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STOCHASTIC ACCELERATION OF GALACTIC COSMIC RAYS BY COMPRESSIBLE PLASMA FLUCTUATIONS IN SUPERNOVA SHELLS

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ABSTRACT

A theory of 2-stage acceleration of Galactic cosmic rays in supernova remnants is proposed. The first stage is accomplished by the supernova shock front, where a power-law spectrum is established up to a certain cutoff energy. It is followed by stochastic acceleration with compressible waves/turbulence in the downstream medium. With a broad $\propto k^{-2}$ spectrum for the compressible plasma fluctuations, the rate of stochastic acceleration is constant over a wide range of particle momentum. In this case, the stochastic acceleration process extends the power-law spectrum cutoff energy of Galactic cosmic rays to the knee without changing the spectral slope. This situation happens as long as the rate of stochastic acceleration is faster than 1/5 of the adiabatic cooling rate. A steeper spectrum of compressible plasma fluctuations that concentrate their power in long wavelengths will accelerate cosmic rays to the knee with a small bump before its cutoff in the comic-ray energy spectrum. This theory does not require a strong amplification of the magnetic field in the upstream interstellar medium in order to accelerate cosmic rays to the knee energy.

Key words: cosmic rays – ISM: magnetic fields – ISM: supernova remnants

1. INTRODUCTION

Supernova remnants are by far the most plausible candidates as dominant sources for the bulk of Galactic cosmic rays. The energy budget for populating and replenishing cosmic rays at the observed intensity level in the Galaxy, the chemical composition of the accelerated material, and the physics of particle acceleration are all consistent with observations. For recent reviews on this subject, interested readers may go for more details in these references (e.g., Drury 2012; Blasi 2013). These review articles may also provide further references on many less-controversial "facts" we use in this paper without a specific citation. Although it is believed that supernova shock waves can accelerate particles up to the knee energy ($\sim 3\times 10^{15}$ eV), such a claim does not come without controversy or difficulty. From the theory of diffusive shock acceleration, the maximum cutoff energy $E_c = cp_c$ of the power-law spectrum achievable by a shock wave in a finite time t is governed by the equation (Lagage & Cesarsky 1983)

$$t = \int_0^{p_c} \frac{3}{U_1 - U_2}\left(\frac{\kappa_1}{U_1} + \frac{\kappa_2}{U_2}\right)\frac{dp}{p}, \tag{1}$$

where U_1 and $U_2 = U_1/R$ are the upstream (subscript 1) and downstream (subscript 2) flow speed relative to the shock front with a compression ratio R (typically $R = 4$ for supernova shocks), and κ's are the particle diffusion coefficient. p is particle momentum, and c is the speed of light. Here, for simplicity, only relativistic cosmic rays are considered. It can be seen from Equation (1), given a limited shock lifetime, that a greater maximum cutoff energy is obtained with higher shock speed or smaller particle diffusion coefficient. The initial shock speed of a typical supernova remnant is quite certain, on the order of $U_{10} = 10^4 (E_{51}/M_{\mathrm{ej},\odot})^{1/2}$ km s^{-1}, where the E_{51} is the total supernova energy in units of 10^{51} erg and $M_{\mathrm{ej},\odot}$ is the amount of ejected mass in units of solar mass. It may slow down in the Sedov–Taylor phase once it has raked enough interstellar material to roughly be equal to the ejected mass, but still its time evolution is more or less certain. The biggest uncertainty in the estimation of the maximum cutoff energy is in the diffusion coefficient. Typically, due to a high level of turbulence in a shocked medium, the particle diffusion in the downstream region κ_2 should be much smaller than in the upstream region κ_1. The rate of particle acceleration by a shock wave is mainly controlled by the diffusion or scattering in the upstream region. One assumption for the diffusion coefficient is the Bohm limit, $\kappa = cp/3ZeB$, where B is the magnetic field strength and Ze is the particle charge. The Bohm diffusion limit is likely to come true when particles are scattered by strong magnetic turbulence generated by accelerated particles both upstream and downstream of extremely strong supernova shocks (Caprioli & Spitkovsky 2014a). Then one can obtain from Equation (1) the maximum cutoff energy as a function of time t

$$E_c = cp_c = \begin{cases} ZeB_1 U_{10}^2\left(1 - \dfrac{1}{R}\right)t & : t \leqslant t_{\mathrm{ST}} \\ ZeB_1 U_{10}^2\left(1 - \dfrac{1}{R}\right)t_{\mathrm{ST}}^{6/5} t^{-1/5} & : t > t_{\mathrm{ST}} \end{cases}, \tag{2}$$

where $t_{\mathrm{ST}} = 207 M_{\mathrm{ej},\odot}^{5/6} E_{51}^{-1/2} (n_{\mathrm{ISM}}/1\ \mathrm{cm}^{-3})^{-1/3}$ years is the time to reach the beginning of the Sedov–Taylor phase, n_{ISM} is the density of installer gas, and B_1 is the strength of magnetic field in the upstream interstellar medium. Given a typical value of $B_1 = 3\ \mu$G, the maximum cutoff energy is 2×10^{14} eV for protons at time t_{ST} and it slowly decreases from that value during the Sedov–Taylor phase. Clearly this maximum cutoff energy is well below the knee. There are other factors that can affect the maximum cutoff energy, such as adiabatic cooling and limited shock radius. We assume these factors are not able to reduce the maximum cutoff energy more than another order of magnitude.

Remedies for overcoming this problem of the maximum cutoff energy have been proposed by many researchers. Particularly notable is Jokipii (1987), who suggested that a nearly perpendicular shock can accelerate particles at higher rates, which might raise the maximum cutoff energy of cosmic rays produced by supernova shocks. This idea was further explored by Giacalone (2005) with a numerical simulation on microscopic scales. However, it is not clear if the high acceleration speed of perpendicular shocks will continue to be experienced by high-energy cosmic rays in the environment around a spherical supernova shock that has only a limited surface of nearly perpendicular magnetic geometry. Further study in this direction is needed. A more widely accepted approach is the amplification of a magnetic field. The amplification must happen to the upstream magnetic field B_1 in order to be effective to the particle acceleration speed by shocks. The amplified field must appear on scales roughly equal to or larger than the particle gyroradius near the knee energy. As one can see from Equation (2), the maximum cutoff is proportional to the upstream magnetic field strength. In order for the maximum cutoff energy to reach the knee energy, the upstream magnetic field needs to enhance by a factor of >10, possibly 10^2 after the consideration of other energy loss mechanisms. It means the magnetic field ahead of supernova shock should almost reach a few hundred μG, which can create many nonlinear effects back on the shock itself. Waves generated by resonating low-energy cosmic rays diffusing upstream (Lee 1982) cannot do the job on that large scale. Bell (2004, 2005) suggested a mechanism in which the magnetic field is amplified by a non-resonant instability generated by the current of highest-energy cosmic rays streaming away from the shock. There are many follow-up studies on this subject (e.g., Giacalone & Jokipii 2007; Caprioli & Spitkovsky 2014b), but how the instability evolves nonlinearly into that much higher field and on that large scale in the upstream interstellar medium has not yet been clearly demonstrated.

In this paper, we go a different route to achieve particle acceleration to the knee energy. Instead of the shock doing the job alone, compressible waves or turbulence downstream might be able to do some additional acceleration. Stochastic acceleration by compressible plasma fluctuations was applied to Galactic cosmic rays propagating in the interstellar medium (Ptuskin 1988; Bykov & Toptygin 1993; Bykov 2001), although Ptuskin (1988) dismissed it as a major mechanism to produce Galactic cosmic rays. Recently, its role in producing energetic particles has been reestablished due to progress in the heliosphere, where in situ observations see particle acceleration in regions where shocks are absent (Fisk & Gloeckler 2006, 2014). The accelerated particles often have a p^{-5} distribution function in the keV to MeV non-relativistic energy regime. One of the ideas is stochastic acceleration by compressible waves or turbulence (Fisk & Gloeckler 2008; Fisk et al. 2010; Zhang 2010, 2011). Zhang & Schlickeiser (2012) and Zhang & Lee (2013) demonstrated how nonlinear interaction between accelerated particles and compressible waves can automatically lead to the p^{-5} distribution. In the initial theoretical calculation of this paper, we show that stochastic acceleration can extend the shock power-law spectrum to a higher energy than the shock cutoff energy. In this away, a maximum cutoff energy at the knee can be achieved with the help of magnetic field amplification only in the downstream medium, which is much easier to achieve and understand than amplification in the upstream interstellar medium.

2. STOCHASTIC COMPRESSION ACCELERATION

Energetic particles are accelerated whenever they go through regions of plasma compression (where the divergence of plasma velocity $\nabla \cdot \boldsymbol{u} < 0$) and they are decelerated in plasma rarefaction regions ($\nabla \cdot \boldsymbol{u} > 0$). In a volume filled with compressible waves or turbulence, particles experience stochastic acceleration/deceleration if they go through them randomly. Quasilinear approximation leads to a diffusion process in the momentum space. The momentum diffusion coefficient can be calculated from the spectrum of longitudinal plasma velocity fluctuation $S(\boldsymbol{k})$ as a function of wavenumber $\boldsymbol{k}$ (Ptuskin 1988; Bykov & Toptygin 1993):

$$D_{\rm pp} = \frac{p^2 2\kappa}{9} \int d^3\boldsymbol{k} \frac{k^2 S(\boldsymbol{k})}{V^2 + \kappa^2 k^2}, \tag{3}$$

where κ is the particle spatial diffusion coefficient, and V is the phase speed of compressible waves. In the medium downstream of the supernova shock, where thermal and cosmic-ray particle pressure dominates magnetic pressure, V should be the sound speed, which is about $\sqrt{5}/4U_1$ immediately after the shock.

We expect the downstream medium to contain strong magnetic fields and associated compressible plasma fluctuations. In situ observations in the heliosphere confirm that these phenomena can happen to many types of shocks, such as coronal mass ejection shocks, solar wind corotating regions, solar wind termination shock, and the bow shock of planets. There, it has been observed that magnetic field amplification several tens of times over the upstream value are quite common. First the magnetic field is compressed by the shock, and gradual plasma compression by the driving medium behind the shock can further increase it. Upstream waves can interact with the shock, and a fraction of waves are converted by the shock to compressible fluctuations, which further increases the magnetic field amplification. Downstream plasma distribution could be unstable through, e.g., Richtmyer–Meshkov instability and Weibel instability or through the magnetic dynamo effect. The instabilities can convert a fraction of flow energy into the magnetic field (e.g., Giacalone & Jokipii 2007; Caprioli & Spitkovsky 2013, 2014b; Huntington et al. 2015). Under the strong magnetic fluctuation, the particle diffusion coefficient κ should more likely approach the Bohm limit. This time it is the amplified magnetic field in the downstream medium that enters into the equation for κ.

The property of velocity fluctuation in the compressible sound waves or turbulence behind shock waves is not well known, so we must guess. Laboratory experiments studying noise emitted from the shocked gas of supersonic jets observe a broadband frequency spectrum with ω^{-2} starting at a frequency of $\sim 0.3u_j/d$, where u_j is jet speed and d is the width of jet (Pao & Seiner 1983). The sound waves are generated when upstream shear waves are converted by the shock formed within the jet (Seiner 1987). A nonlinear wave cascade develops it into the power-law spectrum at high frequencies. Since sound waves have a linear dispersion between ω and k, the spectrum in the k-space has the same shape. A $S(k) \propto k^{-2}$ corresponds to the scaling law of well-developed 1D Burger turbulence (Kida 1979). It is probably a consequence of the steepening of compression into shock-lets in nonlinear fluid mechanics. Fluctuation behind supernova could be very strong. If allowed to evolve, the fluctuation will eventually run into the nonlinear stage, which is one reason for strong magnetic field amplification. Simulations by Caprioli & Spitkovsky (2013) see sheets of amplified magnetic fields, which indicates that the structure is primarily 1D. In 3D, if the shock-lets are randomly orientated, the spectrum of longitudinal velocity fluctuation is isotropic, and it is proportional to $S(k) \propto k^{-4}$ as a function of wavenumber magnitude k (Bykov & Toptygin 1985) . Let us take a 1D velocity fluctuation spectrum,

$$S(\boldsymbol{k}) = \delta(k_y)\delta(k_z)L_0^{-1}\langle u^2\rangle k^{-2}H(kL_0 - 1)H(1 - kL_1), \tag{4}$$

where δ is the Dirac δ-function and H is the Heaviside step function. It has a k^{-2} power law between $k_{\min} = L_0^{-1}$ and $k_{\max} = L_1^{-1}$ with $L_0 \gg L_1$. $\langle u^2\rangle = \int dk S(k)$ is the averaged longitudinal velocity fluctuation square. Substituting Equation (4) into (3), we can integrate it, which yields

$$D_{\rm pp} = D_0 p^2 = p^2 \frac{2\langle u^2\rangle}{9\, VL_0}\left(\mathrm{ArcCot}\frac{VL_0}{\kappa} - \mathrm{ArcCot}\frac{VL_1}{\kappa}\right). \tag{5}$$

When $VL_1 \ll \kappa \ll VL_0$, $D_{\rm pp} = p^2\pi\langle u^2\rangle/9\, VL_0$, its peak value. It tells us that the momentum diffusion rate $D_{\rm pp}/p^2 = D_0$ is essentially constant for all particles with diffusion lengths κ/V greater than the width of compression fronts L_1 but shorter than the average size of the waves L_0. It should cover a wide range of particle momentum if the Bohm limit is used for κ. To relativistic cosmic rays particularly those at high energies near the knee, meeting $VL_1 \ll \kappa$ should be easy, but the condition $\kappa \ll VL_0$ could place an upper limit on the maximum energy of particles able to receive efficient stochastic acceleration. $D_{\rm pp}$ drops quickly outside of this range, rendering an inefficient stochastic acceleration. This result is consistent with the simulation using prescribed 1D compressible wave trains (Zhang 2010, 2011). Ptuskin (1988) got the same answer for randomly oriented weak shocks in 3D interstellar medium turbulence (Bykov & Toptygin 1985).

3. SPECTRUM OF ACCELERATED COSMIC RAYS IN SUPERNOVA SHELLS

For simplicity, let us assume that the spatial diffusion coefficient κ in the medium downstream of supernova shock is small due to the strong magnetic field amplification and turbulence. The small diffusion allows us to neglect its direct effect on particle transport. Cosmic rays already accelerated by the shock subsequently undergo additional stochastic acceleration as they convect downstream from the shock. Their distribution function is governed by

$$\frac{Df}{Dt} = \frac{\partial f}{\partial t} + U_r\frac{\partial f}{\partial r} = \frac{\partial}{p^2\partial p}\left(D_{\rm pp}p^2\frac{\partial f}{\partial p}\right) + \frac{1}{3}\frac{\partial}{r^2\partial r}(r^2 U_r)p\frac{\partial f}{\partial p}. \tag{6}$$

Cosmic rays are injected at the shock at time t_0 with a distribution function $f_0(t_0, p) = Ap^{-\gamma}H(p_c(t_0) - p)$ with a shock spectral slope $\gamma = 3R/(R-1)$, a shock cutoff momentum p_c that can be taken from Equation (2) as function of injection time t_0, and a constant A which represents the intensity at the reference momentum 1 GeV$/c$.

Define $C = \frac{1}{3}\frac{\partial}{r^2\partial r}(r^2 U_r)$, where C is called adiabatic cooling rate. Normally the stochastic acceleration rate $D_0 = D_{\rm pp}/p^2$ should be a function of momentum p; however, if we just look at the particles within the range $VL_1 \ll \kappa \ll VL_0$, D_0 it should be close to a constant. Both C and D_0 can be a function of time t.

3.1. Constant Rates of Adiabatic Cooling and Stochastic Acceleration

For the purpose of demonstrating the overall behaviors of particle acceleration more clearly in an analytical way, we first study simple cases in which D_0 and C are constant. Equation (6) can be simplified to a homogeneous diffusion-convection equation with the change of variable $q = \ln(cp/1\ \mathrm{GeV})$,

$$\frac{Df}{Dt} = D_0\frac{\partial^2 f}{\partial q^2} + (3D_0 + C)\frac{\partial f}{\partial q} \tag{7}$$

with $f_0(t_0, q) = Ae^{-\gamma q}H(q_c - q)$ and $q_c = \ln(cp_c(t_0)/1\ \mathrm{GeV})$. The solution to the initial value problem at an arbitrary time $t > t_0$ in the reference frame comoving with the plasma is

$$f = \int_{-\infty}^{q_c} d\xi A e^{-\gamma\xi}\frac{1}{\sqrt{4\pi D_0\tau}}e^{-\frac{[q-\xi+(3D_0+C)\tau]^2}{4D_0\tau}}, \tag{8}$$

where $\tau = t - t_0$. It can be integrated, yielding

$$f = Ap^{-\gamma}\ e^{(\gamma D_0 - 3D_0 - C)\gamma\tau}\ \mathrm{erfc}(x_c)/2, \tag{9}$$

where $\mathrm{erfc}(x_c)$ is the complementary error function of $x_c = [q - q_c - (2\gamma D_0 - 3D_0 - C)\tau]/\sqrt{4D_0\tau}$.

Figure 1 shows an example of calculated spectra after a period $\tau = 1/C$ (cooling time) of stochastic acceleration in the downstream region. There are several important features that should be pointed out. Below approximately $x_c = 0$ or below a certain energy, all the spectra have a power law. The spectral slope is the same as the initially injected shock-accelerated particles independent of the chosen values of D_0 and C. Above $x_c = 0$ or at high energies the spectra fall off quickly in proportion to $\mathrm{erfc}(x_c)$. If we define the power-law spectra cutoff at $x_c = 0$, where the spectra fall below half the extrapolated power-law level, the new cutoff energy after stochastic acceleration and also adiabatic cooling becomes

$$E_c' = e^{(2\gamma D_0 - 3D_0 - C)\tau}E_c. \tag{10}$$

The intensity level of the power-law spectra can be higher or lower than the initial spectrum, with

$$A' = e^{(\gamma D_0 - 3D_0 - C)\gamma\tau}A, \tag{11}$$

which is the new intensity level at 1 GeV$/c$.

When $(\gamma - 3)D_0 > C$ (a case of strong stochastic acceleration), both the intensity of the entire spectrum and cutoff energy increase exponentially with the time τ. At

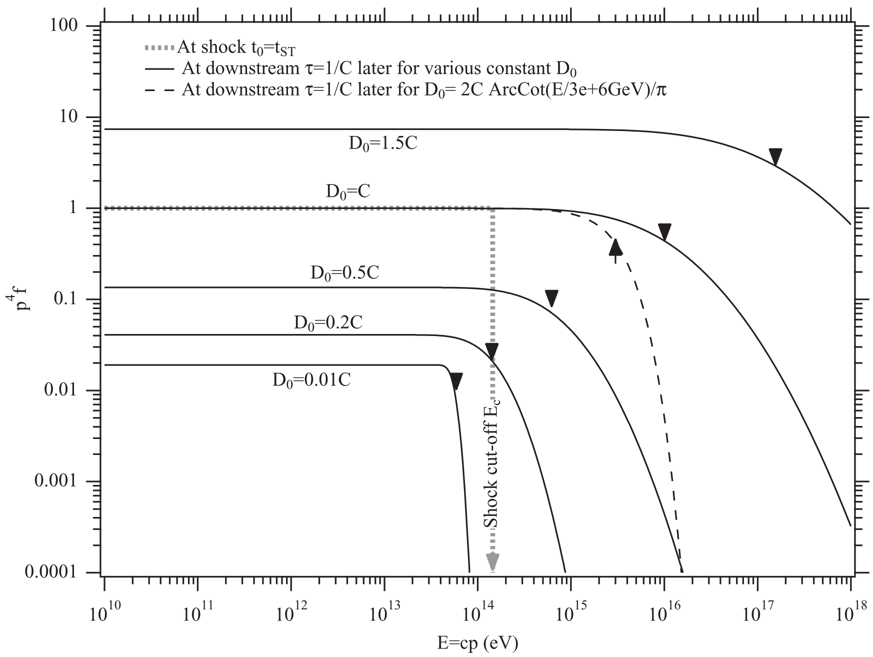

Figure 1. Normalized spectra $p^4f(p)$ of cosmic rays after a period (equal to one adiabatic cooling time) of stochastic acceleration downstream of the supernova shock for various constant values of stochastic acceleration rate D_0 (solid lines) and a momentum-dependent D_0 as given in Equation (5) (dashed line). For comparison, the initial spectrum of injected shock-accelerated cosmic rays is plotted as the dotted line. Cutoff energies where the spectra fall below 1/2 its corresponding power law are indicated by the arrows.

$(\gamma - 3)D_0 = C$ or $D_0 = C$ for $\gamma = 4$ (critical stochastic acceleration), the level of the bulk power-law spectrum does not change with the time, but the maximum cutoff energy increases with the time. In this case, the stochastic acceleration only pumps the lowest-energy particles to the highest energies, of course in a sequential manner through the entire range of energy. When $(2\gamma - 3)D_0 > C > (\gamma - 3)D_0$ or $5D_0 > C > D_0$ for $\gamma = 4$ (weak stochastic acceleration), the bulk power-law level decreases but the maximum cutoff energy still increases with the time. If $(2\gamma - 3)D_0 < C$ (adiabatic cooling dominating), both the spectrum level and maximum cutoff energy decrease.

3.2. *Critical Stochastic Acceleration and Balance through the Dynamics of Cosmic-Ray Pressure*

The rate of stochastic acceleration is determined by compressible plasma fluctuation. How does it fine-tune to be related to adiabatic cooling at least to a level within the same order of magnitude? The key is the pressure of accelerated particles that could affect the dynamics of plasma fluctuations. The supernova shock can convert ~20% of its energy into cosmic-ray energy, almost all of which is in relativistic particles due to a very hard spectrum. Multiplying Equation (6) by $\frac{4\pi}{3}cp^3dp$ and integrating over momentum p, we get an equation for the evolution of pressure $P = \int \frac{4\pi}{3}cp^3 f(p)dp$:

$$\frac{DP}{Dt} = 4(D_0 - C)P, \tag{12}$$

where the solution of P grows or decays exponentially with the time depending on the sign of $D_0 - C$.

If the compressible fluctuations start strong enough so that $D_0 - C > 0$, the cosmic-ray pressure will grow quickly due to particle acceleration across the entire energy spectrum. Since the cosmic rays behind the supernova shock already contain a significant fraction of the total pressure, the increased pressure will have strong effects on the dynamics of plasma and magnetic field. According to the WKB approximation, fluctuation amplitude decreases while wavelength increases as the total pressure increases (Zhang & Lee 2013). No matter what the detailed dependence is, the rate of stochastic acceleration will decrease, eventually to a level of $D_0 = C$, a point where the cosmic-ray pressure does not grow any longer. It is the point at which the stochastic acceleration is at a critical balance with the adiabatic cooling. This point is also what we call the pressure balance condition (Zhang & Schlickeiser 2012; Zhang & Lee 2013). This rate is a quite slow as the adiabatic cooling time is approximately equal to its age since the initial supernova explosion, so such a condition should not be too difficult to reach.

If the initial compressible fluctuations are not large enough, the pressure of cosmic rays will decrease at an exponential rate of $4(C - D_0)$. In the meantime, the pressure of nonrelativistic thermal plasma decays at a faster exponential rate of $5C$ owing to their high ratio of heat capacities and non-participation in the stochastic acceleration process. It makes the system more and more controlled by the cosmic-ray dynamics. Sound speed decreases and the fluctuations lean toward being more supersonic. We suspect that more plasma compression will occur, which can drive up stochastic particle acceleration until eventually reaching its critical point.

Another benefit of the critical stochastic acceleration is that the acceleration and cooling work together to give an eigensolution of p^{-4} distribution for the relativistic cosmic rays. Continuously injected fresh sources of particles during transport in the downstream medium will be accelerated to this distribution, which could blend into the shock-injected cosmic rays without a spectral slope change.

Let us estimate the condition of compressible plasma fluctuations under which the critical stochastic acceleration rate is achieved. Equalizing $D_0 = \pi \langle u^2 \rangle / 9\, VL_0$ and $C = \frac{1}{3}\frac{\partial}{r^2 \partial r}(r^2 U_r)$ behind the supernova shock, we arrive at:

$$\frac{L_0}{r} = \frac{\pi \left\langle u^2 \right\rangle}{6\, VU_r}. \tag{13}$$

Immediately behind the supernova shock, $V \approx \sqrt{5}/4U_1$ and $U_r = U_1 - U_2 = 0.75U_1$, and then $L_0/r = 1.25\langle u^2 \rangle / U_1^2$. If we take a conservative value of $\langle u^2 \rangle / U_1^2 = 0.01$, then $L_0/r = 0.0125$, i.e., the longest wavelength of compressible plasma fluctuations is about 1% of the shock radius. Since the sound waves in the medium downstream of supernova shocks are mainly driven by cosmic-ray pressure, generating such a long wavelength probably should not be a big problem. Perhaps, the quasi-periodic stripes seen in *Chandra* high-resolution X-ray images of Tycho's supernova remnant are a signature of sound waves in critical balance with cosmic rays in the shocked medium (Eriksen et al. 2011).

3.3. Time-dependent Rates of Adiabatic Cooling and Stochastic Acceleration

The adiabatic cooling rate C changes with time as the supernova shell expands. We assume that the plasma speed inside the shell does not change with the radial distance. Then the cooling rate is $C = 2U_r/3r$. The downstream plasma speed inside the shell has the same speed as the plasma immediately behind the shock $U_r = U_1 - U_2 = 0.75U_1$. In the free expansion phase, $r = U_{10}t_0 + U_r(t - t_0) = U_{10}(0.75t + 0.25t_0)$ with $U_1 = 4U_2 = U_{10}$, and then $C = 2/(3t + t_0)$. In the Sedov–Taylor phase, $r = U_{10}t_{\rm ST}(t_0/t_{\rm ST})^{2/5} + \int_{t_0}^{t} U_r dt$ with $U_1 = 4U_2 = U_{10}(t/t_{\rm ST})^{-3/5}$, and then $C = 4/(15t - 7t_0^{2/5}t^{3/5})$. Either way, let us write it as $C = 1/T_c(t)$. Near the critical stochastic acceleration condition, $D_0 = D_{00}/T_c(t)$, where D_{00} is close to 1.

Substituting D_0 and C into Equation (7), we get

$$T_c(t)\frac{Df}{Dt} = D_{00}\frac{\partial^2 f}{\partial q^2} + \left(3D_{00} + 1\right)\frac{\partial f}{\partial q}. \tag{14}$$

If we change the time variable to $\eta = \int T_c(t)^{-1}dt$, the equation is again a homogeneous diffusion-convection equation. The solution to its initial value problem can be obtained in the same fashion:

$$f = Ap^{-\gamma}\ e^{(\gamma D_{00} - 3D_{00} - 1)\gamma\eta}\ \mathrm{erfc}(y_c)/2, \tag{15}$$

where $y_c = [q - q_c - (2\gamma D_{00} - 3D_{00} - 1)\eta]/\sqrt{4D_{00}\eta}$. Setting $y_c = 0$, we obtain a new maximum cutoff energy,

$$E_c' = e^{(2\gamma D_{00} - 3D_{00} - 1)\eta}E_c(t_0) \tag{16}$$

$$\eta = \begin{cases} \ln\left(\dfrac{3t + t_0}{4t_0}\right)^{2/3} & : t_0 < t < t_{\rm ST} \\ \ln\left(\dfrac{3t_{\rm ST} + t_0}{4t_0}\right)^{2/3} + \ln\left(\dfrac{15t^{2/5} - 7t_{\rm ST}^{2/5}}{8t_{\rm ST}^{2/5}}\right)^{2/3} & : t_0 < t_{\rm ST} < t \\ \ln\left(\dfrac{15t^{2/5} - 7t_0^{2/5}}{8t_0^{2/5}}\right)^{2/3} & : t > t_0 > t_{\rm ST}. \end{cases} \tag{17}$$

Substitute E_c of Equation (2) into (16), and we have:

$$E_c' = \begin{cases} ZeB_1U_{10}^2\left(1 - \dfrac{1}{R}\right)t_0 \times \left[\left(\dfrac{3t + t_0}{4t_0}\right)^{2/3}\right]^{2\gamma D_{00} - 3D_{00} - 1} & : t_0 < t < t_{\rm ST} \\ ZeB_1U_{10}^2\left(1 - \dfrac{1}{R}\right)t_0 \times \left[\left(\dfrac{3t_{\rm ST} + t_0}{4t_0}\right)^{2/3}\left(\dfrac{15t^{2/5} - 7t_{\rm ST}^{2/5}}{8t_{\rm ST}^{2/5}}\right)^{2/3}\right]^{2\gamma D_{00} - 3D_{00} - 1} & : t_0 < t_{\rm ST} < t \\ ZeB_1U_{10}^2\left(1 - \dfrac{1}{R}\right)t_{\rm ST}^{6/5}t_0^{-1/5} \times \left[\left(\dfrac{15t^{2/5} - 7t_0^{2/5}}{8t_0^{2/5}}\right)^{2/3}\right]^{2\gamma D_{00} - 3D_{00} - 1} & : t > t_0 > t_{\rm ST}. \end{cases} \tag{18}$$

The terms inside the square brackets are the amount of the boost of maximum cutoff energy from the stochastic acceleration downstream of the supernova shock.

3.4. Maximum Cutoff Energy

The cosmic-ray production rate is proportional to $4\pi R_{\rm sh}(t_0)^2 U_1(t_0)$, which rises as $\propto t_0^2$ in the free expansion phase with $R_{\rm sh} = U_{10}t_0$ and $\propto t_0^{1/5}$ in the Sedov–Taylor phase. On the other hand, the maximum cutoff energy by the shock in Equation (2) peaks at $t_{\rm ST}$. The most likely maximum cutoff energy of all cosmic rays produced by a supernova shell is from those particles initially injected into the shock at the time around $t_{\rm ST}$ and then subsequently got further boosted in energy by the stochastic acceleration downstream of the shock. Using the third equation in (18), we find that the maximum cutoff energy E_c' is 58 times $E_c(t_{\rm ST})$ or 10^{16} eV for protons at time $t = 10t_{\rm ST}$, with the critical stochastic acceleration rate $D_{00} = 1$. Even at $t = 5t_{\rm ST}$, E_c' is $14E_c(t_{\rm ST})$ or 3×10^{15} eV. Therefore, from the aspect of particle acceleration time limitation we conclude that supernova shells should be able to accelerate cosmic rays to the knee energy. Since the acceleration to the highest energy is done by stochastic acceleration in the downstream medium, we do not require strong magnetic field amplification in the upstream interstellar medium.

On the other hand, our theory does require magnetic field amplification beyond the regular shock compression in the downstream region. According to Equation (5), if κ is too large $\kappa \gg VL_0 \gg VL_1$ at high energies, the stochastic acceleration rate falls off quickly as $D_0 = 2\langle u^2\rangle/9\kappa$. Using the Bohm limit for κ, we can obtain the maximum energy for an efficient stochastic acceleration to take place:

$$E_c'' = 3ZeB_2VL_0, \tag{19}$$

where B_2 is the magnetic field strength in the downstream medium. Let us take $B_2 = 10^3\,\mu$G, sound speed $V = 5600\,\mathrm{km\,s^{-1}}$, and $L_0 = 0.06$ pc (10^{-2} of a 6 pc shock radius at $t = 10t_{\mathrm{ST}}$). That will give a $E_c'' = 3 \times 10^{15}$ eV for protons. The gyroradius of 3×10^{15} eV protons, a scale size for the particles to be confined, is 3×10^{-3} pc in a $10^3\,\mu$G magnetic field, which is well below L_0.

4. DISCUSSION

In the above analysis, we have made two major assumptions without which our solution could change significantly. One of them is the spectrum of velocity fluctuation, and the other is the neglect of spatial diffusion in the cosmic-ray transport Equation (6). Now let us examine these assumptions in more detail.

First, it should be emphasized that the velocity fluctuations that enter into the calculation of the particle momentum diffusion coefficient in (3) are only the longitudinal part, in which the variation of plasma velocity is parallel to the wavenumber vector. Incompressible shear or Alfvén waves/turbulence do not contribute. In fact, MHD simulations (e.g., Cho & Lazarian 2003) show that the incompressible fluctuations dominate in most space plasmas. Other simulations that do not distinguish modes of compression would get a spectrum mainly for the incompressible ones. Notice that the magnetic field variation of compressible fluctuations does not linearly correlate with plasma velocity variation, so the spectrum of velocity fluctuations should have a different shape than the magnetic field fluctuation spectrum. Cho & Lazarian (2003) found a $k^{-3/2}$ spectrum in the inertial range for the magnetic field fluctuations of the fast magnetosonic mode, but its velocity fluctuation spectrum appears to be much steeper. According to our examination of Cho & Lazarian's (2003) results, the spectrum of fast-mode velocity fluctuations is very close to k^{-2} in the inertial range, and it becomes steeper at high k numbers. Slow mode fluctuations are also compressible. In Cho & Lazarian (2003) they are called pseudo-Alfvén waves. In high-β plasmas, slow-mode fluctuations have little compressibility. In the extremely high-β plasma downstream of a supernova shock, they probably do not exist anymore.

Let us examine what compressible fluctuations will do to the stochastic acceleration of energetic particles if their longitudinal velocity fluctuation spectrum in 1D is not $\propto k^{-2}$ but instead has a different power-law slope of $-2 + \alpha$, i.e.,

$$S(k) = (1 - \alpha)L_0^{-1+\alpha}\langle u^2\rangle k^{-2+\alpha}H\big(kL_0 - 1\big)H\big(1 - kL_1\big). \tag{20}$$

For a Kolmogorov spectrum (mainly for incompressible velocity fluctuations), $\alpha = 1/3$. Substituting (20) into (3), we have the following integration

$$D_{\mathrm{pp}} = p^2\frac{2(1-\alpha)\left\langle u^2\right\rangle}{9L_0^{1-\alpha}}\int_{1/L_0}^{1/L_1}\frac{\kappa k^{\alpha}dk}{V^2 + \kappa^2k^2}. \tag{21}$$

Under certain conditions of κ, the integration can be carried out analytically. The rate of stochastic acceleration can be written as:

$$D_0 = \frac{D_{\mathrm{pp}}}{p^2} \approx \begin{cases} \dfrac{2(1-\alpha)\left\langle u^2\right\rangle\kappa}{9(1+\alpha)V^2L_0^{1-\alpha}L_1^{1+\alpha}} & : \kappa \ll VL_1 \\ \dfrac{(1-\alpha)\pi\left\langle u^2\right\rangle}{9\cos(\pi\alpha/2)\kappa^{\alpha}\left(VL_0\right)^{1-\alpha}} & : VL_1 \ll \kappa \ll VL_0, \\ \dfrac{2\left\langle u^2\right\rangle}{9\kappa} & : \kappa \gg VL_0 \end{cases} \tag{22}$$

which is in general a function of p if the diffusion coefficient κ is momentum-dependent.

If we use the values from Section 3.4 for the parameters in Equation (22) on a Kolmogorov spectrum ($\alpha = 1/3$), we get $D_0 = C(E/3 \times 10^6\,\mathrm{GeV})^{-\alpha}$. The stochastic acceleration rate D_0 at low energies of GeV to TeV is much larger than the adiabatic cooling rate. In fact, the value is so large that the calculated distribution function there quickly approaches to very large numbers that only the computer can handle. In Figure 2, we just calculate for a few spectra of compressible fluctuations that are slightly different from k^{-2}. In the case $D_0 = C(E/3 \times 10^6\,\mathrm{GeV})^{-0.1}H_c(E)$, which corresponds to a $k^{-1.9}$ velocity fluctuation spectrum and has a high-energy cutoff $H_c(E)$ approximated by $\mathrm{ArcCot}(E/3 \times 10^6\,\mathrm{GeV})\ 2/\pi$, the distribution function at 10 GeV is enhanced by 10^9 times. Even if we reduce $\langle u^2\rangle$ by several orders of magnitude so that $D_0 = C(E/1 \times 10^3\,\mathrm{GeV})^{-0.1}H_c(E)$, we still see large enhancements of particles at low energies. To understand this behavior, we look at the integrand in Equation (21), which peaks at $k_m = \sqrt{\alpha/(2-\alpha)}\,V/\kappa$. It means that the stochastic acceleration of particles with a spatial diffusion coefficient κ broadly "resonates" with compressible waves at this wavenumber. A spectrum flatter than k^{-2} contains high levels of plasma fluctuation at large wavenumbers that accelerate low-energy particles much faster than the critical rate of adiabatic cooling, thus resulting in a rapid increase of the particle distribution function. These levels of particle intensity enhancement at low energies are impossible because they contain much higher pressure than the initial input energy density of the system, which is already dominated by the cosmic rays.

Because of the efficient mechanism of stochastic acceleration by compressible plasma fluctuations together with high levels of cosmic-ray intensity in supernova shells, the interaction between cosmic rays and compressible fluctuations must be tightly binding with a broad "resonance." When compressible fluctuations are generated, e.g., probably through conversion from incompressible fluctuations, their growth is limited or saturated if there are already enough particles involved in the acceleration process. Below the maximum cutoff energy, where the spectrum already contains ~20% of the total energy

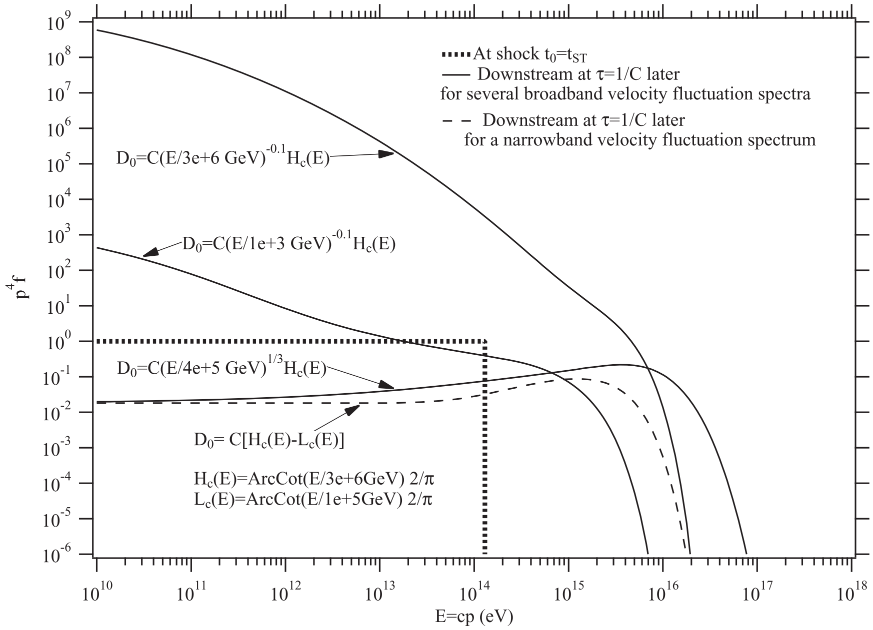

Figure 2. Normalized spectra $p^4f(p)$ of cosmic rays after a period of stochastic acceleration downstream of the supernova shock for several cases of D_0 based on different assumptions of compressible plasma fluctuation spectrum.

density, any increase of particle intensity will increase the cosmic-ray pressure proportionally. According to Equation (11), such an increase exponentially grows with the time if $D_0 > C$. If the growth of cosmic-ray pressure tends to damp the compressible fluctuations, then the fluctuations cannot grow easily. The compressible plasma fluctuation spectrum is upbounded by k^{-2} if the stochastic acceleration rate of cosmic rays at all energies must remain below the critical level.

The only region where compressible plasma fluctuations can grow more easily is at long wavelengths where there are few cosmic-ray particles to "resonate" with, i.e., above the maximum cutoff energy. It is these compressible fluctuations that "resonate" with the particles around the cutoff energy and push the cutoff energy higher and higher. There is a possibility the compressible fluctuations are significantly higher in a relatively narrow wavelength band than other wavelengths. Perhaps, the stripes of X-ray emissions seen in Tycho's supernova remnant are just an example. In Figure 2, we also show a calculated cosmic-ray spectrum resulting from a narrowband k^{-2} spectrum of compressible fluctuations that cuts off the stochastic acceleration slightly below the cutoff energy of the injected shock particle spectrum and above the knee energy with a $D_0 = 2C\,[\mathrm{ArcCot}(E/3 \times 10^6\,\mathrm{GeV}) - \mathrm{ArcCot}(E/1 \times 10^5\,\mathrm{GeV})]/\pi$. The spectrum is bumped up slightly from the power law between the shock cutoff energy ($\sim 1 \times 10^{14}$ eV) and the knee energy. Below $\sim 1 \times 10^{14}$ eV it has a power law of the original shock slope and its level is reduced due to adiabatic cooling. The location of the bump is particle species-dependent, which could lead to swift changes of cosmic-ray composition in the region as a function of energy. The bump probably is not as prominent if we sum up all cosmic rays injected during the entire lifetime of a supernova remnant.

Figure 2 also shows a cosmic-ray spectrum resulting from a $D_0 = C\,(E/4 \times 10^5\,\mathrm{GeV})^{1/3} H_c(E)$, which corresponds to a steeper $k^{-7/3}$ velocity fluctuation spectrum. The rest of the parameters used to derive this D_0 are taken from Section 3.4. The spectrum is slightly flatter than E^{-4} and extends beyond the knee energy, with a bump before its cutoff.

In all our calculations, we have neglected the spatial diffusion of cosmic rays in the downstream region. It does not mean that the particles do not diffuse. In fact, we rely on the spatial diffusion through random compressions and rarefactions to achieve the stochastic acceleration in momentum space. The spatial diffusion is dropped out only when we coarse-grain average the cosmic-ray transport over a distance greater than the particle diffusion length κ/U_2. Using the Bohm limit for κ, we found the diffusion length is $0.04(E/10^6\,\mathrm{GeV})\,\mathrm{pc}$. For comparison, $L_0 = 0.06$ pc in the above estimate. At energies significantly below the knee, the distance back to the shock is much longer than the diffusion length, so the neglect of spatial diffusion is justifiable. The spatial profile of cosmic ray density can be obtained by convective mapping of the temporal profile. If the stochastic acceleration rate is comparable to the adiabatic cooling rate, we would see a slower decrease (maybe even flat) of cosmic ray density away from the downstream of the shock than the pure adiabatic cooling predicts. It is only for those cosmic rays close to the knee energy that we need to consider their spatial diffusion in the downstream region. In this case, the shock acceleration and stochastic acceleration in the downstream medium are directly connected. The full solution to this problem needs to treat the spatial diffusion and momentum diffusion simultaneously. Above the knee energy, the spatial diffusion is too large to hold them in the acceleration region. It perhaps is another reason for the cutoff at the knee energy.

5. CONCLUSION

We have proposed a theory of 2-stage acceleration of Galactic cosmic rays by supernova remnants. Particles accelerated by the supernova shock front are further accelerated stochastically by compressible plasma fluctuations in the downstream region. Our calculations demonstrate that the stochastic acceleration can extend the maximum cutoff energy to the knee. If the compressible plasma fluctuations have a broad k^{-2} spectrum, the rate of stochastic acceleration is constant over a wide range of particle momentum. In this case, the accelerated cosmic-ray energy spectrum still has the same power-law slope as the shock spectrum up to the knee. It happens under a wide range of parameters as long as the stochastic acceleration rate is faster than one-fifth of the adiabatic cooling rate. A more preferred situation in the supernova shell is that the stochastic acceleration is in a critical balance with the adiabatic cooling, under which condition the maximum cutoff energy grows with the time while the bulk spectrum does not change. If the compressible plasma fluctuations have a steeper spectrum that concentrates its power in long wavelengths, the accelerated cosmic-ray energy spectrum will show a small bump before its cutoff at the knee. Our analysis also shows that in supernova shells it is impossible to have a considerably strong compressible plasma fluctuation spectrum flatter than k^{-2} extending to high wavenumbers; otherwise, it would accelerate low-energy cosmic rays so efficiently that it would violate energy conservation. Since the acceleration to the knee energy is done in the downstream region, the theory does not require magnetic field amplification in the upstream interstellar medium.

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