 1)S_1 - 6S_2 &\leq k(j + M + 2) + j(M - 1) - 2 \ . \end{aligned} \tag{4.43}$$

The corresponding upper bound for $P(\xi = k)$ is obtained as

$$P(\xi = k) \leq \frac{(j + 1)(j + 2M + 2) - 2(2j + M + 1)S_1 + 6S_2}{(j - k + 1)(j - k + 2)(M - k + 1)} \ . \tag{4.44}$$

• The basis $B_{max} = \{k, M, M + 1\}$ is also primal feasible if only if the following conditions are satisfied.

$$\begin{aligned} 2MS_1 - 2S_2 &\leq M(M + 1) \\ 2(2M + k)S_1 - 6S_2 &\geq (M + 1)(2k + M) \\ 2(2M + k - 2)S_1 - 6S_2 &\leq M(M + 2k - 1) \ . \end{aligned} \tag{4.45}$$

The upper bound for $P(\xi = k)$ is given by the formula

$$P(\xi = k) \leq \frac{3M(M + 1) - 6MS_1 + 6S_2}{(M - k)(M - k + 1)(M - k + 2)} \ . \tag{4.46}$$

4.4 Closed form bounds for $P(\xi = k)$, $M+1 \leq k \leq n$, based on the binomial moments S_1, S_2

Below we present closed form bounds for the probability that exactly k events occur in case of problem (3.6) when $m = 2$. As shown in Prékopa-Subasi-Subasi (2008) if only the first two binomial moments S_1 and S_2 are known, the relaxed version of problem (3.6), when $M + 1 \leq k \leq n$, can be written in the form:

$$\min(\max)\ \{v_k + ... + v_n\}$$

subject to

$$\begin{gathered}
\sum_{i=0}^{M}(M-i+1)v_i + \sum_{i=M+1}^{n}(i-M)v_i = 1 \\
\sum_{i=0}^{M}\left[\binom{M+1}{2} - \binom{i}{2}\right]v_i + \sum_{i=M+1}^{n}\left[\binom{i+1}{2} - \binom{M+1}{2}\right]v_i = S_1 \\
\sum_{i=0}^{M}\left[\binom{M+1}{3} - \binom{i}{3}\right]v_i + \sum_{i=M+1}^{n}\left[\binom{i+1}{3} - \binom{M+1}{3}\right]v_i = S_2 \\
v_i \geq 0\ , \quad i = 0, \cdots, n\ .
\end{gathered} \tag{4.47}$$

Theorem 4 implies that a dual feasible basis for the minimization problem (4.47), presented in terms of the subscripts, is of the form

$$B_{min} = \begin{cases} \{s,u,t\} & \text{if } \ 0 \leq s < u < t \leq k-1 \\ \{k-1, i, i+1\} & 0 \leq i \leq n-1 \\ \{0, k-1, n\} & \end{cases}$$

and a dual feasible basis for the maximization problem (4.47) is

$$B_{max} = \begin{cases} \{0, k, n\} & \\ \{k, j, j+1\} & 0 \le j \le n-1 \end{cases} .$$

4.4.1 Lower bounds for $P(\xi = k)$, $M + 1 \le k \le n$

Optimality conditions for B_{min} and the corresponding lower bounds

• If $B_{min} = \{s, u, t\}$ is optimal, then the optimum value of (4.47) is 0, so is the lower bound for $P(\xi = k)$.

• The basis $B_{min} = \{0, k-1, n\}$ is primal feasible if and only if the following conditions are satisfied:

$$\begin{aligned} 2(n+M-1)S_1 - 6S_2 &\ge nM \\ 2(k+M-2)S_1 - 6S_2 &\le (k-1)M \ , \\ 2(n+k+M-1)S_1 - 6S_2 &\le k(n+1) + M(n+k) \ . \end{aligned} \tag{4.48}$$

In this case the lower bound formula is given as

$$P(\xi = k) \ge \frac{6S_2 - 2(k+M-2)S_1 + M(k-1)}{(n-M)(n+1)(n-k+1)} \ . \tag{4.49}$$

• If $k = M$, then $B_{min} = \{0, M-1, n\}$ is primal feasible if and only if

$$\begin{aligned} 2(n+M-1)S_1 - 6S_2 &\geq n(M+2) \ , \\ 4(M-1) - 6S_2 &\leq M(M-1) \ , \\ 2(n+2M-1)S_1 - 6S_2 &\leq M(2n+M-1) \ . \end{aligned} \tag{4.50}$$

If conditions (4.50) are satisfied, then the lower bound for $P(\xi = k)$ can be obtained by the use of the formula:

$$P(\xi = k) \geq \frac{6S_2 - 4(M-1)S_1 + M(M-1)}{(n+1)(n-M)(n-M+1)} \ . \tag{4.51}$$

Next we present the primal feasibility conditions for the basis $B_{min} = \{k-1, i, i+1\}$.

• If $0 \leq i \leq k-3$, then the lower bound corresponding to $B_{min} = \{k-1, i, i+1\}$ is 0.

• If $k \leq i \leq n-1$, the basis $B_{min} = \{k-1, i, i+1\}$ is primal feasible if and only if

$$\begin{aligned} 2(i+k+M)S_1 - 6S_2 &\geq k(i+M+2) + (i+1)M \ , \\ 2(i+k+M-1)S_1 - 6S_2 &\leq k(i+M+1) + iM \ , \\ 2(M+1)S_1 - 6S_2 &\leq (i+1)(i+2M+2) \ , \end{aligned} \tag{4.52}$$

and the corresponding lower bound for $P(\xi = k)$ is

$$\begin{aligned} P(\xi = k) \geq &\frac{k(i+M+1) + iM}{(i-M+1)(i-k+2)} - \frac{k(i+M+2) + (i+1)M}{(i-M)(i-k+1)} \\ &+ \frac{2[(i-k+1)(2i+k) + (i+k+M)(i-M+1)]S_1 - 6(2i-k-M+2)S_2}{(i-M)(i-M+1)(i-k+1)(i-k+2)} \ . \end{aligned} \tag{4.53}$$

• Let $k = M$. Basis $B_{min} = \{M-1, i, i+1\}$ is also primal feasible if i is determined by the following inequalities:

$$\begin{aligned} 2(i+2M)S_1 - 6S_2 &\geq M(2i+M+3) \ , \\ 2(i+2M-1)S_1 - 6S_2 &\leq M(2i+M+1) \ , \\ 2(2i+M+1)S_1 - 6S_2 &\leq M(4i-M+5) + (i-M+1)(i-M+2) \ . \end{aligned} \tag{4.54}$$

In this case the lower bound is

$$P(\xi = k) \geq \frac{6(i+M)S_1 - 12S_2 + (i-M)(i-M+2) - 6M(i+1)}{(i-M)(i-M+1)(i-M+2)} \ . \tag{4.55}$$

4.4.2 Upper bounds for $P(\xi = k)$, $M+1 \leq k \leq n$

Optimality conditions for B_{max} and the corresponding upper bounds

We give primal feasibility conditions for dual feasible basis B_{max}.

• Basis $B_{max} = \{0, k, n\}$ is also primal feasible if

$$\begin{aligned} 2(n+M-1)S_1 - 6S_2 &\geq nM \ , \\ 2(k+M-1)S_1 - 6S_2 &\leq kM \ , \\ 2(n+k+M)S_1 - 6S_2 &\leq (k+1)(n+1) + M(n+k+1) \ . \end{aligned} \tag{4.56}$$

The corresponding upper bound for $P(\xi = k)$ is

$$\begin{aligned} P(\xi = k) \leq & \frac{2(n^2+k^2-M^2+nk+M-1)S_1 - 6(n+k-M+1)S_2}{(n-M)(n+1)(k-M)(k+1)} \\ & - \frac{M(n^2+nk+k^2+n+k-nM-kM-M)}{(n-M)(n+1)(k-M)(k+1)} \ . \end{aligned} \tag{4.57}$$

• Basis $B_{max} = \{k, j, j+1\}, \quad 0 \leq j \leq M-2$ is also primal feasible if j is determined by the inequalities

$$
\begin{aligned}
2(k+M+j-1)S_1 - 6S_2 &\geq k(M+j) + j(M+1) \ , \\
2(M+2j-1)S_1 - 6S_2 &\leq j(2M+j+1) \ , \\
2(k+j+M)S_1 - 6S_2 &\leq (M+1)(k+j+1) + kj \ .
\end{aligned}
\tag{4.58}
$$

The upper bound for $P(\xi = k)$ is given by the formula

$$
P(\xi = k) \leq \frac{j(j+2M+1) - 2(2j+M-1)S_1 + 6S_2}{(k-M)(k-j)(k-j+1)} \ . \tag{4.59}
$$

• The basis $B_{max} = \{k, j, j+1\}, \quad M \leq j \leq k-2$ is also primal feasible if the following conditions are satisfied:

$$
\begin{aligned}
2(k+j+M)S_1 - 6S_2 &\geq (M+1)(k+j+1) + kj \ , \\
2(2j+M+1)S_1 - 6S_2 &\leq (j+1)(2M+j+2) \ , \\
2(k+j+M+1)S_1 - 6S_2 &\leq (M+1)(k+j+2) + k(j+1) \ .
\end{aligned}
\tag{4.60}
$$

Then the upper bound for $P(\xi = k)$ is given by the following formula:

$$
\frac{(j+1)(j+2M+2) - 2(2j+M+1)S_1 + 6S_2}{(k-M)(k-j)(k-j-1)} \ . \tag{4.61}
$$

• Basis $B_{max} = \{k, j, j+1\}, \quad k+1 \le j \le n-1$ is also primal feasible if j is determined by

$$\begin{aligned} 2(k+j+M)S_1 - 6S_2 &\le (M+1)(k+j+1) + kj \ , \\ 2(2j+M+1)S_1 - 6S_2 &\le (j+1)(2M+j+2) \ , \\ 2(k+j+M+1)S_1 - 6S_2 &\ge (M+1)(k+j+2) + k(j+1) \ . \end{aligned} \tag{4.62}$$

In this case we have the following upper bound for $P(\xi = k)$:

$$\begin{aligned} P(\xi = k) \le \frac{1}{k-M} &+ \frac{2k(j+1) + (M+1)(2j-M+2)}{(k-M)(j-M)(j-M+1)} \\ &- \frac{2(2j+k+1)S_1 - 6S_2}{(k-M)(j-M)(j-M+1)} \ . \end{aligned} \tag{4.63}$$

We remark that if we use the relaxed version of problems (2.5)-(2.16) and problems (3.3)-(3.7), rather than that of problems (2.4)-(2.15) and (3.2)-(3.6), then the lower and upper bounds change in such a way that we have to replace $M-1$ for M in the formulas of Sections 4.1-4.4.

4.5 Algorithmic bounds

In Chapter 4 we have derived closed form bounds for the probabilities that at least k out of n events occur by the use of the relaxed problems (2.4), (2.15) and for the probability that exactly k out of n events occur by the use of problems (3.2), (3.6) when $m = 2$. For larger m values the solution of the relaxed problems can be obtained by specially designed dual algorithms of linear programming. Once an algorithm of this kind terminates, the solutions for the non-relaxed problem can be continued again by the dual algorithm. In fact, as it is well known in linear programming, the dual algorithm can efficiently be used, as a reoptimization technique, whenever the optimal basis has already been found but a further constraint is introduced into the problem.

The algorithm presented below works in this way and is applicable to cases with consecutive and non-consecutive moments. Our algorithm is a customized dual-type simplex algorithm. We remark that dual degeneracy comes up quite frequently, in the course of computation. In fact, if we encounter a basis such that the basic components of the objective function are all zero, then it is dual degenerate, the pivoting is no longer unique and there is no guarantee for the finiteness of the algorithm. If in the course of the procedure dual degeneracy occurs, then we apply Bland's rule in the pivoting: the smallest subscripted candidate vector goes out of the basis and the smallest subscripted candidate comes in. If we look at the structures of the dual feasible bases, then it is clear that whenever there is no dual degeneracy, our pivoting rule is the same as Bland's rule, at least in case of the incoming vector. The outgoing vector can be chosen arbitrarily if there is no dual degeneracy because the objective function value will not be the same in that case. Thus, if we carry out the algorithm this way, then it will be finite and in fact either terminates with all dual degenerate bases or we encounter a nondegenerate one which necessarily has one of the structures in Theorems 1-4.

***Algorithmic solutions of problems* (2.4), (2.15), (3.2), and (3.6)**

Step 0. Find an initial dual feasible basis B to the relaxed problem. Any basis that has the structure presented in Theorems 1-4 is suitable.

Step 1. Check for primal feasibility. If $B^{-1}b \geq 0$, then the solution of the relaxed problem terminates. Go to Step 4. Otherwise go to Step 2.

Step 2. If $(B^{-1}b)_j < 0$ for some j, then the jth vector in B (not necessarily equal to a_j) is a candidate to leave the basis. Choose the smallest j among the candidates to leave the basis. Go to Step 3.

Step 3. Include the vector a_l into the basis that restores the dual feasible basis structure. Go to Step 1.

Step 4. If the constraint "$v_0 + ... + v_M \geq v_{M+1} + ... + v_n$" (or "$v_M + ... + v_n \geq v_0 + ... + v_{M-1}$") is satisfied, then the solution of problem terminates. Otherwise go to Step 5.

Step 5. Reoptimize the problem with the additional constraint "$v_0 + ... + v_M \geq v_{M+1} + ... + v_n$" (or "$v_M + ... + v_n \geq v_0 + ... + v_{M-1}$"): introduce slack variable into the additional inequality constraint, prescribe nonnegativity relation for the slack variable, set up the new dual tableau and carry out the dual method.

If the sequence of probabilities $p_0, ..., p_n$ is increasing or decreasing, i.e., if $M = n$ or $M = 0$, then the solution of the problems terminates with Step 3. Reoptimization is not needed. In this case the relaxed problem is equivalent to the original problem. The critical point of the algorithm described above is to discover infeasibility of the problem that can occur in case of inaccurate moments due to rounding off or measurement errors. The solution of the formulated LP problems of higher order moments ($m > 10$) requires high precision float arithmetic. In this case to ensure the numerical stability of the problem one can apply special freely available software packages such as GNU (`http://www.gnu.org`) and MPFUN++ (`http://www.tucows.com/preview/8966/MPFUN`).

Chapter 5

Numerical Examples

In this chapter we present numerical examples that show the contribution of the use of shape constraint (unimodality) in the improvement of bounds for the probabilities that at least k or exactly k-out-of-n events occur. We proceed as follows

(a) Given an n and M with $0 \leq M \leq n$, randomly generate a unimodal probability distribution $\{p_i\}$ satisfying

- $\sum_{i=0}^{n} p_i = 1,$
- $p_i \leq p_{i+1}, i = 0, 1..., M,$
- $p_i \geq p_{i+1}, i = M, M+1, ..., n,$
- $p_i \geq 0, i = 0, 1, ..., n.$

(b) Compute the binomial moments $S_1, ..., S_m$ by the use of

$$\sum_{i=0}^{n} \binom{i}{j} p_i = S_j, j = 1, ..., m.$$

(c) Assume that the distribution $\{p_i\}$ is unknown and solve problem (2.4) or (2.15) to obtain bounds for $P(\xi \geq k)$ and problem (3.2) or (3.6) to obtain bounds for $P(\xi = k)$.

5.1 Numerical examples for $P(\xi \geq k)$

Tables 5.1, 5.2-5.4, and 5.5-5.7 give the lower and upper bounds for the probability that at least k-out-of-n events occur for the cases of $n = 10, 20, 30$, respectively. In those tables LB_j and UB_j are the lower and upper bounds obtained from problem (1.4) which do not use the shape information of the underlying probability distribution and where the first j binomial moments are known for the cases $j = 2, 3$. LB_j^* and UB_j^* designate the lower and upper bounds obtained from problem (2.1) in which the unimodality constraint (the underlying probability distribution is assumed to be unimodal with mode M) is prescribed and the first j binomial moments are known for the cases $j = 2, 3$. Tables 5.1-5.7 also present the improvement rates in bounds, calculated as $\Delta_j/\Delta_j^* = (UB_j - LB_j)/(UB_j^* - LB_j^*),\ j = 2, 3$, to show the contribution of the unimodality constraint.

Note that the lower and upper bounds presented in all numerical examples are obtained by using the first j binomial moments where $j = 2, 3$ and they are sharp bounds for $P(\xi \geq k)$, i.e., no better bounds can be found because they are the optimal solutions of the LP's involved. We also remark that the bounds become much tighter (with an improvement rate ranges from 1.70 to > 10) if they are obtained by our proposed bounding approach that uses the shape information (unimodal with mode M) of the underlying unknown probability distribution. When a lower bound is obtained as smaller than 10^{-5} we report it as 0 in the tables. For example, in case of the example for the probability that at least k-out-of-n events occur, where $n = 30$, $M = 8$, and $k = 25$, the lower bound $LB_2 = 0$ is improved to $LB_2^* = 8.33 \times 10^{-17}$ when the unimodality constraint is prescribed and the overall improvement rate is around 2.24.

In Table 5.16 we present bounds for $P(\xi \geq k)$ based on the first j moments for $j = 2, ..., 6$, $n = 10$, and $k = 1, 3, 5, 7, 9$.

5.2 Numerical examples for $P(\xi = k)$

Tables 5.9, 5.10-5.12, and 5.13-5.15 present the lower and upper bounds for the probability that exactly k-out-of-n events occur when $n = 10, 20, 30$, respectively. In those tables LB_j and UB_j, $j = 2, 3$, are the lower and upper bounds obtained from Problem Problem (1.5) which do not use the shape information of the underlying probability distribution and where the first j binomial moments are known for the cases $j = 2, 3$. LB_j^* and UB_j^*, $j = 2, 3$ designate the lower and upper bounds obtained from problem (3.1), where the underlying probability distribution is assumed to be unimodal with mode M, is prescribed and the first j binomial moments are known for the cases $j = 2, 3$. Tables 5.9-5.15 also present the improvement rates in bounds, calculated as $\Delta_j/\Delta_j^* = (UB_j - LB_j)/(UB_j^* - LB_j^*)$, $j = 2, 3$, to show the contribution of the unimodality constraint.

Note that the lower and upper bounds presented in all numerical examples are obtained by using the first j binomial moments where $j = 2, 3$ and they are sharp bounds for $P(\xi = k)$, i.e., no better bounds can be found because they are the optimal solutions of the LP's involved. We also remark that the bounds are significantly improved if they are obtained by our proposed bounding approach that uses the shape information (unimodal with mode M) of the underlying unknown probability distribution.

In Table 5.16 we present bounds for the probability that exactly k-out-of-n events occur, $P(\xi = k)$, based on the first j moments for $j = 2, ..., 6$, $n = 30$, and $k = 1, 3, 5, 8, 9, 15, 18, 19, 22, 25, 27, 29$.

Table 5.1: Bounds for $P(\xi \geq k)$ when $n = 10$

$n = 10$										
M=2	Without the use of unimodality				With Unimodality				Improvement	
k	LB_2	UB_2	LB_3	UB_3	LB_2^*	UB_2^*	LB_3^*	UB_3^*	Δ_2/Δ_2^*	Δ_3/Δ_3^*
1	0.738	1	0.828	1	0.894	1	0.918	1	2.48	2.09
3	0.332	0.932	0.364	0.873	0.408	0.655	0.512	0.655	2.43	3.52
5	0.058	0.651	0.068	0.510	0.191	0.358	0.239	0.331	3.56	4.85
8	0	0.171	0	0.119	0	0.092	0.015	0.062	1.87	2.56
9	0	0.122	0	0.071	0	0.046	0	0.029	2.67	2.42
M=4	Without the use of unimodality				With Unimodality				Improvement	
k	LB_2	UB_2	LB_3	UB_3	LB_2^*	UB_2^*	LB_3^*	UB_3^*	Δ_2/Δ_2^*	Δ_3/Δ_3^*
1	0.845	1	0.901	1	0.932	1	0.948	1	2.28	1.92
3	0.571	1	0.592	0.955	0.750	0.931	0.753	0.883	2.37	2.79
5	0.066	0.745	0.083	0.632	0.150	0.461	0.179	0.408	2.18	2.39
8	0	0.153	0	0.099	0	0.079	0	0.054	1.93	1.84
9	0	0.102	0	0.056	0	0.040	0	0.026	2.58	2.17
M=6	Without the use of unimodality				With Unimodality				Improvement	
k	LB_2	UB_2	LB_3	UB_3	LB_2^*	UB_2^*	LB_3^*	UB_3^*	Δ_2/Δ_2^*	Δ_3/Δ_3^*
1	0.844	1	0.890	1	0.934	1	0.947	1	2.37	2.08
3	0.632	1	0.633	0.943	0.789	0.894	0.791	0.866	3.52	4.12
5	0.162	0.873	0.323	0.846	0.499	0.703	0.551	0.664	3.49	4.63
8	0	0.266	0	0.168	0	0.116	0.003	0.087	2.28	1.99
9	0	0.177	0	0.090	0	0.058	0	0.036	3.04	2.47

Table 5.2: Bounds for $P(\xi \geq k)$ when $n = 20$, $M = 3$

$n = 20$										
M=3	Without the use of unimodality				With Unimodality				Improvement	
k	LB_2	UB_2	LB_3	UB_3	LB_2^*	UB_2^*	LB_3^*	UB_3^*	Δ_2/Δ_2^*	Δ_3/Δ_3^*
1	0.717	1	0.802	1	0.933	1	0.945	1	4.22	3.59
3	0.571	1	0.628	1	0.796	1	0.811	0.941	2.11	2.85
5	0.338	1	0.374	0.907	0.506	0.698	0.573	0.686	3.44	4.73
8	0.099	0.805	0.159	0.804	0.284	0.482	0.390	0.464	3.56	8.73
9	0.074	0.758	0.077	0.686	0.247	0.419	0.329	0.396	3.97	9.14
15	0	0.256	0	0.209	0.026	0.123	0.046	0.098	2.66	4.05
18	0	0.152	0	0.091	0	0.049	0	0	3.1	> 10
19	0	0.130	0	0.072	0	0.025	0	0.015	5.31	4.85

Table 5.3: Bounds for $P(\xi \geq k)$ when $n = 20$, $M = 9$

$n = 20$										
M=9	Without the use of unimodality				With Unimodality				Improvement	
k	LB_2	UB_2	LB_3	UB_3	LB_2^*	UB_2^*	LB_3^*	UB_3^*	Δ_2/Δ_2^*	Δ_3/Δ_3^*
1	0.860	1	0.927	1	0.972	1	0.981	1	5.07	3.92
3	0.795	1	0.870	1	0.917	1	0.944	1	2.47	2.31
5	0.686	1	0.750	1	0.858	1	0.882	0.962	2.22	3.14
8	0.350	1	0.405	0.957	0.650	0.886	0.669	0.868	2.75	2.77
9	0.191	0.955	0.306	0.938	0.516	0.841	0.575	0.836	2.35	2.43
15	0	0.393	0.010	0.371	0.050	0.192	0.110	0.185	2.76	4.84
18	0	0.203	0	0.144	0	0.075	0	0.062	2.7	2.32
19	0	0.167	0	0.109	0	0.038	0	0.031	4.45	3.51

Table 5.4: Bounds for $P(\xi \geq k)$ when $n = 20$, $M = 15$

$n = 20$										
M=15	Without the use of unimodality				With Unimodality				Improvement	
k	LB_2	UB_2	LB_3	UB_3	LB_2^*	UB_2^*	LB_3^*	UB_3^*	Δ_2/Δ_2^*	Δ_3/Δ_3^*
1	0.829	1	0.897	1	0.967	1	0.976	1	5.1	4.31
3	0.764	1	0.826	1	0.900	0.993	0.928	0.968	2.51	4.29
5	0.659	1	0.691	1	0.832	0.910	0.850	0.899	4.43	6.31
8	0.375	0.993	0.480	0.931	0.694	0.784	0.698	0.764	6.88	6.90
9	0.259	0.955	0.409	0.916	0.635	0.742	0.647	0.707	6.39	8.53
15	0	0.576	0.064	0.571	0.223	0.453	0.253	0.356	2.5	4.90
18	0	0.315	0	0.235	0	0.109	0.015	0.093	2.89	2.99
19	0	0.263	0	0.175	0	0.055	0	0.047	4.82	3.74

Table 5.5: Bounds for $P(\xi \geq k)$ when $n = 30$, $M = 8$

$n = 30$										
M=8	Without the use of unimodality				With Unimodality				Improvement	
k	LB_2	UB_2	LB_3	UB_3	LB_2^*	UB_2^*	LB_3^*	UB_3^*	Δ_2/Δ_2^*	Δ_3/Δ_3^*
1	0.829	1	0.904	1	0.979	1	0.985	1	8.28	6.54
3	0.779	1	0.858	1	0.938	1	0.956	1	3.58	3.23
5	0.704	1	0.782	1	0.897	1	0.922	1	2.87	2.79
8	0.521	1	0.556	1	0.761	1	0.764	0.950	2	2.38
9	0.435	1	0.442	0.982	0.564	0.847	0.625	0.817	2	2.81
15	0.061	0.854	0.071	0.752	0.250	0.456	0.315	0.422	3.85	6.40
18	0	0.619	0.044	0.524	0.161	0.298	0.170	0.293	4.53	3.92
19	0	0.526	0.033	0.472	0.131	0.253	0.134	0.253	4.31	3.70
22	0	0.323	0	0.297	0.042	0.154	0.069	0.139	2.88	4.24
25	0	0.208	0	0.159	0	0.093	0.007	0.065	2.24	2.74
27	0	0.161	0	0.110	0	0.056	0	0.038	2.88	2.90
29	0	0.128	0	0.078	0	0.019	0	0.013	6.88	6.18

Table 5.6: Bounds for $P(\xi \geq k)$ when $n = 30$, $M = 14$

$n = 30$										
M=14	Without the use of unimodality				With Unimodality				Improvement	
k	LB_2	UB_2	LB_3	UB_3	LB_2^*	UB_2^*	LB_3^*	UB_3^*	Δ_2/Δ_2^*	Δ_3/Δ_3^*
1	0.812	1	0.887	1	0.971	1	0.978	1	6.56	5.09
3	0.757	1	0.833	1	0.914	1	0.933	1	2.83	2.50
5	0.679	1	0.746	1	0.857	1	0.886	0.956	2.25	3.63
8	0.491	0.758	0.506	0.988	0.761	0.907	0.768	0.866	1.83	4.92
9	0.405	1	0.426	0.964	0.715	0.868	0.715	0.835	3.89	4.50
15	0.072	0.844	0.079	0.763	0.216	0.530	0.222	0.463	2.46	2.84
18	0	0.648	0.049	0.552	0.131	0.323	0.160	0.293	3.39	3.79
19	0	0.556	0.037	0.503	0.103	0.266	0.140	0.256	3.43	4.00
22	0	0.350	0	0.321	0.018	0.156	0.0786	0.1473	2.54	4.68
25	0	0.230	0	0.173	0	0.095	0.020	0.0755	2.41	3.12
27	0	0.178	0	0.119	0	0.057	0	0.045	3.12	2.62
29	0	0.142	0	0.084	0	0.019	0	0.015	7.44	5.56

Table 5.7: Bounds for $P(\xi \geq k)$ when $n = 30$, $M = 21$

$n = 30$										
M=21	Without the use of unimodality				With Unimodality				Improvement	
k	LB_2	UB_2	LB_3	UB_3	LB_2^*	UB_2^*	LB_3^*	UB_3^*	Δ_2/Δ_2^*	Δ_3/Δ_3^*
1	0.867	1	0.924	1	0.965	1	0.965	1	3.81	2.18
3	0.835	1	0.894	1	0.944	1	0.942	1	2.94	1.82
5	0.789	1	0.847	1	0.906	1	0.916	0.996	1.7	1.92
8	0.682	1	0.718	1	0.848	0.972	0.850	0.936	2.56	3.29
9	0.632	1	0.651	1	0.824	0.956	0.819	0.915	2.78	3.62
15	0.184	0.946	0.289	0.933	0.575	0.886	0.593	0.681	2.44	7.36
18	0.089	0.882	0.110	0.786	0.425	0.769	0.415	0.544	2.31	5.23
19	0.046	0.857	0.093	0.721	0.373	0.698	0.351	0.497	2.5	4.31
22	0	0.566	0.029	0.564	0	0.321	0.109	0.300	1.76	2.81
25	0	0.355	0	0.293	0	0.161	0.028	0.122	2.21	3.11
27	0	0.266	0	0.191	0	0.097	0	0.061	2.76	3.14
29	0	0.205	0	0.129	0	0.032	0	0.023	6.36	5.58

Table 5.8: Bounds for $P(\xi \geq k)$ when $n = 10$, $M = 2$, and $2 \leq m \leq 6$

$n = 10,\ M = 2$											
		Without the use of unimodality					With Unimodality				
		$m = 2$	$m = 3$	$m = 4$	$m = 5$	$m = 6$	$m = 2$	$m = 3$	$m = 4$	$m = 5$	$m = 6$
$k = 1$	LB	0.737	0.828	0.901	0.930	0.951	0.894	0.918	0.935	0.956	0.968
	UB	1	1	1	1	0.987	1	1	0.985	0.976	0.973
$k = 3$	LB	0.332	0.364	0.426	0.440	0.469	0.408	0.510	0.538	0.552	0.580
	UB	0.932	0.873	0.817	0.749	0.737	0.655	0.655	0.626	0.617	0.616
$k = 5$	LB	0.058	0.068	0.161	0.164	0.212	0.191	0.239	0.264	0.285	0.286
	UB	0.651	0.510	0.489	0.453	0.441	0.358	0.331	0.329	0.323	0.318
$k = 7$	LB	0	0	0	0.012	0.015	0	0.015	0.025	0.028	0.031
	UB	0.171	0.119	0.087	0.082	0.080	0.092	0.062	0.058	0.055	0.050
$k = 9$	LB	0	0	0	0	0	0	0	0	0.005	0.009
	UB	0.122	0.071	0.036	0.026	0.018	0.046	0.029	0.019	0.016	0.013

Table 5.9: Bounds for $P(\xi = k)$ when $n = 10$

$n = 10$										
M=2	Without the use of unimodality				With Unimodality				Improvement	
k	LB_2	UB_2	LB_3	UB_3	LB_2^*	UB_2^*	LB_3^*	UB_3^*	Δ_2/Δ_2^*	Δ_3/Δ_3^*
1	0	0.415	0	0.348	0	0.224	0.050	0.219	1.86	2.05
3	0	0.819	0	0.665	0.066	0.345	0.096	0.248	2.94	4.39
5	0	0.651	0	0.465	0.038	0.180	0.059	0.179	4.61	3.88
8	0	0.171	0	0.119	0	0.072	0.008	0.061	2.37	2.21
9	0	0.122	0	0.071	0	0.046	0	0.029	2.67	2.42
M=4	Without the use of unimodality				With Unimodality				Improvement	
k	LB_2	UB_2	LB_3	UB_3	LB_2^*	UB_2^*	LB_3^*	UB_3^*	Δ_2/Δ_2^*	Δ_3/Δ_3^*
1	0	0.257	0	0.186	0	0.123	0.018	0.113	2.09	1.96
3	0	0.762	0	0.610	0.048	0.373	0.051	0.299	2.34	2.45
5	0	0.745	0	0.601	0.030	0.345	0.042	0.306	2.36	2.27
8	0	0.153	0	0.100	0	0.066	0	0.054	2.32	1.87
9	0	0.102	0	0.056	0	0.040	0	0.026	2.58	2.17
M=6	Without the use of unimodality				With Unimodality				Improvement	
k	LB_2	UB_2	LB_3	UB_3	LB_2^*	UB_2^*	LB_3^*	UB_3^*	Δ_2/Δ_2^*	Δ_3/Δ_3^*
1	0	0.234	0	0.195	0.005	0.105	0.033	0.105	2.34	2.72
3	0	0.603	0	0.416	0.050	0.186	0.052	0.157	4.45	3.93
5	0	0.800	0	0.762	0.065	0.312	0.094	0.283	3.24	4.05
8	0	0.266	0	0.168	0	0.112	0.002	0.087	2.37	1.96
9	0	0.177	0	0.090	0	0.058	0	0.036	3.04	2.47

Table 5.10: Bounds for $P(\xi = k)$ when $n = 20$, $M = 3$

$n = 20$										
M=3	Without the use of unimodality				With Unimodality				Improvement	
k	LB_2	UB_2	LB_3	UB_3	LB_2^*	UB_2^*	LB_3^*	UB_3^*	Δ_2/Δ_2^*	Δ_3/Δ_3^*
1	0	0.346	0	0.269	0	0.102	0	0.095	3.39	2.85
3	0	0.535	0	0.512	0.065	0.402	0.072	0.306	1.59	2.19
5	0	0.804	0	0.592	0.039	0.195	0.048	0.127	5.17	7.52
8	0	0.732	0	0.726	0.024	0.113	0.042	0.088	8.21	15.79
9	0	0.715	0	0.625	0.022	0.098	0.039	0.083	9.52	14.08
15	0	0.256	0	0.209	0.005	0.059	0.009	0.053	4.77	4.72
18	0	0.152	0	0.091	0	0.030	0	0	4.98	> 10
19	0	0.130	0	0.071	0	0.025	0	0.015	5.31	4.85

Table 5.11: Bounds for $P(\xi = k)$ when $n = 20$, $M = 9$

$n = 20$										
M=9	Without the use of unimodality				With Unimodality				Improvement	
k	LB_2	UB_2	LB_3	UB_3	LB_2^*	UB_2^*	LB_3^*	UB_3^*	Δ_2/Δ_2^*	Δ_3/Δ_3^*
1	0	0.169	0	0.097	0	0.035	0	0.027	4.76	3.60
3	0	0.251	0	0.177	0	0.062	0.002	0.058	4.05	3.15
5	0	0.398	0	0.357	0.004	0.117	0.012	0.096	3.53	4.25
8	0	0.809	0	0.652	0.018	0.273	0.018	0.242	3.17	2.91
9	0	0.817	0	0.751	0.072	0.520	0.074	0.500	1.82	1.76
15	0	0.393	0	0.371	0.010	0.106	0.022	0.079	4.08	6.64
18	0	0.203	0	0.144	0	0.048	0	0.044	4.23	3.28
19	0	0.167	0	0.109	0	0.038	0	0.031	4.45	3.51

Table 5.12: Bounds for $P(\xi = k)$ when $n = 20$, $M = 15$

$n = 20$										
M=15	Without the use of unimodality				With Unimodality				Improvement	
k	LB_2	UB_2	LB_3	UB_3	LB_2^*	UB_2^*	LB_3^*	UB_3^*	Δ_2/Δ_2^*	Δ_3/Δ_3^*
1	0	0.200	0	0.133	0	0.041	0	0.034	4.86	3.71
3	0	0.283	0	0.231	0.012	0.060	0.017	0.053	5.95	6.28
5	0	0.417	0	0.412	0.022	0.064	0.023	0.061	9.88	10.83
8	0	0.733	0	0.523	0.029	0.077	0.037	0.073	15.29	14.74
9	0	0.717	0	0.597	0.030	0.084	0.041	0.076	13.36	17.26
15	0	0.576	0	0.565	0.059	0.453	0.064	0.231	1.46	3.40
18	0	0.315	0	0.235	0	0.078	0.007	0.074	4.06	3.53
19	0	0.2626	0	0.1748	0	0.0545	0	0.0467	4.82	3.74

Table 5.13: Bounds for $P(\xi = k)$ when $n = 30$, $M = 8$

$n = 30$										
M=8	Without the use of unimodality				With Unimodality				Improvement	
k	LB_2	UB_2	LB_3	UB_3	LB_2^*	UB_2^*	LB_3^*	UB_3^*	Δ_2/Δ_2^*	Δ_3/Δ_3^*
1	0	0.193	0	0.116	0	0.026	0	0.019	7.53	5.99
3	0	0.255	0	0.175	0	0.043	0	0.036	5.9	4.81
5	0	0.345	0	0.275	0.001	0.080	0.003	0.077	4.33	3.62
8	0	0.565	0	0.558	0.015	0.436	0.056	0.326	1.46	2.05
9	0	0.664	0	0.610	0.017	0.255	0.038	0.183	2.98	4.20
15	0	0.821	0	0.690	0.014	0.088	0.026	0.079	11.6	13.07
18	0	0.619	0	0.491	0.011	0.068	0.015	0.067	11.37	9.50
19	0	0.526	0	0.450	0.009	0.064	0.012	0.060	10.15	9.34
22	0	0.323	0	0.297	0.002	0.055	0.009	0.047	6.48	7.80
25	0	0.208	0	0.159	0	0.036	0.001	0.035	5.82	4.80
27	0	0.161	0	0.110	0	0.026	0	0.021	6.25	5.19
29	0	0.128	0	0.078	0	0.019	0	0.013	6.88	6.18

Table 5.14: Bounds for $P(\xi = k)$ when $n = 30$, $M = 14$

$n = 30$										
M=14	Without the use of unimodality				With Unimodality				Improvement	
k	LB_2	UB_2	LB_3	UB_3	LB_2^*	UB_2^*	LB_3^*	UB_3^*	Δ_2/Δ_2^*	Δ_3/Δ_3^*
1	0	0.213	0	0.137	0	0.034	0	0.029	6.32	4.79
3	0	0.279	0	0.205	0	0.048	0.004	0.046	5.86	4.79
5	0	0.373	0	0.318	0	0.068	0.012	0.057	5.72	7.13
8	0	0.595	0	0.562	0.008	0.091	0.018	0.075	7.82	9.98
9	0	0.690	0	0.587	0.010	0.101	0.020	0.085	8.27	8.98
15	0	0.799	0	0.694	0.017	0.236	0.015	0.209	3.6	3.57
18	0	0.648	0	0.515	0.018	0.105	0.013	0.080	6.9	7.68
19	0	0.556	0	0.479	0.019	0.091	0.013	0.067	6.85	8.80
22	0	0.350	0	0.321	0	0.065	0.010	0.046	5.62	8.80
25	0	0.230	0	0.173	0	0.037	0.004	0.036	6.27	5.33
27	0	0.178	0	0.119	0	0.026	0	0.024	6.86	5.05
29	0	0.142	0	0.084	0	0.019	0	0.015	7.44	5.56

Table 5.15: Bounds for $P(\xi = k)$ when $n = 30$, $M = 21$

$n = 30$										
M=21	Without the use of unimodality				With Unimodality				Improvement	
k	LB_2	UB_2	LB_3	UB_3	LB_2^*	UB_2^*	LB_3^*	UB_3^*	Δ_2/Δ_2^*	Δ_3/Δ_3^*
1	0	0.148	0	0.089	0.965	0.043	0	0.015	3.48	5.86
3	0	0.186	0	0.127	0.944	0.029	0	0.026	6.34	4.94
5	0	0.241	0	0.186	0.906	0.047	0.004	0.039	5.18	5.30
8	0	0.368	0	0.349	0.848	0.055	0.009	0.048	7.9	9.11
9	0	0.430	0	0.430	0.824	0.059	0.011	0.052	8.82	10.46
15	0	0.789	0	0.727	0.575	0.118	0.027	0.076	7.74	15.04
18	0	0.836	0	0.693	0.425	0.192	0.033	0.101	4.81	10.08
19	0	0.857	0	0.647	0.373	0.233	0.034	0.122	4.01	7.36
22	0	0.566	0	0.564	0	0.254	0.016	0.153	2.23	4.11
25	0	0.355	0	0.293	0	0.079	0.007	0.074	4.5	4.41
27	0	0.266	0	0.191	0	0.049	0	0.041	5.49	4.68
29	0	0.205	0	0.129	0	0.032	0	0.023	6.36	5.58

Table 5.16: Bounds for $P(\xi = k)$ when $n = 30$, $M = 8$, and $2 \leq m \leq 6$

$n = 30$, $M = 8$											
		Without the use of unimodality					With Unimodality				
		$m = 2$	$m = 3$	$m = 4$	$m = 5$	$m = 6$	$m = 2$	$m = 3$	$m = 4$	$m = 5$	$m = 6$
k=1	LB	0	0	0	0	0	0	0	0	0	0
	UB	0.193	0.116	0.060	0.043	0.031	0.026	0.019	0.016	0.0141	0.0129
k=3	LB	0	0	0	0	0	0	0	0	0.004	0.005
	UB	0.255	0.175	0.114	0.097	0.088	0.043	0.036	0.035	0.032	0.027
k=5	LB	0	0	0	0	0	0.001	0.003	0.006	0.007	0.007
	UB	0.345	0.275	0.231	0.228	0.196	0.080	0.077	0.065	0.054	0.050
k=8	LB	0	0	0	0	0	0.015	0.056	0.053	0.058	0.060
	UB	0.565	0.558	0.491	0.450	0.427	0.436	0.326	0.264	0.260	0.201
k=9	LB	0	0	0	0	0	0.017	0.038	0.044	0.044	0.050
	UB	0.664	0.610	0.557	0.555	0.456	0.255	0.183	0.161	0.154	0.126
k=15	LB	0	0	0	0	0	0.014	0.026	0.028	0.032	0.033
	UB	0.821	0.690	0.535	0.478	0.440	0.088	0.079	0.071	0.069	0.065
k=18	LB	0	0	0	0	0	0.011	0.015	0.022	0.023	0.025
	UB	0.619	0.491	0.476	0.393	0.330	0.068	0.067	0.059	0.056	0.055
k=19	LB	0	0	0	0	0	0.009	0.012	0.019	0.021	0.022
	UB	0.526	0.450	0.408	0.386	0.300	0.064	0.060	0.056	0.052	0.051
k=22	LB	0	0	0	0	0	0.002	0.009	0.009	0.013	0.015
	UB	0.323	0.297	0.289	0.233	0.225	0.055	0.047	0.044	0.043	0.039
k=25	LB	0	0	0	0	0	0	0.001	0.004	0.005	0.005
	UB	0.208	0.159	0.124	0.115	0.114	0.036	0.035	0.032	0.028	0.027
k=27	LB	0	0	0	0	0	0	0	0	0.001	0.002
	UB	0.161	0.110	0.067	0.053	0.043	0.026	0.021	0.019	0.018	0.018
k=29	LB	0	0	0	0	0	0	0	0	0	0
	UB	0.128	0.078	0.039	0.025	0.016	0.019	0.013	0.009	0.007	0.005

Chapter 6

Applications

Our bounding technique which takes into account the shape of the underlying probability distribution can be used to approximate (i) the distribution of critical path length in PERT, (ii) the k-out-of-n type system reliability in reliability, and (iii) the distribution of the European Call Option Price [47, 55, 56, 57].

In this Chapter we apply our bounding methodology to obtain lower and upper bounds for the distribution of the critical, i.e., the largest path in "Example 1" presented by Prékopa, Long, and Szántai (2004) [44], where an approximation of the cumulative distribution function (c.d.f.) of the critical path under moment information. In that paper first an enumeration algorithm finds those paths that are candidates to become critical. Then probability distribution of the path lengths are approximated by a multivariate normal distribution that serves a basis for the bounding procedure.

Below we briefly outline the approach proposed by Prékopa, Long, and Szántai (2004) [44] and give details of our approach which provide us with improved bounds for the c.d.f of the critical path length in "Example 1" presented in [44].

6.1 Application in PERT

ERT is a project management tool used to schedule, organize, and coordinate tasks within a project. PERT stands for **P**rogram **E**valuation **R**eview **T**echnique. It is developed by the U.S. Navy in the 1950s to manage the Polaris submarine missile program. A PERT chart presents a graphic illustration of a project as a network diagram consisting of numbered nodes representing events, or milestones in the project linked by labelled/weighted arcs representing tasks in the project, where weight of an arc corresponds to the duration of the associated task. The PERT method is used to determine how long a project should take to complete, and which steps in the project planning are most critical – that is, which steps would act as bottlenecks if delayed.

The critical path is determined by adding the times for the activities in each sequence and determining the longest path in the project. The critical path determines the shortest time required for the completion of the project. If activities outside the critical path speed up or slow down (within limits), the total project time does not change. A good reference text for PERT and critical path problem is Hillier and Lieberman (2001) [26].

Consider a project network, G, which consists of τ tasks (arcs), an origin (a node representing the start of the project) with no arcs leading into it, a terminal node (representing the completion of the project) with no arcs leaving it, and N paths from the origin to the terminal node. Let $A = (a_{ij})$ denote the $N \times \tau$ path-arc incidence matrix of network G defined as

$$a_{ij} = \begin{cases} 1 & \text{if path i contains arc j} \\ 0 & \text{otherwise} \end{cases}$$

Let $\nu_t, t = 1, ..., \tau$ be the duration of task t and define $\nu = (\nu_1, ..., \nu_\tau)^T$, that is the random vector of task durations. Then the length of a critical path, i.e., the maximum length path from the origin to the terminal node, can be given as

$$L(\nu) = \max_{1 \le i \le N} a_i^T \nu \tag{6.1}$$

where a_i^T is the ith row of the path-arc incidence matrix A. Note that the critical path length (6.1) can also be written as

$$L(\nu) = \max_{1 \le i \le N} \sum_{j \in P_i} \nu_j$$

where $P_i, i = 1, ..., N$ are the paths in network G. Note that $L(\nu)$ is the convolution (sum) of the task durations on the critical path length and hence, is a random variable. Our goal then is to find or approximate the c.d.f of the critical path length, that is,

$$F(x) = P(L(\nu) \le x) \tag{6.2}$$

Since the problem is of practical several research efforts have been channelled to this area. We refer to [44] and the references therein. In what follows we describe how the problem of approximation of the c.d.f. of the critical path length can be formulated as a binomial moment problem similar to problem (1.4).

6.1.1 Approximation of the distribution of critical path length

Let $\mu_t = E(\nu_t)$ and $\sigma_t^2 = Var(\nu_t)$ be the expected value and variance of task durations $\nu_t, t = 1, ..., \tau$, respectively and $\mu = (\mu_1, ..., \mu_\tau)^T$ the vector of expected durations, C the covariance matrix of random vector $\nu = (\nu_1, ..., \nu_\tau)^T$. Then we can rewrite the c.d.f of $L(\nu)$ as

$$\begin{aligned} F(x) &= P(\max_{1\le i\le N} a_i^T\nu \le x) \\ &= P(a_1^T\nu \le x, \cdots, a_N^T\nu \le x) \\ &= P\left(\frac{a_1^T\nu - a_1^T\mu}{\sqrt{a_1^TCa_1}} \le \frac{x - a_1^T\mu}{\sqrt{a_1^TCa_1}}, \cdots, \frac{a_N^T\nu - a_N^T\mu}{\sqrt{a_N^TCa_N}} \le \frac{x - a_N^T\mu}{\sqrt{a_N^TCa_N}}\right) \end{aligned} \tag{6.3}$$

where the random variable $\frac{a_i^T\nu - a_i^T\mu}{\sqrt{a_i^TCa_i}}, i = 1, ..., N$ have the standard normal distribution.

If we define A_i as the event that

$$\frac{a_i^T\nu - a_i^T\mu}{\sqrt{a_i^TCa_i}} \le \frac{x - a_i^T\mu}{\sqrt{a_i^TCa_i}}, \quad i = 1, ..., N,$$

then the c.d.f (6.3) is equivalent to

$$\begin{aligned} F(x) &= P(A_1 \cap \cdots \cap A_N) \\ &= 1 - P(\bar{A}_1 \cup \cdots \cup \bar{A}_N) \end{aligned} \tag{6.4}$$

where $\bar{A}_i$ is the complement of event A_i.

In order to use a binomial moment based probability bounding approach to approximate the c.d.f of the critical path length, $L(\nu)$, more specifically the probability of the union of events $\bar{A}_1, ..., \bar{A}_N$ in (6.4) we introduce a random variable ξ as the number of those events among $\bar{A}_1, ..., \bar{A}_N$ occur. Then we have the binomial moments

$$\bar{S}_j = \sum_{1\le t_1<...<t_j\le N} P(\bar{A}_{t_1} \cap \cdots \cap \bar{A}_{t_j}) = E\left[\binom{\xi}{j}\right], \quad j = 1, ..., N \tag{6.5}$$

and $\bar{S}_0 = 1$.

Let $p_i = P(\xi = i), i = 0, ..., N$. Then the following equations can uniquely determine the jth binomial moments:

$$\sum_{i=0}^{N} \binom{i}{j} p_i = \bar{S}_j, \quad j = 1, ..., N$$

and hence, the two optimum values of the binomial moment problem

$$\begin{aligned} & \min(\max) \quad p_1 + ... + p_N \\ \text{subject to} \quad & \\ & \sum_{i=0}^{N} \binom{i}{j} p_i = \bar{S}_j, \quad j = 0, \cdots, m \\ & p_i \geq 0, \quad i = 0, \cdots, N \end{aligned} \tag{6.6}$$

provide us with sharp lower and upper bounds for $P(\xi \geq 1) = P(\bar{A}_1 \cup \cdots \cup \bar{A}_N)$ based on the first m binomial moments of the random variable ξ as discussed in Chapter 1.

For the case of $m = 2$, i.e., the first two binomial moments are known, Dawson and Sankoff [13, 44] presented the following sharp lower bound for the probability of the union of events:

$$P(\xi \geq 1) \geq \frac{2}{i+1}\bar{S}_1 - \frac{2}{i(i+1)}\bar{S}_2 \tag{6.7}$$

where $i = 1 + \lfloor \frac{2\bar{S}_2}{\bar{S}_1} \rfloor$.

The upper bound for the probability of the union of events based on the first two binomial moments was presented by Kwerel [32] and Sathe, Pradhan, and Shah [54] and is given by

$$P(\xi \geq 1) \leq \bar{S}_1 - \frac{2}{N}\bar{S}_2. \tag{6.8}$$

Subasi (2007) [55] presented the following lower bounds based on the first two binomial moments for the probability of the union of events, where the shape of the distribution is unimodal with mode M, where $0 \leq M \leq N$.

Case 1: $1 \leq i \leq M-2$

$$P(\xi \geq 1) \geq \frac{(M-1)(i-1)}{M(i+1)} + \frac{2(M+2i-2)\bar{S}_1 - 6(2M-1)\bar{S}_2}{Mi(i+1)} \tag{6.9}$$

where

$$2(M+i-2)\bar{S}_1 - 6\bar{S}_2 \geq (M-1)i$$

$$2(M+i-3)\bar{S}_1 - 6\bar{S}_2 \leq (M-1)(i-1)$$

$$2(M+2i-2)\bar{S}_1 - 6\bar{S}_2 \leq i(2M+i-1)$$

Case 2: $M \leq i \leq N-1$

$$P(\xi \geq 1) \geq \frac{(M-1)i}{M(i+2)} + \frac{2(M+2i)\bar{S}_1 - 6\bar{S}_2}{M(i+1)(i+2)} \tag{6.10}$$

where

$$2(M+i-1)\bar{S}_1 - 6\bar{S}_2 \geq (M-1)(i+1)$$

$$2(M+i-2)\bar{S}_1 - 6\bar{S}_2 \leq (M-1)i$$

$$2(M+2i)\bar{S}_1 - 6\bar{S}_2 \leq (i+1)(2M+i)$$

Case 3: $i = M-1$

$$P(\xi \geq 1) \geq \frac{M-2}{M+1} + \frac{8(M-1)\bar{S}_1 - 6\bar{S}_2}{(M-1)M(M+1)} \tag{6.11}$$

where

$$2(2M-2)\bar{S}_1 - 6\bar{S}_2 \geq M(M-1)$$

$$2(2M-4)\bar{S}_1 - 6\bar{S}_2 \le (M-2)(M-1)$$

$$2(4M-4)\bar{S}_1 - 6\bar{S}_2 \le 3M(M-1).$$

Subasi (2007) [55] also gave the following upper bound based on the first two binomial moments for the probability of the union of events, where the shape of the distribution is unimodal with mode M, where $0 \le M \le N$.

$$P(\xi \ge 1) \le \frac{2(N+M)\bar{S}_1 - 6\bar{S}_2}{(N+1)(M+1)} \tag{6.12}$$

where

$$2(N+M)\bar{S}_1 - 6\bar{S}_2 \le (N+1)(M+1)$$

$$2(N+M-1)\bar{S}_1 - 6\bar{S}_2 \ge NM$$

$$(M-1)\bar{S}_1 \le 3\bar{S}_2.$$

We remark that the lower and upper bounds presented by Subasi (2007) [55] can be obtained as special cases of the bounds given in Sections 4.1-4.2: Closed form bounds for $P(\xi \ge k)$ in Chapter 4 by choosing $k = 1$.

Also, note that due to the relation in (6.4), the inequalities (6.7), (6.9)-(6.11) give upper bounds for $P(\xi \ge 1)$ and the inequalities (6.8) and (6.12) generate lower bounds for $P(\xi \ge 1)$.

6.1.2 Finding the mode of the c.d.f of the critical path length

If the probability distribution of the critical path length is known to be unimodal, but we do not know where the mode is, then we can use the closed form bounds in (6.9)-(6.12) and reoptimize them with respect to the unknown mode M. Let $LB(M)$ and $UB(M)$ denote the lower and upper bounds as functions of M, respectively. Since M is unknown we can find lower and upper bounds for $P(\xi \geq 1)$ as follows:

$$\min_{0 \leq M \leq N} LB(M) \leq P(\xi \geq 1) \leq \max_{0 \leq M \leq N} UB(M)$$

More specifically, we solve the following mixed nonlinear integer programming problem to find an upper bound for $P(\xi \geq 1)$:

$$\begin{aligned}
&\text{maximize} \quad \frac{2(N+M)\bar{S}_1 - 6\bar{S}_2}{(N+1)(M+1)} \\
&\text{subject to} \\
&\qquad 2(N+M)\bar{S}_1 - 6\bar{S}_2 \leq (N+1)(M+1) \\
&\qquad 2(N+M-1)\bar{S}_1 - 6\bar{S}_2 \geq NM \\
&\qquad (M-1)\bar{S}_1 \leq 3\bar{S}_2 \\
&\qquad 0 \leq M \leq N, \quad M \in \mathbb{Z}^+
\end{aligned} \tag{6.13}$$

Once the value of M is determined from the optimization problem (6.13), we solve reoptimize the lower bound closed forms given in (6.9)-(6.12) based on the value of M.

For Case 1 we solve

$$\begin{aligned}
&\text{minimize} \quad \frac{(M-1)(i-1)}{M(i+1)} + \frac{2(M+2i-2)\bar{S}_1 - 6(2M-1)\bar{S}_2}{Mi(i+1)} \\
&\text{subject to} \\
&\qquad 2(M+i-2)\bar{S}_1 - 6\bar{S}_2 \geq (M-1)i \\
&\qquad 2(M+i-3)\bar{S}_1 - 6\bar{S}_2 \leq (M-1)(i-1) \\
&\qquad 2(M+2i-2)\bar{S}_1 - 6\bar{S}_2 \leq i(2M+i-1) \\
&\qquad 1 \leq i \leq M-2, \quad i \in \mathbb{Z}^+
\end{aligned} \tag{6.14}$$

For Case 2 we have

$$\begin{aligned}
&\text{minimize} \quad \frac{(M-1)i}{M(i+2)} + \frac{2(M+2i)\bar{S}_1 - 6\bar{S}_2}{M(i+1)(i+2)} \\
&\text{subject to} \\
&\qquad 2(M+i-1)\bar{S}_1 - 6\bar{S}_2 \geq (M-1)(i+1) \\
&\qquad 2(M+i-2)\bar{S}_1 - 6\bar{S}_2 \leq (M-1)i \\
&\qquad 2(M+2i)\bar{S}_1 - 6\bar{S}_2 \leq (i+1)(2M+i) \\
&\qquad M \leq i \leq N-1, \quad i \in \mathbb{Z}^+
\end{aligned} \tag{6.15}$$

As for Case 3, we simply substitute the value of M in the closed form bound (6.11) if the corresponding optimality conditions are satisfied.

Below we adopt an example ("Example 1") presented by [44] and apply our proposed approach to obtain lower and upper bounds for the c.d.f of the critical path length. We also compare the new bounds we find with those obtained from (6.7) and (6.8).

6.2 Computational results

Consider the network G given in Figure 6.1 with 28 nodes and $\tau = 66$ arcs, where arcs represents the task durations which are assumed to be independent discrete random variables, node #1 is the origin and node #28 is the terminal node. Network G contains $N = 1623$ paths from the origin to the terminal node.

Tables 6.1-6.2 give the lower and upper bounds on the 66 task durations in network G. Tables 6.3-6.6 show the assumed discrete probability distribution of each task duration (arc) in Network G. Note that the probability function of each task duration in Tables 6.3-6.6 is unimodal and hence, the critical path length which the convolution of unimodal discrete random variables is also unimodal. In fact, probability functions obtained by the discretizations, using equal length subintervals, are logconcave sequences. Convolution of logconcave sequences are also logconcave and any logconcave sequence is unimodal. Proof of this assertion can be found in Prékopa (1995) [43].

Prékopa, Long, and Szántai (2004) [44] used a path elimination technique to eliminate redundant paths in network G. The procedure resulted in a network $\hat{G}$, consisting on $n = 8$ paths, where only 21 arcs (tasks) are involved. Network $\hat{G}$ of non-eliminated paths are shown in Figure 6.2.

Figure 6.1: PERT Network G

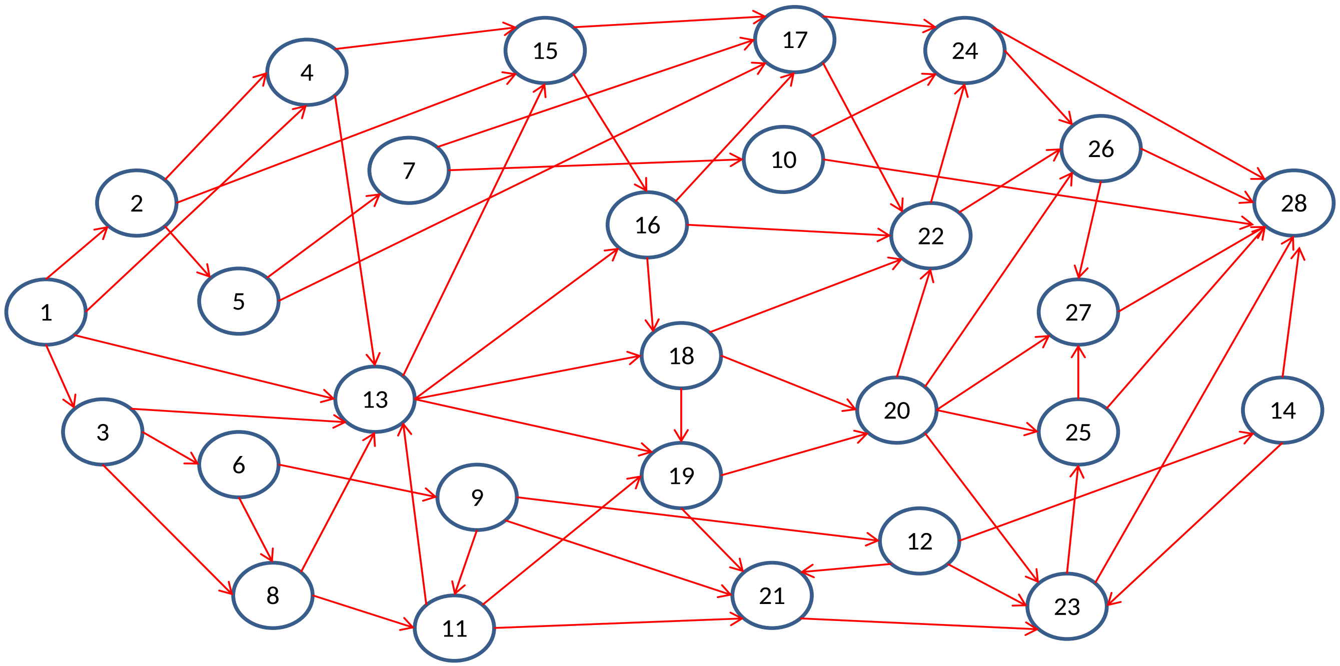


Table 6.1: Lower and upper bounds on task durations 1 through 33

No	Arcs	Lower Bound	Upper Bound
1	(1,2)	24	32
2	(1,3)	48	56
3	(1,4)	49	57
4	(1,13)	24	32
5	(2,4)	21	29
6	(2,5)	43	51
7	(2,15)	30	38
8	(3,6)	14	22
9	(3,8)	28	36
10	(3,13)	29	37
11	(4,13)	36	44
12	(4,15)	19	27
13	(5,7)	49	57
14	(5,17)	12	20
15	(6,8)	35	43
16	(6,9)	28	36
17	(7,10)	15	23
18	(7,17)	26	34
19	(8,11)	33	41
20	(8,13)	46	54
21	(9,11)	41	49
22	(9,12)	47	55
23	(9,21)	42	50
24	(10,24)	40	48
25	(10,28)	37	45
26	(11,13)	27	35
27	(11,19)	26	34
28	(11,21)	31	39
29	(12,14)	38	46
30	(12,21)	48	56
31	(12,23)	29	37
32	(13,15)	32	40
33	(13,16)	20	28

Table 6.2: Lower and upper bounds on task durations 34 through 66

No	Arcs	Lower Bound	Upper Bound
34	(13,18)	47	55
35	(13,19)	44	52
36	(14,23)	11	19
37	(14,28)	36	44
38	(15,16)	39	47
39	(15,17)	18	26
40	(16,17)	13	21
41	(16,18)	41	49
42	(16,22)	42	50
43	(17,22)	38	46
44	(17,24)	27	35
45	(18,19)	26	34
46	(18,20)	39	47
47	(18,22)	25	33
48	(19,20)	13	21
49	(19,21)	16	24
50	(20,22)	29	37
51	(20,23)	42	50
52	(20,25)	33	41
53	(20,26)	43	51
54	(20,27)	44	52
55	(21,23)	22	30
56	(22,24)	46	54
57	(22,26)	19	27
58	(23,25)	33	41
59	(23,28)	39	47
60	(24,26)	15	23
61	(24,28)	48	56
62	(25,27)	27	35
63	(25,28)	26	34
64	(26,27)	29	37
65	(26,28)	22	30
66	(27,28)	20	28

Table 6.3: Probability distributions of task durations 1 through 15

1	(1,2)	24	26	28	30	32
		0.007	0.115	0.569	0.287	0.022
2	(1,3)	48	50	52	54	56
		0.02	0.159	0.385	0.362	0.074
3	(1,4)	49	51	53	55	57
		0.085	0.187	0.444	0.191	0.093
4	(1,13)	24	26	28	30	32
		0.067	0.122	0.462	0.278	0.071
5	(2,4)	21	23	25	27	29
		0.041	0.19	0.416	0.246	0.107
6	(2,5)	43	45	47	49	51
		0.032	0.178	0.348	0.276	0.166
7	(2,15)	30	32	34	36	38
		0.012	0.067	0.705	0.171	0.045
8	(3,6)	14	16	18	20	22
		0.043	0.136	0.498	0.199	0.124
9	(3,8)	28	30	32	34	36
		0.041	0.274	0.323	0.285	0.077
10	(3,13)	29	31	33	35	37
		0.007	0.157	0.495	0.282	0.059
11	(4,13)	36	38	40	42	44
		0.015	0.057	0.475	0.412	0.041
12	(4,15)	19	21	23	25	27
		0.031	0.104	0.7	0.134	0.031
13	(5,7)	49	51	53	55	57
		0.1	0.138	0.349	0.282	0.131
14	(5,17)	12	14	16	18	20
		0.096	0.179	0.298	0.283	0.144
15	(6,8)	35	37	39	41	43
		0.05	0.152	0.427	0.227	0.144

Table 6.4: Probability distributions of task durations 16 through 32

16	(6,9)	28	30	32	34	36
		0.042	0.244	0.339	0.266	0.109
17	(7,10)	15	17	19	21	23
		0.075	0.233	0.259	0.246	0.187
18	(7,17)	26	28	30	32	34
		0.008	0.271	0.38	0.316	0.025
19	(8,11)	33	35	37	39	41
		0.04	0.137	0.425	0.307	0.091
20	(8,13)	46	48	50	52	54
		0.029	0.077	0.694	0.139	0.061
21	(9,11)	41	43	45	47	49
		0.118	0.178	0.338	0.246	0.12
22	(9,12)	47	49	51	53	55
		0.036	0.157	0.588	0.166	0.053
23	(9,21)	42	44	46	48	50
		0.048	0.266	0.286	0.27	0.13
24	(10,24)	40	42	44	46	48
		0.026	0.198	0.36	0.354	0.062
25	(10,28)	37	39	41	43	45
		0.061	0.197	0.439	0.227	0.121
26	(11,13)	27	29	31	33	35
		0.021	0.17	0.362	0.327	0.12
27	(11,19)	26	28	30	32	34
		0.025	0.135	0.654	0.145	0.041
28	(11,21)	31	33	35	37	39
		0.001	0.054	0.537	0.378	0.03
29	(12,14)	38	40	42	44	46
		0.074	0.149	0.464	0.181	0.132
30	(12,21)	48	50	52	54	56
		0.007	0.15	0.486	0.302	0.055
31	(12,23)	29	31	33	35	37
		0.027	0.108	0.474	0.293	0.098
32	(13,15)	32	34	36	38	40
		0.011	0.036	0.685	0.232	0.036

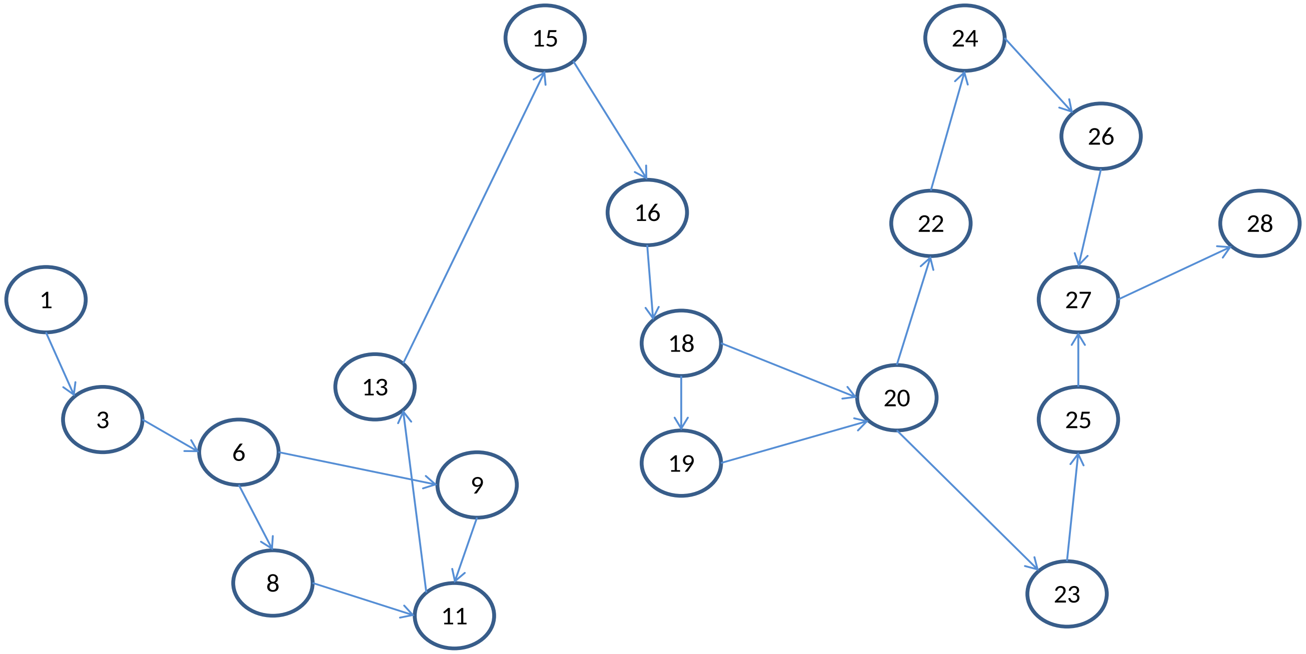
Table 6.5: Probability distributions of task durations 33 through 49

33	(13,16)	20	22	24	26	28
		0.054	0.269	0.328	0.278	0.071
34	(13,18)	47	49	51	53	55
		0.064	0.164	0.484	0.173	0.115
35	(13,19)	44	46	48	50	52
		0.129	0.156	0.318	0.245	0.152
36	(14,23)	11	13	15	17	19
		0.127	0.188	0.331	0.216	0.138
37	(14,28)	36	38	40	42	44
		0.023	0.044	0.827	0.079	0.027
38	(15,16)	39	41	43	45	47
		0.065	0.217	0.365	0.248	0.105
39	(15,17)	18	20	22	24	26
		0.041	0.186	0.35	0.344	0.079
40	(16,17)	13	15	17	19	21
		0.004	0.157	0.451	0.255	0.133
41	(16,18)	41	43	45	47	49
		0.006	0.207	0.544	0.213	0.03
42	(16,22)	42	44	46	48	50
		0.075	0.112	0.47	0.244	0.099
43	(17,22)	38	40	42	44	46
		0.115	0.207	0.328	0.209	0.141
44	(17,24)	27	29	31	33	35
		0.044	0.128	0.385	0.336	0.107
45	(18,19)	26	28	30	32	34
		0.069	0.122	0.561	0.13	0.118
46	(18,20)	39	41	43	45	47
		0.052	0.084	0.684	0.107	0.073
47	(18,22)	25	27	29	31	33
		0.02	0.162	0.495	0.212	0.111
48	(19,20)	13	15	17	19	21
		0.014	0.133	0.564	0.174	0.115
49	(19,21)	16	18	20	22	24
		0.061	0.16	0.445	0.186	0.148

Table 6.6: Probability distributions of task durations 50 through 66

50	(20,22)	29	31	33	35	37
		0.036	0.095	0.652	0.146	0.071
51	(20,23)	24	26	28	30	32
		0.129	0.144	0.416	0.18	0.131
52	(20,25)	33	35	37	39	41
		0.034	0.188	0.426	0.221	0.131
53	(20,26)	43	45	47	49	51
		0.074	0.174	0.396	0.252	0.104
54	(20,27)	44	46	48	50	52
		0.005	0.145	0.535	0.286	0.029
55	(21,23)	22	24	26	28	30
		0.084	0.132	0.36	0.334	0.09
56	(22,24)	46	48	50	52	54
		0.133	0.175	0.273	0.265	0.154
57	(22,26)	19	21	23	25	27
		0.097	0.124	0.534	0.133	0.112
58	(23,25)	33	35	37	39	41
		0.002	0.145	0.571	0.2	0.082
59	(23,28)	39	41	43	45	47
		0.043	0.224	0.331	0.277	0.125
60	(24,26)	15	17	19	21	23
		0.071	0.219	0.344	0.233	0.133
61	(24,28)	48	50	52	54	56
		0.01	0.107	0.491	0.299	0.093
62	(25,27)	27	29	31	33	35
		0.08	0.205	0.395	0.227	0.093
63	(25,28)	26	28	30	32	34
		0.002	0.146	0.456	0.38	0.016
64	(26,27)	29	31	33	35	37
		0.093	0.126	0.381	0.278	0.122
65	(26,28)	22	24	26	28	30
		0.044	0.192	0.465	0.216	0.083
66	(27,28)	20	22	24	26	28
		0.073	0.204	0.381	0.223	0.119

Figure 6.2: Network $\hat{G}$ of the 8 non-eliminated paths



The 21 arcs involved in the 8 non-eliminated paths are

- $\nu_2 = (1,3),\ \nu_8 = (3,6),\ \nu_{15} = (6,8),\ \nu_{16} = (6,9),\ \nu_{19} = (8,11),$
- $\nu_{21} = (9,11),\ \nu_{26} = (11,13),\ \nu_{32} = (13,15),\ \nu_{38} = (15,16),$
- $\nu_{41} = (16,18),\ \nu_{45} = (18,19),\ \nu_{46} = (18,20),\ \nu_{48} = (19,20),$
- $\nu_{50} = (20,22),\ \nu_{51} = (20,23),\ \nu_{56} = (22,24),\ \nu_{58} = (23,25),$
- $\nu_{60} = (24,26),\ \nu_{62} = (25,27),\ \nu_{64} = (26,27),\ \nu_{66} = (27,28).$

Hence, the path-arc incidence matrix corresponding to network $\hat{G}$ is of size 8×21 and is given below:

$$A = \begin{pmatrix} 1 & 1 & 0 & 1 & 0 & 1 & 1 & 1 & 1 & 1 & 1 & 0 & 1 & 1 & 0 & 1 & 0 & 1 & 0 & 1 & 1 \\ 1 & 1 & 1 & 0 & 1 & 0 & 1 & 1 & 1 & 1 & 1 & 0 & 1 & 1 & 0 & 1 & 0 & 1 & 0 & 1 & 1 \\ 1 & 1 & 0 & 1 & 0 & 1 & 1 & 1 & 1 & 1 & 0 & 1 & 0 & 1 & 0 & 1 & 0 & 1 & 0 & 1 & 1 \\ 1 & 1 & 1 & 0 & 1 & 0 & 1 & 1 & 1 & 1 & 0 & 1 & 0 & 1 & 0 & 1 & 0 & 1 & 0 & 1 & 1 \\ 1 & 1 & 0 & 1 & 0 & 1 & 1 & 1 & 1 & 1 & 1 & 0 & 1 & 0 & 1 & 0 & 1 & 0 & 1 & 0 & 1 \\ 1 & 1 & 1 & 0 & 1 & 0 & 1 & 1 & 1 & 1 & 1 & 0 & 1 & 0 & 1 & 0 & 1 & 0 & 1 & 0 & 1 \\ 1 & 1 & 0 & 1 & 0 & 1 & 1 & 1 & 1 & 1 & 0 & 1 & 0 & 0 & 1 & 0 & 1 & 0 & 1 & 0 & 1 \\ 1 & 1 & 1 & 0 & 1 & 0 & 1 & 1 & 1 & 1 & 0 & 1 & 0 & 0 & 1 & 0 & 1 & 0 & 1 & 0 & 1 \end{pmatrix}$$

Note that Paths 1&2 contain 15, Paths 3-6 contain 14, and Paths 7&8 contain 13 arcs (tasks). In Table 6.7 we present the lower and upper bounds for the c.d.f. of the critical path length in network $\bar{G}$ in Figure 6.2, where all values are tabulated for different argument values $x \in [480, 509]$. Table 6.8 shows the lower and upper bounds for the c.d.f. of the critical path length in network $\bar{G}$ in Figure 6.2, where all values are tabulated for different argument values $x \in [510, 539]$.

In Tables 6.7-6.8, LB and UB represent the lower and upper bounds for the c.d.f of the critical path length obtained from closed form bounds (6.8) and (6.7), respectively. LB^* and UB^* denote the bounds on obtained by our bounding approach where we solve the optimization problems (6.13) and (6.14)-(6.15) or use (6.11) based on the value of mode M found as the optimal solution of problem (6.13). The exact values of the c.d.f of the critical path length are also included in Tables 6.7-6.8 to compare the bounds obtained by our bounding methodology which takes into account the shape of the underlying distribution with those that do not.

Figure 6.3 shows the exact value and approximations of the c.d.f of the critical path length based on the first two binomial moments presented in Table **??**. Note that the bounds obtained by our bounding methodology are significantly tighter than those obtained without using the shape information of the probability function of the critical path length.

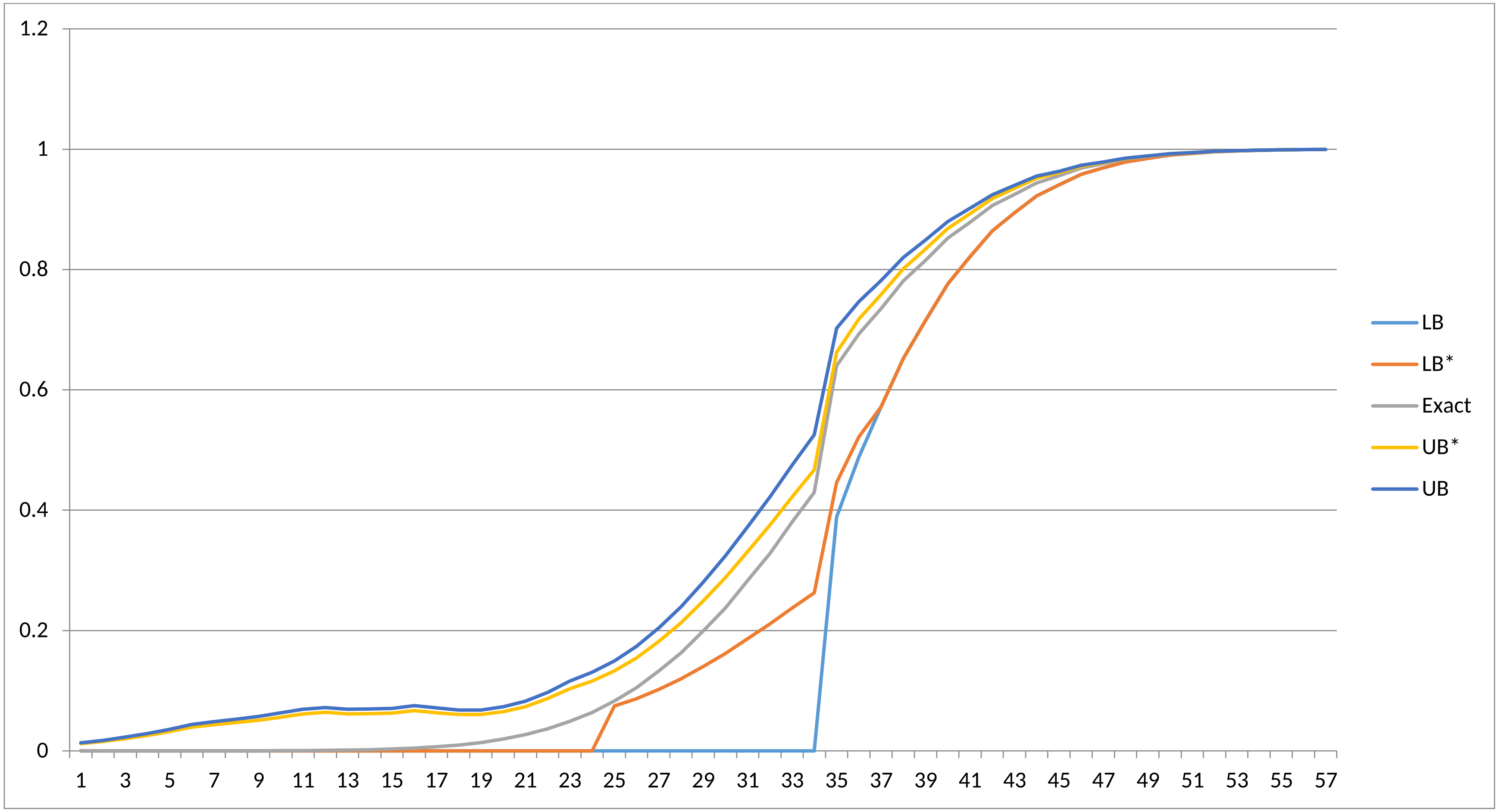
Table 6.7: Bounds for the c.d.f of the critical path length when $x \in [480, 509]$

x	LB	LB^*	Exact	UB^*	UB
480	0	0	0	0.0119	0.0134
481	0	0	0	0.0156	0.0176
482	0	0	0	0.0202	0.0227
483	0	0	0	0.0256	0.0288
484	0	0	0	0.0319	0.0359
485	0	0	0	0.0393	0.0442
486	0	0	0.0001	0.0434	0.0488
487	0	0	0.0001	0.0470	0.0529
488	0	0	0.0002	0.0508	0.0571
489	0	0	0.0003	0.0561	0.0631
490	0	0	0.0005	0.0615	0.0692
491	0	0	0.0008	0.0641	0.0721
492	0	0	0.0013	0.0616	0.0693
493	0	0	0.002	0.0620	0.0697
494	0	0	0.003	0.0628	0.0707
495	0	0	0.0045	0.0668	0.0751
496	0	0	0.0067	0.0635	0.0714
497	0	0	0.0097	0.0604	0.0679
498	0	0	0.0139	0.0604	0.0680
499	0	0	0.0195	0.0652	0.0734
500	0	0	0.027	0.0734	0.0826
501	0	0	0.0365	0.0869	0.0972
502	0	0	0.049	0.1029	0.1158
503	0	0	0.0639	0.1162	0.1307
504	0	0.0748	0.0832	0.1330	0.1496
505	0	0.0868	0.1052	0.1543	0.1736
506	0	0.1023	0.1328	0.1819	0.2046
507	0	0.1198	0.1627	0.2129	0.2395
508	0	0.1404	0.1996	0.2495	0.2807
509	0	0.1620	0.2375	0.2880	0.3240

Table 6.8: Bounds for the c.d.f of the critical path length when $x \in [510, 539]$

x	LB	LB^*	Exact	UB^*	UB
510	0	0.1865	0.2833	0.3315	0.3729
511	0	0.2109	0.3278	0.3748	0.4217
512	0	0.2375	0.3807	0.4221	0.4749
513	0	0.2627	0.4295	0.4670	0.5254
517	0.3886	0.4459	0.6402	0.6626	0.7025
518	0.4888	0.5220	0.6933	0.7180	0.7467
519	0.5715	0.5715	0.7349	0.7583	0.7817
520	0.6522	0.6522	0.7810	0.8014	0.8200
521	0.7153	0.7153	0.8150	0.8341	0.8489
522	0.7761	0.7761	0.8523	0.8681	0.8796
523	0.8211	0.8211	0.8782	0.8927	0.9016
524	0.8640	0.8640	0.9062	0.9177	0.9244
525	0.8939	0.8939	0.9245	0.9347	0.9398
526	0.9222	0.9222	0.9441	0.9516	0.9553
527	0.9408	0.9408	0.9561	0.9607	0.9632
528	0.9582	0.9582	0.9688	0.9716	0.9733
529	0.9690	0.9690	0.9761	0.9777	0.9788
530	0.9790	0.9790	0.9837	0.9847	0.9854
531	0.9848	0.9848	0.9878	0.9884	0.9888
532	0.9901	0.9901	0.9921	0.9924	0.9927
533	0.9930	0.9930	0.9942	0.9943	0.9945
534	0.9957	0.9957	0.9964	0.9965	0.9966
535	0.9970	0.9970	0.9975	0.9975	0.9976
536	0.9982	0.9982	0.9985	0.9986	0.9986
537	0.9988	0.9988	0.9990	0.9990	0.9990
538	0.9993	0.9993	0.9994	0.9994	0.9994
539	0.9996	0.9996	0.9996	0.9996	0.9996

Figure 6.3: Exact value and approximations of the c.d.f of the critical path length



References

[1] R. Barlow, F. Proschan, Statistical theory of reliability and life testing probability models, Holt, Rinehartand Winston Inc., 1975.

[2] R. Bettancourt, L. Szczecinski, R. Feick, BER evaluation of BICM-ID via Bonferroni-type bounds, Vehicular Technology, IEEE Transactions 5 (2008), 2815–2821.

[3] J. Battersby, Network Analysis for Planning and Scheduling (3rd edn.), Macmillan, 1970.

[4] G. Boole, Laws of thought, American Reprint of 1854 Edition, Dover, New York, 1854.

[5] C. E. Bonferroni, Teoria Statistica delle classi e calcoro delle probabilit, Volume in onore di Riccardo Dalla Volta, Universit di Frenze 2, 1937, pp. 1–62.

[6] E. Boros, A. Prékopa, Closed form two-sided bounds for probabilities that exactly r and at least r out of n events occur, Math. Oper. Res., 14 (1989), 317–347.

[7] E. Boros, A. Prékopa, Probabilistic bounds and algorithms for the maximum satisfiability problem, Annals of Operations Research 21 (1989), 109–126.

[8] E. Boros, A. Scozzari, F. Tardella, P. Veneziani, Polynomially computable bounds for the probability of the union of events, Mathematics of Operations Research 39 (2014), 1311–1329.

[9] J. Bukszár, A. Prékopa, Probability bounds with cherry trees, Math. Oper. Res., 26 (2001), 174–192.

[10] J. Bukszár, Hypermultitrees and Bonferroni inequalities, Mathemtical Inequalities and Applications, 6(2003), 727–745.

[11] J. Bukszár, G. Mádi-Nagy, T. Szántai, Computing bounds for the probability of the union of events by different methods, Annals of Operations Research 201(1) (2012), 63–81.

[12] S.H. Cox, Y. Lin, R. Tian, L.F. Zuluaga, Bounds for Probabilities of Extreme Events Defined by Two Random Variables, Variance Casualty Actuarial Society - Arlington, Virginia 4(10) (2010), 47–65.

[13] D.A. Dawson, A. Sankoff, An inequality for probabilities, Proceedings of the American Mathematical Society, 18 (1967), 504–507.

[14] K. Dohmen, P. Tittmann, Improved Bonferroni inequalities and binomially bounded functions, Electronic Notes in Discrete Mathematics 28 (2007), 91–93.

[15] J.H. Dulá, A multidimensional semi-parametric upper bound for pricing a general class of complex call options, University of Mississippi MS38677, 1995.

[16] S. Eryilmaz, On system relability in stress-strength setup, Statistics & Probability Letters 80 (2010), 34–839.

[17] M. Fréchet, Les Probabilités Associées a un Systéme d'Événement Compatibles et Dépendants, Actualités Scientifique et Industrielles, Nos. 859, 942, Paris, 1940/43.

[18] J. Galambos, R. Mucci, Inequalities for linear combinations of binomial moments, Publ. Math. 27 (1980), 263–269.

[19] J. Galambos, I. Simonelli, Bonferroni-type inequalities with applications, Springer, New York, 1996.

[20] L. Gao, A. Prékopa, On performance prediction of cellular telephone networks, RUTCOR Research Report RRR 41-2001.

[21] L. Gao, A. Prékopa, Lower and Upper Bounds for the Probability that at least r and Exactly r out of n Events Occur, Mathematical Inequalities and Applications 5 (2002), 315–333.

[22] J. Gessel, G. Viennot, Binomial determinants, paths, and hook length formulae, Advences in Mathematics, 58 (1985) 300–321.

[23] S. Giglio, Credit default swap spreads and systemic financial risk, Job Market Paper, Harvard University, 2011.

[24] B.D. Grundy, Option prices and underlying asset's return, The Journal of Finance 46 (1991), 1045–1069.

[25] T. Hailperin, Best possible inequalities for the probability of a logical function of events, Am. Math. Monthly 72 (1965), 343–359.

[26] F.S. Hillier and G.J. Lieberman, Introduction to Operations Research, New York: McGraw Hill, 2001.

[27] F.M. Hoppe, M. Nediak, Frechet optimal bounds on the probability of a union with supplementary information, Statistics & Probability Letters 78 (3) (2008), 311–319.

[28] A. Horváth, Lower and upper bounds on the probability that hazardous materials can be transported without incident from one point to another, Pollack Periodica 3 (2008), 114–120.

[29] A. Horváth, Investigation of Failure Systems, Acta Polytechnica Hungarica 5 (2) (2008), 127–132.

[30] D. Hunter, Bounds for the probability of a union, J. of Applied Probability 13 (1976), 597–603.

[31] K. Jordan, Foundations of probability theory (in Hungarian), Matematikai és Fizikai Lapok 34 (1927), 109–136.

[32] S.M. Kwerel, Most Stringent bounds on aggregated probabilities of partially specified dependent probability systems, J. Amer. Stat. Assoc. 70 (1975), 472–479.

[33] A.W. Lo, Semi-definite upper bounds for option prices and expected payoffs, Journal of Financial Economics 19 (1987), 373–387.

[34] B.B. Mandelbrot, The fractal geometry of nature, Freeman, New York, 1977.

[35] T.F. Móri, G.J. Székely, A Note on the Background of Several Bonferroni-Galambos-Type Inequalities, Journal of Applied Probability 22(4) (1985), 836–843.

[36] V.V. Petrov, A generalization of the Chung-Erds inequality for the probability of a union of events, Journal of Mathematical Sciences 147(4) (2007), 6932–6934.

[37] A. Prékopa, Boole-Bonferroni inequalities and linear programming, Operations Research 36 (1988), 145–162.

[38] A. Prékopa, Totally positive linear programming problems, in: L.V. Kantorovich Memorial Volume, Oxford Univ. Press, New York, 1989, pp. 197–207.

[39] A. Prékopa, Sharp bounds on probabilities using linear programming, Operations Research, 38 (1990), 227–239.

[40] A. Prékopa, The discrete moment problem and linear programming, Discrete Applied Mathematics, 27 (1990) 235–254.

[41] A. Prékopa, E. Boros, On the existence of a feasible flow in a stochastic transportation network, Operations Research 39 (1991), 119–129.

[42] A. Prékopa, E. Boros, K.W. Lih . The use of binomial moments for bounding network reliability, in: Reliability of Computer and Communication Networks, F. Roberts, F. Hwang, C. Monma, (Eds.). DIMACS Series in Discrete Mathematics and Theoretical Computer Science, Vol. 5, AMS/ACM, 1991, pp. 197–212.

[43] A. Prékopa, Stochastic Programming, Kluwer Academic Publishers, Dordtecht, Boston, 1995.

[44] A. Prékopa, J. Long, T. Szántai, New bounds and approximations for the probability distribution of the length of critical path, in: K. Marti, Yu. Ermoliev, G. Pflug (Eds.), Dynamic Stochastic Optimization, Lecture Notes in Economics and Mathematical Systems 532, Springer, Berlin, Heidelberg, 2004, pp. 293–320.

[45] A. Prékopa, Discrete higher order convex functions and their applications, in: N. Hadjisavvas, J.E. Martinez-Legaz, J-P. Penot (Eds.), Generalized Convexity and Monotonicity, Lecture Notes in Economics and Mathematical Systems, Springer, 2001, pp. 21–47.

[46] A. Prékopa, L. Gao, Bounding the probability of the union of events by the use of aggregation and disaggregation in linear programs, Discrete Applied Mathematics 145 (2005), 444–454.

[47] A. Prékopa, M. Subasi, E. Subasi, Sharp bounds for the probability of the union of events under unimodality condition, European Journal of Pure and Applied Mathematics 1 (2008), 60–81.

[48] A. Prékopa, A. Ninh, G. Alexe, On the relationship between the discrete and continuous bounding moment problems and their numerical solutions, Annals of Operations Research 238 (2016), 521–575.

[49] A. Prékopa, K. Yoda, Improved bounds on the probability of the union of events, some of whose intersections are empty, Operations Research Letters 44 (2016) 39–43.

[50] T. Radwan, A. Habib, R. Alseedy, A. Elsherbeny, Bounds for increasing multi-state consecutive k-out-of-r-from-n: F system with equal components probabilities, Applied Mathematical Modelling 35(5) (2011), 2366–2373.

[51] P. Ritchken, On option pricing bounds, The Journal of Finance 40 (1985), 1219–1233.

[52] F. Roberts, F. Hwang, C. Monma (Eds.), Reliability of Computer and Communication Networks, DIMACS Series in Discrete Math and Theoretical Computer Science 5, 1991.

[53] S.M. Samuels, W.J. Studden, Bonferroni-type probability bounds as an application of the theory of Tchebycheff system, Probability, Statistics and Mathematics, Papers in Honor of Samuel Karlin, Academic Press, 1989.

[54] Y.S. Sathe, M. Pradhan, S.P. Shah, Inequalities of the probability of occurrence at least m out of n events, Journal of Applied Probability 17 (1980), 1127–1132.

[55] E. Subasi, Sharp bounds for expectations and probabilities in a distribution with given shape information, Dissertation, RUTCOR, Rutgers Center for Operations Research, January 2007.

[56] E. Subasi, M.M. Subasi, A. Prékopa, Discrete moment problems with distributions known to be unimodal, Mathematical Inequalities and Applications, 12(3) (2009), 587-610.

[57] M.M. Subasi, E. Subasi, A. Binmahfoudh, A. Prékopa, New Bounds for the probability that at least k-out-of-n events occur with unimodal distributions, Discrete Applied Mathematics. Accepted for publication on March 2017.

[58] L. Takács, On the general probability theorem, Communications of the Dept. Of Math. and Physics of the Hungarian Acad. Sci. 5 (1955), 467–476.

[59] P. Veneziani, Graph-based upper bounds for probability of the union of events, The Electronic Journal of Combinatorics 15 (2008), #R28.

[60] P. Veneziani, Upper bounds of degree 3 for the probability of the union of events via linear programming, Discrete Applied Mathematics 157 (2009), 858–863.

Vita

Ahmed M. Binmahfoudh

Education

August 2012 – Present	Ph.D. in Operations Research Department of Mathematical Sciences Florida Institute of Technology, Melbourne, FL
February 2007 – August 2008	M.A. in Engineering Management Department of Engineering Systems Florida Institute of Technology, Melbourne, FL
May 1999 – July 2005	B.S. in Computer Engineering King Fahad University of Petroleum & Minerals Dhahran, KSA

Work Experience

January 2010 – Present	Lecturer Computers and Information Technology College Taif University

Awards

May 2015	***United States President SILVER Community Service*** Florida Institute of Technology, Melbourne, FL
May 2015	***Florida Tech President BRONZE Campus Service*** Florida Institute of Technology, Melbourne, FL
May 2015	***Florida Tech President GOLD Philanthropy*** Florida Institute of Technology, Melbourne, FL
April 2014	***Outstanding Student Award*** Florida Institute of Technology, Melbourne, FL
April 2008	***Outstanding Achievement Award*** Saudi Student Association Club Florida Institute of Technology, Melbourne, FL

Publications

- New bounds for the probability that at least k-out-of-n events occur with unimodal distributions. Discrete Applied Mathematics (2017), http://dx.doi.org/10.2016/j.da,/2017.03.011 (joint with M.M. Subasi, E. Subasi, and A. Prékopa)
- New bounds for the probability that exactly k-out-of-n events occur with unimodal distributions. Annals of Operations Research. To be submitted (joint with M.M. Subasi, E. Subasi, and A. Prékopa)
- Improved bounds for the distribution of length of critical path in PERT. Working paper

Conference Presentations

- New bounds for the probability that at least k-out-of-n events occur with unimodal distributions. SIAM Annual Conference, July 10-14, 2017, Pittsburgh, PA, USA (joint with M.M. Subasi, E. Subasi, and A. Prékopa)